PHYSICS 110A : MECHANICS 1
PROBLEM SET #8 SOLUTIONS

[**1**] Starting with the Hamiltonian for a charged particle in an electromagnetic field,

$$H = \frac{1}{2m}\Big(\boldsymbol{p} - \frac{q}{c}\,\boldsymbol{A}(\boldsymbol{x},t)\Big)^2 + q\phi(\boldsymbol{x},t) \quad ,$$

use Hamilton's equations of motion to derive the Lorentz force law.

Solution :

We have

$$\dot x_i = +\frac{\partial H}{\partial p_i} = \frac{1}{m}\Big(p_i - \frac{q}{c}\,A_i(\boldsymbol{x},t)\Big)$$

$$\dot p_i = -\frac{\partial H}{\partial x_i} = -q\,\frac{\partial\phi(\boldsymbol{x},t)}{\partial x_i} + \frac{q}{mc}\sum_j \overbrace{\Big(p_j - \frac{q}{c}A_j(\boldsymbol{x},t)\Big)}^{m\dot x_i}\,\frac{\partial A_j(\boldsymbol{x},t)}{\partial x_i} \quad .$$

Now take the time derivative of $\dot x_i$ and multiply by m:

$$\begin{aligned}
m\ddot x_i &= \dot p_i - \frac{q}{c}\,\frac{dA_i}{dt} = \dot p_i - \frac{q}{c}\bigg(\frac{\partial A_i}{\partial t} + \sum_j \frac{\partial A_i}{\partial x_j}\,\dot x_j\bigg) \\
&= -q\,\frac{\partial\phi}{\partial x_i} - \frac{q}{c}\,\frac{\partial A_i}{\partial t} + \frac{q}{c}\sum_j \overbrace{\bigg(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\bigg)}^{\sum_k \epsilon_{ijk}B_k}\,\dot x_j \\
&= -q\,\frac{\partial\phi}{\partial x_i} - \frac{q}{c}\,\frac{\partial A_i}{\partial t} + \frac{q}{c}\sum_{j,k}\epsilon_{ijk}\,\dot x_j B_k \quad ,
\end{aligned}$$

which is the Lorentz force law $m\ddot{\boldsymbol{x}} = q\boldsymbol{E} + \frac{q}{c}\,\dot{\boldsymbol{x}}\times\boldsymbol{B}$ in component notation.

[**2**] A particle moves in an elliptical orbit in an inverse square force field. If the ratio of its maximum angular velocity to its minimum angular velocity is λ, show that the orbit has eccentricity

$$\varepsilon = \frac{\sqrt{\lambda}-1}{\sqrt{\lambda}+1} \quad .$$

Solution :

The shape of the orbit is given by

$$r(\phi) = \frac{r_0}{1+\varepsilon\,\cos\phi} \quad .$$

Thus, $r_{\rm max} = r_0/(1-\varepsilon)$ and $r_{\rm min} = r_0/(1+\varepsilon)$. We also know that $\ell = mr^2\dot\phi$ is conserved, hence $\dot\phi = \ell/mr^2$. Accordingly,

$$\lambda = \frac{\dot\phi_{\rm max}}{\dot\phi_{\rm min}} = \frac{\ell/mr^2_{\rm min}}{\ell/mr^2_{\rm max}} = \frac{r^2_{\rm max}}{r^2_{\rm min}} = \frac{(1+\varepsilon)^2}{(1-\varepsilon)^2} \quad ,$$

and thus

$$\frac{1+\varepsilon}{1-\varepsilon} = \sqrt{\lambda} \quad \Rightarrow \quad \varepsilon = \frac{\sqrt{\lambda}-1}{\sqrt{\lambda}+1} \quad .$$

[3] Two point particles of masses m_1 and m_2 interact via the central potential

$$U(r) = U_0 \, \ln\!\left(\frac{r^2}{r^2+b^2}\right) \ ,$$

where b is a constant with dimensions of length.

(a) For what values of the relative angular momentum ℓ does a circular orbit exist? Find the radius r_0 of the circular orbit. Is it stable or unstable?

(b) For the case where a circular orbit exists, sketch the phase curves for the radial motion in the $(r, \dot r)$ half-plane. Identify the energy ranges for bound and unbound orbits.

(c) Suppose the orbit is nearly circular, with $r = r_0 + \eta$, where $|\eta| \ll r_0$. Find the equation for the shape $\eta(\phi)$ of the perturbation.

(d) What is the angle $\Delta\phi$ through which periapsis changes each cycle? For which value(s) of ℓ does the perturbed orbit not precess?

Solution :

(a) The effective potential is

$$U_{\rm eff}(r) = \frac{\ell^2}{2\mu r^2} + U(r) = \frac{\ell^2}{2\mu r^2} + U_0 \, \ln\!\left(\frac{r^2}{r^2+b^2}\right) \quad .$$

where $\mu = m_1 m_2/(m_1 + m_1)$ is the reduced mass. For a circular orbit, we must have $U'_{\rm eff}(r) = 0$, or

$$\frac{l^2}{\mu r^3} = U'(r) = \frac{2U_0 b^2}{r\,(r^2+b^2)} \quad .$$

The solution is

$$r_0^2 = \frac{b^2\ell^2}{2\mu b^2 U_0 - \ell^2} \quad .$$

Since $r_0^2 > 0$, the condition on ℓ is

$$\ell < \ell_{\rm c} \equiv \sqrt{2\mu b^2 U_0} \quad .$$

For large r, we have

$$U_{\rm eff}(r) = \left(\frac{\ell^2}{2\mu} - U_0\, b^2\right) \cdot \frac{1}{r^2} + \mathcal{O}(r^{-4}) \quad .$$

Thus, for $\ell < \ell_{\rm c}$ the effective potential is negative for sufficiently large values of r. Thus, over the range $\ell < \ell_{\rm c}$, we must have $U_{\rm eff,min} < 0$, which must be a global minimum, since $U_{\rm eff}(0^+) = \infty$ and $U_{\rm eff}(\infty) = 0$. Therefore, the circular orbit is stable whenever it exists.

(b) Let $\alpha = \ell^2/\ell_{\rm c}^2$. The effective potential is then

$$U_{\rm eff}(r) = U_0\, f(r/b) \ ,$$

where the dimensionless effective potential is

$$f(s) = \frac{\alpha}{s^2} - \ln(1 + s^{-2}) \quad .$$

The phase curves are plotted in Fig. 1.

(c) The energy is

$$\begin{aligned} E &= \tfrac{1}{2}\mu\dot{r}^2 + U_{\rm eff}(r) \\ &= \frac{\ell^2}{2\mu r^4}\left(\frac{dr}{d\phi}\right)^2 + U_{\rm eff}(r) \ , \end{aligned}$$

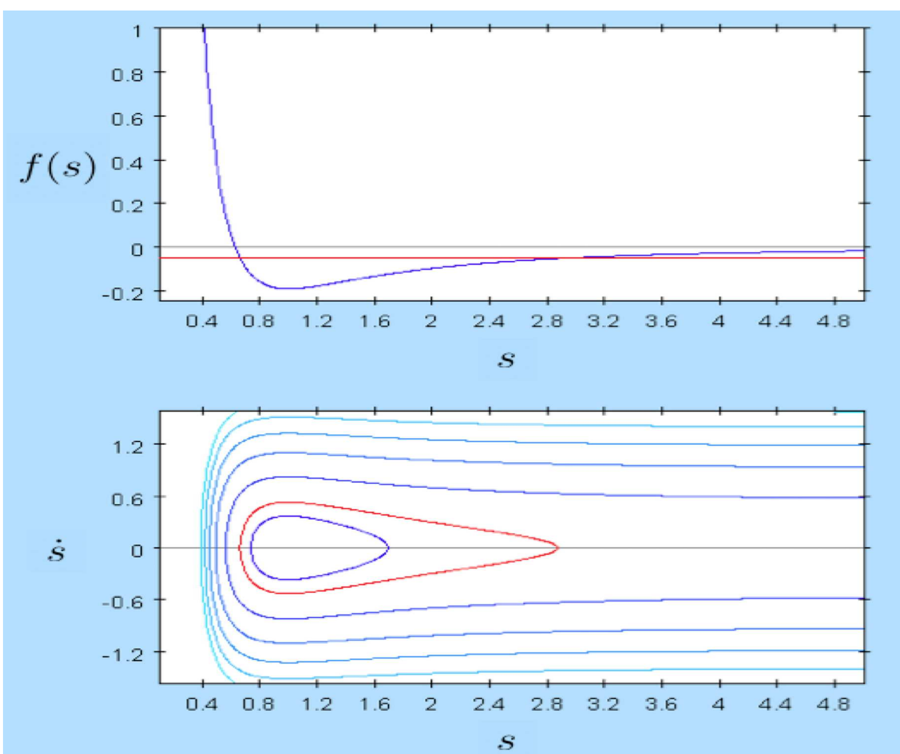

Figure 1: Phase curves for the scaled effective potential $f(s) = \alpha\, s^{-2} - \ln(1 + s^{-2})$, with $\alpha = \ell^2/\ell_{\rm c}^2 = 2^{-1/2}$. The dimensionless time variable is $\tau = t \cdot \sqrt{U_0/mb^2}$.

where we've used $\dot{r} = \dot{\phi}\, r'$ along with $\ell = \mu r^2 \dot{\phi}$. Writing $r = r_0 + \eta$ and differentiating E with respect to ϕ, we find

$$\eta'' = -\beta^2 \eta \qquad , \qquad \beta^2 = \frac{\mu r_0^4}{\ell^2}\, U''_{\rm eff}(r_0) \quad .$$

For our potential, we have $U_{\rm eff}(r) = U_0\, f(s)$ where $s = r/b$. Note that

$$f'(s) = -\frac{2\alpha}{s^3} + \frac{2}{s} - \frac{2s}{s^2+1}$$
$$f''(s) = \frac{6\alpha}{s^4} - \frac{2}{s^2} - \frac{2}{s^2+1} + \frac{4s^2}{(s^2+1)^2} \quad .$$

Setting $f'(s) = 0$ yields

$$s_0^2 = \frac{\alpha}{1-\alpha}$$

and furthermore we have

$$f''(s_0) = \frac{4(1-\alpha)^3}{\alpha} \quad .$$

We now have $U''_{\rm eff}(r_0) = U_0\, b^{-2}\, f''(s_0)$ and thus

$$\beta^2 = \frac{\mu b^4 s_0^4}{\ell^2} \times U_0\, b^{-2}\, f''(s_0) = 2(1-\alpha) \quad .$$

The solution is

$$\eta(\phi) = A\, \cos(\beta\phi + \delta) \quad .$$

where A and δ are constants.

(d) The change of periapsis per cycle is

$$\Delta\phi = 2\pi\big(\beta^{-1} - 1\big) \quad .$$

If $\beta > 1$ then $\Delta\phi < 0$ and periapsis *advances* each cycle (*i.e.*it comes sooner with every cycle). If $\beta < 1$ then $\Delta\phi > 0$ and periapsis *recedes*. For $\beta = 1$, which means $\ell = \sqrt{\mu b^2 U_0}$, there is no precession and $\Delta\phi = 0$.

[4] Two objects of masses m_1 and m_2 move under the influence of a central potential $U = k\,\big|\boldsymbol{r}_1 - \boldsymbol{r}_2\big|^{1/4}$.

(a) Sketch the effective potential $U_{\rm eff}(r)$ and the phase curves for the radial motion. Identify for which energies the motion is bounded.

(b) What is the radius r_0 of the circular orbit? Is it stable or unstable? Why?

(c) For small perturbations about a circular orbit, the radial coordinate oscillates between two values. Suppose we compare two systems, with $\ell'/\ell = 2$, but $\mu' = \mu$ and $k' = k$. What is the ratio ω'/ω of their frequencies of small radial oscillations?

(d) Find the equation of the shape of the slightly perturbed circular orbit: $r(\phi) = r_0 + \eta(\phi)$. That is, find $\eta(\phi)$. Sketch the shape of the orbit.

(e) What value of n would result in a perturbed orbit shaped like that in fig. 4?

Solution :

(a) The effective potential is

$$U_{\rm eff}(r) = \frac{\ell^2}{2\mu r^2} + kr^n \quad ,$$

with $n = \frac{1}{4}$. In sketching the effective potential, I have rendered it in dimensionless form,

$$U_{\rm eff}(r) = E_0 \, \mathcal{U}_{\rm eff}(r/r_0) \ ,$$

where $r_0 = (\ell^2/nk\mu)^{(n+2)^{-1}}$ and $E_0 = \left(\frac{1}{2} + \frac{1}{n}\right)\ell^2/\mu r_0^2$, which are obtained from the results

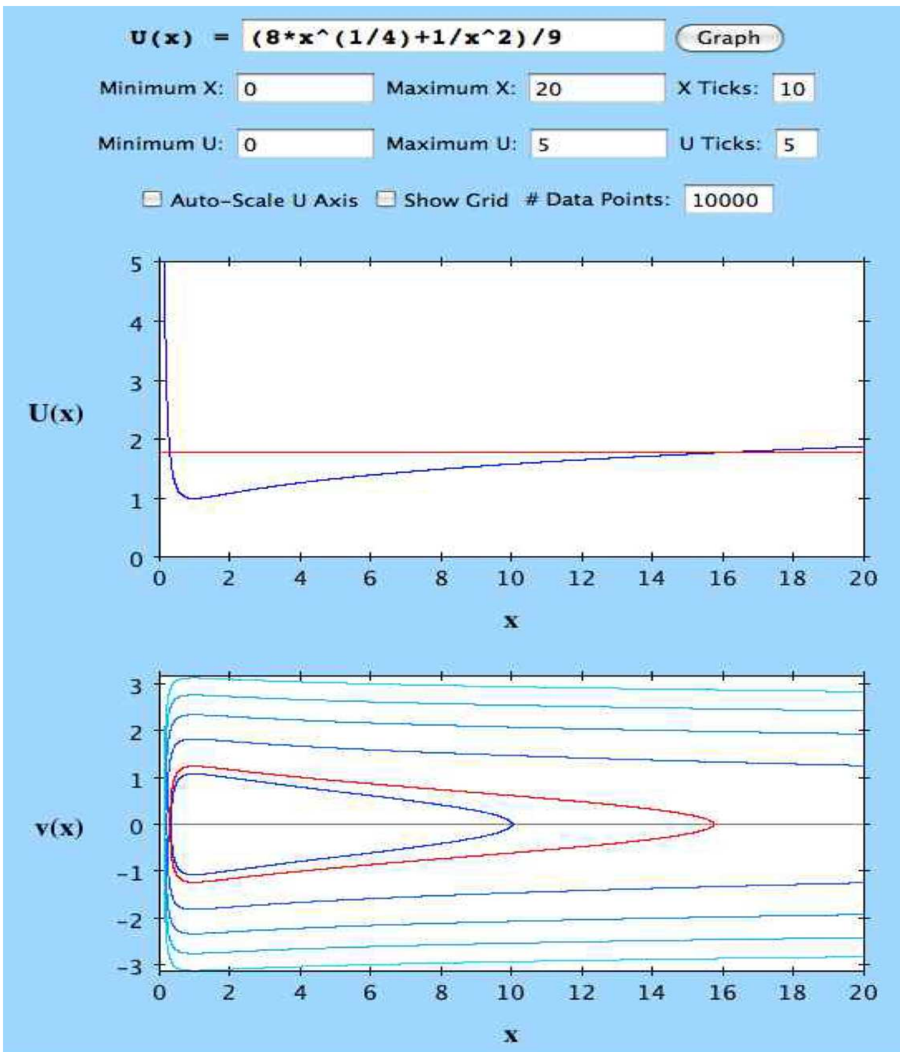

Figure 2: The effective $U_{\rm eff}(r) = E_0 \, \mathcal{U}_{\rm eff}(r/r_0)$, where r_0 and E_0 are the radius and energy of the circular orbit.

of part (b). One then finds

$$\mathcal{U}_{\text{eff}}(x) = \frac{n\,x^{-2} + 2\,x^n}{n+2} \quad .$$

Although it is not obvious from the detailed sketch in fig. 2, the effective potential does diverge, albeit slowly, for $r \to \infty$. Clearly it also diverges for $r \to 0$. Thus, the relative coordinate motion is bounded for all energies; the allowed energies are $E \geq E_0$.

(b) For the general power law potential $U(r) = kr^n$, with $nk > 0$ (attractive force), setting $U'_{\text{eff}}(r_0) = 0$ yields

$$-\frac{\ell^2}{\mu r_0^3} + nkr_0^{n-1} = 0 \quad .$$

Thus,

$$r_0 = \left(\frac{\ell^2}{nk\mu}\right)^{1/(n+2)} = \left(\frac{4\ell^2}{k\mu}\right)^{\frac{4}{9}} \quad .$$

The orbit $r(t) = r_0$ is stable because the effective potential has a local minimum at $r = r_0$, *i.e.* $U''_{\text{eff}}(r_0) > 0$. This is obvious from inspection of the graph of $U_{\text{eff}}(r)$ but can also be computed explicitly:

$$U''_{\text{eff}}(r_0) = \frac{3\ell^2}{\mu r_0^4} + n(n-1)kr_0^n = (n+2)\,\frac{\ell^2}{\mu r_0^4} \quad .$$

Thus, provided $n > -2$ we have $U''_{\text{eff}}(r_0) > 0$.

(c) From the radial coordinate equation $\mu\ddot{r} = -U'_{\text{eff}}(r)$, we expand $r = r_0 + \eta$ and find

$$\mu\ddot{\eta} = -U''_{\text{eff}}(r_0)\,\eta + \mathcal{O}(\eta^2) \quad .$$

The radial oscillation frequency is then

$$\omega = (n+2)^{1/2}\,\frac{\ell}{\mu r_0^2} = (n+2)^{1/2}\,n^{2/(n+2)}\,k^{2/(n+2)}\,\mu^{-n/(n+2)}\,\ell^{(n-2)/(n+2)} \quad .$$

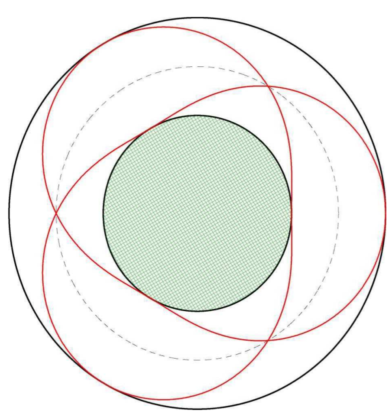

Figure 3: Radial oscillations with $\beta = \frac{3}{2}$

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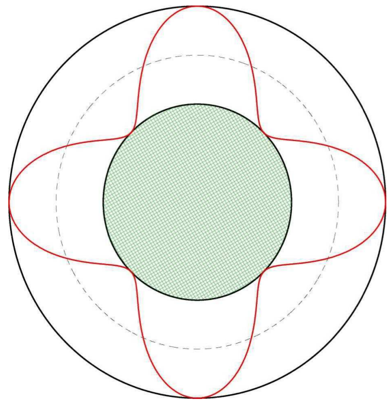

Figure 4: Closed precession in a central potential $U(r) = kr^n$.

The ℓ dependence is what is key here. Clearly $\omega'/\omega = (\ell'/\ell)^{(n-2)/(n+2)}$. In our case, with $n = \frac{1}{4}$, we have $\omega \propto \ell^{-7/9}$ and thus $\omega'/\omega = 2^{-7/9}$.

(d) We have that $\eta(\phi) = \eta_0 \cos(\beta\phi + \delta_0)$, with

$$\beta = \frac{\omega}{\dot{\phi}} = \frac{\mu r_0^2}{\ell} \cdot \omega = \sqrt{n+2} \quad .$$

With $n = \frac{1}{4}$, we have $\beta = \frac{3}{2}$. Thus, the radial coordinate makes three oscillations for every two rotations. The situation is depicted in fig. 3.

(e) Clearly $\beta = \sqrt{n+2} = 4$, in order that $\eta(\phi) = \eta_0 \cos(\beta\phi + \delta_0)$ executes four complete periods over the interval $\phi \in [0, 2\pi]$. This means $n = 14$.