**PHYSICS 110A : MECHANICS 1**
**PROBLEM SET #5 SOLUTIONS**

[**1**] Extremize the functional

$$F\big[y(x)\big] = \int\limits_0^{\ln 2} \! dx \, \Big(\tfrac{1}{2}y'^2 + ayy' + \tfrac{1}{2}y^2 + y\Big)$$

subject to the boundary conditions $y(0) = y_0$ and $y(\ln 2) = y_1$ .

Solution:

The variation of $F$ is

$$\delta F = \big(y' + ay\big)\,\delta y\,\bigg|_0^{\ln 2} + \int\limits_0^{\ln 2} \! dx \, \Big(-y'' + y + 1\Big)\,\delta y \ .$$

Note that $a$ is the coefficient of a total derivative, hence it does not appear in the kernel $\delta F/\delta y(x)$ for $0 < x < \ln 2$. We now set

$$y'' = y + 1 \qquad \Longrightarrow \qquad y(x) = A\,e^x + B\,e^{-x} - 1 \ ,$$

with $A$ and $B$ as yet unknown constants. The boundary conditions yield

$$\begin{aligned} y(0) &= A + B - 1 = y_0 \\ y(\ln 2) &= 2A + \tfrac{1}{2}B - 1 = y_1 \quad , \end{aligned}$$

the solution of which is

$$A = \tfrac{1}{3} - \tfrac{1}{3}\,y_0 + \tfrac{2}{3}\,y_1 \qquad , \qquad B = \tfrac{2}{3} + \tfrac{4}{3}\,y_0 - \tfrac{2}{3}\,y_1 \quad .$$

Working out the integral, we obtain the extremal value of $F$ as

$$F^* = \tfrac{3}{2}(1+a)A^2 + \tfrac{3}{8}(1-a)B^2 - aA + \tfrac{1}{2}aB - \tfrac{1}{2}\ln 2 \quad ,$$

with $A$ and $B$ given above.

[**2**] Extremize the functional

$$F\big[y(x), z(x)\big] = \int\limits_0^{\frac{\pi}{2}} \! dx \, \Big(y'^2 + z'^2 + 2yz\Big)$$

subject to the boundary conditions

$$y(0) = z(0) = 0 \qquad , \qquad y(\tfrac{\pi}{2}) = z(\tfrac{\pi}{2}) = 1 \ .$$

Solution:

We have

$$\delta F = \big(2y'\,\delta y + 2z'\,\delta z\big)\bigg|_0^{\pi/2} + \int\limits_0^{\frac{\pi}{2}} dx\,\Big(2\big(z-y''\big)\,\delta y + 2\big(y-z''\big)\,\delta z\Big)$$

Since the values of $y(x)$ and $z(x)$ are fixed at the endpoints, the first term above vanishes. Setting $\delta F = 0$ then yields the coupled ODEs

$$y'' = z \qquad , \qquad z'' = y \ .$$

Taking two derivatives of the first equation, we arrive at

$$y'''' = y \ ,$$

which has four solutions. Thus, we find

$$\begin{aligned} y(x) &= A\cosh(x) + B\sinh(x) + C\cos(x) + D\sin(x) \\ z(x) &= A\cosh(x) + B\sinh(x) - C\cos(x) - D\sin(x) \ . \end{aligned}$$

We now invoke the boundary conditions, which yield

$$\begin{aligned} A + C &= 0 \\ A - C &= 0 \\ A\cosh\big(\tfrac{\pi}{2}\big) + B\sinh\big(\tfrac{\pi}{2}\big) + D &= 1 \\ A\cosh\big(\tfrac{\pi}{2}\big) + B\sinh\big(\tfrac{\pi}{2}\big) - D &= 1 \ , \end{aligned}$$

the solution to which is $A = C = D = 0$ and $B = \operatorname{csch}\big(\frac{\pi}{2}\big)$. Thus,

$$y(x) = z(x) = \frac{\sinh x}{\sinh\big(\frac{\pi}{2}\big)} \ .$$

The integrand of $F$ is then

$$y'^2 + z'^2 + 2yz = \frac{2\cosh(2x)}{\sinh^2\big(\frac{\pi}{2}\big)} \ ,$$

and the extremal value of $F$ is

$$F = \int\limits_0^{\pi/2} dx\, \frac{2\cosh(2x)}{\sinh^2\big(\frac{\pi}{2}\big)} = 2\operatorname{ctnh}\big(\tfrac{\pi}{2}\big) \ .$$

[**3**] Derive the equations of motion for the Lagrangian

$$L = e^{\gamma t}\Big[\tfrac{1}{2}m\dot{q}^2 - \tfrac{1}{2}kq^2\Big] \ ,$$

where $\gamma > 0$. Compare with known systems. Rewrite the Lagrangian in terms of the new variable $Q \equiv q\,\exp(\gamma t/2)$, and from this obtain a constant of the motion.

Solution:

We have

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}\,e^{\gamma t} \qquad , \qquad F = \frac{\partial L}{\partial q} = -kq\,e^{\gamma t} \ .$$

Newton then says

$$\dot{p} = F \qquad \Rightarrow \qquad m\ddot{q} + \gamma m\dot{q} = -kq \ ,$$

which is the equation of a damped harmonic oscillator. The phase curves all collapse to the origin, which is a stable spiral if $\gamma < 2\sqrt{k/m}$ and a stable node if $\gamma > 2\sqrt{k/m}$.

In general, there is no reason for there to be a conserved quantity in a dissipative system like this, but consider the coordinate transformation $Q \equiv q\,\exp(\gamma t/2)$, which is inverted trivially to yield $q = Q\,\exp(-\gamma t/2)$. We have

$$\dot{q} = \big(\dot{Q} - \tfrac{1}{2}\gamma\,Q\big)\,e^{-\gamma t/2}$$

and therefore

$$\begin{aligned} L &= \tfrac{1}{2}m\,\big(\dot{Q} - \tfrac{1}{2}\,\gamma\,Q\big)^2 - \tfrac{1}{2}kQ^2 \\ &= \tfrac{1}{2}m\,\dot{Q}^2 - \tfrac{1}{2}\gamma\,m\,Q\,\dot{Q} - \tfrac{1}{2}\big(k - \tfrac{1}{4}m\gamma^2\big)\,Q^2 \ . \end{aligned}$$

Since $L(Q,\dot{Q},t)$ is independent of $t$, we have that $H$ is conserved:

$$\begin{aligned} H &= \dot{Q}\,\frac{\partial L}{\partial \dot{Q}} - L \\ &= \tfrac{1}{2}m\,\dot{Q}^2 + \tfrac{1}{2}\big(k - \tfrac{1}{4}m\gamma^2\big)\,Q^2 \ . \\ &= \Big[\tfrac{1}{2}m\dot{q}^2 + \tfrac{1}{2}\gamma m q\dot{q} + \tfrac{1}{2}kq^2\Big]\,e^{\gamma t} \ . \end{aligned}$$