PHYSICS 110A : MECHANICS 1
PROBLEM SET #4 SOLUTIONS

[1] An electrical circuit consists of a resistor $R$ and a capacitor $C$ connected in series to an emf $V(t)$.

(a) Write down the differential equation for the charge $Q(t)$ on one of the capacitor plates.

(b) Solve the homogeneous equation for $Q(t)$, i.e. find $Q(t)$ when $V(t) = 0$ subject to arbitrary initial value of $Q(0)$.

(c) Solve for the current $I(t)$ flowing in the circuit when $V(t) = V_0 \Theta(t)$. Assume $Q(0) = 0$.

(d) Solve for $I(t)$ when $V(t) = V_0 \sin(\Omega t) \Theta(t)$ and $Q(0)=0$.

For parts (c) and (d), you should use the Green’s function formalism in the time domain. The following integral may prove useful:

$$
\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega s}}{1 - i\omega \tau} = \frac{1}{\tau} e^{-s/\tau} \Theta(s) .
$$

Solution:

(a) Kirchoff’s law applied to the circuit yields $RI + C^{-1}Q = V(t)$, i.e.

$$
\dot{Q} + \frac{Q}{RC} = \frac{V(t)}{R} .
$$

We identify $RC \equiv RC$ as the time constant.

(b) Solving when $V(t) = 0$, we obtain

$$
Q(t) = Q(0) e^{-t/RC} .
$$

(c) Taking the Fourier transform of the equation in part (a), we have

$$
\hat{Q}(\omega) = \frac{C\hat{V}(\omega)}{1 - i\omega RC} .
$$

Thus, using the integral given in the problem statement, we have

$$
Q(t) = Q(0) e^{-t/RC} + \frac{1}{R} \int_0^t dt' V(t') e^{-(t-t')/RC} .
$$

For $V(t') = V_0 \Theta(t')$ and $Q(0) = 0$, we have

$$
Q(t) = CV_0 \left(1 - e^{-t/RC}\right) \Theta(t) .
$$
The current is then
\[ I(t) = \dot{Q}(t) = \frac{V_0}{R} e^{-t/RC} \Theta(t). \]

Another approach is to use an integrating factor:
\[ e^{-t/RC} \frac{d}{dt} [Q(t) e^{t/RC}] = \frac{V(t)}{R} \Rightarrow \frac{d}{dt} [Q(t) e^{t/RC}] = \frac{V(t)}{R} e^{t/RC}. \]

Now integrate:
\[ \int_0^t \frac{d}{dt'} \left[ Q(t') e^{t'/RC} \right] dt' = \frac{1}{R} \int_0^t \frac{d}{dt'} V(t') e^{t'/RC} dt' = \frac{V_0}{R} \int_0^t e^{t'/RC} dt' \]

\[ Q(t) e^{t/RC} - Q(0) = CV_0 \left( e^{t/RC} - 1 \right) \Theta(t), \]

which yields
\[ Q(t) = Q(0) e^{-t/RC} + CV_0 \left( 1 - e^{-t/RC} \right) \Theta(t). \]

(d) We have
\[ Q(t) = \frac{V_0}{R} \int_0^t \sin(\Omega t') e^{-(t-t')/\tau} dt' \]
\[ = \frac{V_0}{R} e^{-t/\tau} \text{Im} \int_0^t e^{(\tau-1+i\Omega) t'} dt' \]
\[ = CV_0 \text{Im} \left\{ \frac{1}{1+i\Omega \tau} [e^{i\Omega t} - e^{-t/\tau}] \right\} \]
\[ = CV_0 \left\{ \frac{\sin(\Omega t) - \Omega t \cos(\Omega t)}{1 + \Omega^2 \tau^2} + \frac{\Omega \tau}{1 + \Omega^2 \tau^2} e^{-t/\tau} \right\}. \]

[2] Do either of the following:

(a) A forced, damped harmonic oscillator obeys the equation of motion
\[ \ddot{x} + 2\beta \dot{x} + \omega_0^2 x = f_0 e^{-\gamma t} \Theta(t). \]

Compute \( x(t) \) assuming \( x(0) = \dot{x}(0) = 0. \)

(b) A forced, damped harmonic oscillator obeys the equation of motion
\[ \left( \frac{d}{dt} + \alpha \right) \left( \frac{d}{dt} + \beta \right) x = f_0 e^{-\gamma t} \Theta(t). \]

Compute \( x(t) \) assuming \( x(0) = \dot{x}(0) = 0. \)
Solution:

(a) The Green’s function is

\[ G(s) = \nu^{-s} e^{-\beta s} \sin(\nu s) \Theta(s) \, . \]

Thus,

\[ x(t) = f_0 \nu e^{-\beta t} \Im \left\{ \int_0^t e^{\beta t'} e^{-\gamma t'} \sin(\nu(t-t')) e^{-\gamma t'} \, dt' \right\} \]

(b) In fact the two equations are equivalent provided we identify \( 2\beta = \alpha + \tilde{\beta} \) and \( \omega_0^2 = \alpha \tilde{\beta} \), where \( \tilde{\beta} \) is our temporarily renamed \( \beta \) from part (b). But let’s try a different solution. At the level of differential operators, we have

\[ \frac{d}{dt} + \alpha = e^{-\alpha t} \frac{d}{dt} e^{\alpha t} \, , \]

which means

\[ \left( \frac{d}{dt} + \alpha \right) \left( \frac{d}{dt} + \beta \right) x = e^{-\alpha t} \frac{d}{dt} \left[ e^{\alpha t} (\dot{x} + \beta x) \right] \, . \]

Thus, our second order ODE may be recast as

\[ \frac{d}{dt} \left[ e^{\alpha t} (\dot{x} + \beta x) \right] = f_0 e^{(\alpha - \gamma)t} \Theta(t) \, . \]

Now replace \( t \) in the above equation by \( t' \) and integrate over the interval \( t' \in [0, t] \), resulting in

\[ e^{\alpha t} (\dot{x}(t) + \beta x(t)) - (\dot{x}(0) + \beta x(0)) = f_0 \int_0^t e^{(\alpha - \gamma)t'} = \frac{f_0}{\alpha - \gamma} \left( e^{(\alpha - \gamma)t} - 1 \right) \, . \]

This is equivalent to

\[ e^{-\beta t} \frac{d}{dx} \left[ e^{\beta t} x(t) \right] = \dot{x}(t) + \beta x(t) \]

\[ = \left( \beta x(0) + \dot{x}(0) \right) e^{-\alpha t} + \frac{f_0}{\alpha - \gamma} (e^{-\gamma t} - e^{-\alpha t}) \, . \]
Now multiply both sides by $e^{\beta t}$, send $t \to t'$, and then integrate over $t' \in [0, t]$, yielding

$$e^{\beta t} x(t) - x(0) = \left( \beta x(0) + \dot{x}(0) \right) \frac{e^{(\beta - \alpha) t} - 1}{\beta - \alpha} + \frac{f_0}{\alpha - \gamma} \left[ \frac{e^{(\beta - \gamma) t} - 1}{\beta - \gamma} \right] - \frac{f_0}{\alpha - \gamma} \left[ \frac{e^{(\beta - \alpha) t} - 1}{\beta - \alpha} \right],$$

which says

$$x(t) = \left( \frac{\beta e^{-\alpha t} - \alpha e^{\beta t}}{\beta - \alpha} \right) x(0) + \frac{e^{-\alpha t} - e^{-\beta t}}{\beta - \alpha} \dot{x}(0)$$

$$- \frac{f_0}{(\beta - \alpha)(\gamma - \beta)(\alpha - \gamma)} \left[ (\gamma - \beta) e^{-\alpha t} + (\alpha - \gamma) e^{-\beta t} + (\beta - \alpha) e^{-\gamma t} \right].$$

Using L'Hospital’s rule, one can check that the RHS remains finite in the limits $\alpha \to \beta$, $\beta \to \gamma$, and $\gamma \to \alpha$. With the initial conditions $x(0) = \dot{x}(0) = 0$ we have

$$x(t) = -\frac{f_0}{(\beta - \alpha)(\gamma - \beta)(\alpha - \gamma)} \left[ (\gamma - \beta) e^{-\alpha t} + (\alpha - \gamma) e^{-\beta t} + (\beta - \alpha) e^{-\gamma t} \right].$$