Thus, if \( m(\infty) > 0 > m(-\infty) \), we have normalizable solutions \( \Phi_1 \) and \( \Phi_4 \), while if \( m(\infty) < 0 < m(-\infty) \), we have normalizable solutions \( \Phi_2 \) and \( \Phi_3 \). The time-dependence is

(i) \( e^{iky} e^{-iE t/\hbar} = e^{iky(y-ct)} : \text{up-mover, } Y=+1 \)

(ii) \( e^{iky} e^{-iE t/\hbar} = e^{iky(y+ct)} : \text{down-mover, } Y=-1 \)

**Lecture 4 (Jan 14):** Adiabatic theorem and Berry's phase

Consider a Hamiltonian \( H(\vec{\lambda}) \) dependent on a set of parameters \( \vec{\lambda} = \{\lambda_1, ..., \lambda_K\} \), with eigenfunctions \( \{\Phi_n(\vec{\lambda})\} \):

\[
H(\vec{\lambda})|\Phi_n(\vec{\lambda})\rangle = E_n(\vec{\lambda})|\Phi_n(\vec{\lambda})\rangle
\]

Now let \( \vec{\lambda} = \vec{\lambda}(t) \) be time-dependent. The adiabatic theorem says that if \( \vec{\lambda}(t) \) evolves very slowly, such that \( \Delta E_n \cdot \tau \gg \hbar \), where \( \tau \) is the time scale of the variation, i.e. \( \tau = |\vec{\lambda}|/|\dot{\vec{\lambda}}| \), and \( \Delta E_n = E_{n+1} - E_n \) is the gap between consecutive levels, then the solutions to the time-dependent Schrödinger equation

\[
i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = H(\vec{\lambda}(t))|\Psi(t)\rangle
\]

are proportional to the instantaneous adiabatic WFs, with

\[
|\Psi_n(t)\rangle = e^{i\int_{t_0}^{t} E_n(\vec{\lambda}(t')) dt'} e^{-i\int_{t_0}^{t} E_n(\vec{\lambda}(t')) dt'} |\Phi_n(\vec{\lambda}(t))\rangle
\]
with corrections which vanish exponentially in $\Delta E_n t / \hbar$. We recognize the $\exp[ -i \hbar^{-1} \int dt' E_n(\lambda(t')) ]$ term as the dynamical phase accrued. What is $\gamma_n(t)$? Taking the time derivative and then the overlap with $\langle \Psi_n(\lambda) \rangle$, we find

$$\frac{d\gamma_n}{dt} = i \langle \Psi_n(\lambda(t)) | \frac{d}{dt} | \Psi_n(\lambda(t)) \rangle$$

$$\equiv \vec{A}_n(\lambda(t)) \cdot \frac{d\lambda(t)}{dt} = A_n(t)$$

where

$$A_n^\mu(\lambda) = i \langle \Psi_n(\lambda) | \frac{\partial}{\partial \lambda^\mu} | \Psi_n(\lambda) \rangle$$

is the Berry (or geometric) connection. If $\lambda(t)$ traverses a closed loop $C$ in the space of parameters, then $|\Psi_n(t)\rangle$ will accrue a geometric phase (also called Berry’s phase),

$$\gamma_n(C) = \oint_C d\lambda \cdot \vec{A}_n(\lambda)$$

We can also eliminate the dynamical phase entirely by defining $\tilde{H}_n(\lambda) \equiv H - E_n(\lambda)$. Then if

$$i \hbar \frac{\partial}{\partial t} |\tilde{\Psi}_n(t)\rangle = \tilde{H}_n(\lambda(t)) |\tilde{\Psi}_n(t)\rangle$$

in the adiabatic limit we have $|\tilde{\Psi}_n(t)\rangle = e^{i\gamma_n(t)} |\Psi_n(\lambda(t))\rangle$. Note that the geometric phase is invariant under time reparameterization and depends only on the path traversed by $\lambda$ in the parameter space manifold $M$. 

...
Mathematical setting: Hermitian line bundle over $M$. The parameter space manifold $M$ is the base space, and the adiabatic WFs $|\psi_n(\lambda)\rangle$ are the fibers, which are projections of a Hilbert space $\mathcal{H}$. The adiabatic theorem furnishes us with a definition of parallel transport of $|\psi_n(\lambda)\rangle$ along a curve $C$. The object $\tilde{A}_n(\lambda)$ is the connection and the Berry phase $\gamma_n(C)$ is the holonomy, which for a closed loop does not depend on the starting point. The curvature tensor of the bundle is given by

$$\Omega_n^{\mu\nu}(\lambda) = \frac{\partial A_n^\nu}{\partial \lambda^\mu} - \frac{\partial A_n^\mu}{\partial \lambda^\nu}$$

$$= i \left< \frac{\partial \psi_n}{\partial \lambda^\mu} \left| \frac{\partial \psi_n}{\partial \lambda^\nu} \right> - i \left< \frac{\partial \psi_n}{\partial \lambda^\nu} \left| \frac{\partial \psi_n}{\partial \lambda^\mu} \right> \right>$$
Using completeness of the adiabatic basis, we may write

$$\Sigma_{\mu}^{\nu}(\vec{x}) = i \sum_{l \neq n} \frac{\langle \Phi_l | \frac{\partial H}{\partial \lambda^\mu} | \Phi_n \rangle - \langle \Phi_n | \frac{\partial H}{\partial \lambda^\nu} | \Phi_l \rangle}{(E_l(\vec{x}) - E_n(\vec{x}))^2}$$

Note that the connection is gauge-covariant, viz.

$$\Phi_n(\vec{x}) \rightarrow e^{i f_n(\vec{x})} \Phi_n(\vec{x}) \Rightarrow A_n^\mu(\vec{x}) \rightarrow A_n^\mu(\vec{x}) - \frac{\partial f_n(\vec{x})}{\partial \lambda^\mu}$$

The curvature, however, is gauge-invariant. Can we fix a gauge and give an unambiguous definition of the connection $A_n^\mu(\vec{x})$? One way might be to choose a particular point in space $\vec{r}_0$ and demand $<\vec{r}_0 | \Phi_n(\vec{x})> \in \mathbb{R}_+$ for all $\vec{x} \in M$. But this prescription fails if there exists a value of $\vec{x}$ where $<\vec{r}_0 | \Phi_n(\vec{x})> = 0$.

The integral of the curvature $\Sigma_{n}^{12}(\vec{x})$ over a closed two-dimensional base space is a topological invariant. Using Stokes' theorem, we may turn an area integral of the curvature into a line integral of the connection:

$$\int_{\partial M} d^2\lambda \; \Sigma_{n}^{12}(\vec{x}) = - \sum_i \oint_{C_i} d\vec{\lambda} \cdot \vec{A}_n(\vec{x})$$

where $C_i$ encloses the $i$th singularity of $\vec{A}_n(\vec{x})$ in a
counterclockwise fashion ($M$ is assumed orientable). Singularities occur at points $\tilde{\lambda}_i$ where $|\Psi_n(\tilde{\lambda}_i)|$ is ill defined (using, for example, the prescription $<\tilde{\nu}_0 | \Psi_n(\tilde{\lambda})> \in \mathbb{R}_+^+$). In the vicinity of a given singularity, the connection has a vortex, behaving as

$$\tilde{A}_n(\tilde{\lambda}) = -q_i \Omega^2 \tan^{-1} \left( \frac{\lambda_2 - \lambda_i \tilde{\lambda}}{\lambda_1 - \lambda_i \tilde{\lambda}} \right) + \tilde{A}^{\text{reg}}_n(\tilde{\lambda})$$

which can be “unwound” by a singular gauge transformation,

$$|\Psi_n(\tilde{\lambda})> = e^{i q_i \cdot \Omega^2 (\tilde{\lambda} - \tilde{\lambda}_i)} |\tilde{\Psi}_n(\tilde{\lambda}_i)>$$

with $\Omega^2 (\tilde{\lambda} - \tilde{\lambda}_i) = \tan^{-1} (|\langle \lambda_2 - \lambda_i \tilde{\lambda}/(\lambda_1 - \lambda_i \tilde{\lambda}) \rangle|$. Thus,

$$C_n = \frac{1}{2\pi} \int_M d^2 \lambda \Omega^2(\tilde{\lambda}) = \sum_i q_i \in \mathbb{Z}$$

In mathematical parlance, $C_n$ is the Chern number of the Hermitian line bundle corresponding to the adiabatic wavefunction $|\Psi_n(\tilde{\lambda})>$. 

- Example: $S = \frac{1}{2}$ object in a magnetic field

  The Hamiltonian is

$$H(\vec{B}(t)) = g \mu_B \vec{B} \cdot \vec{\sigma} = g \mu_B B \begin{pmatrix} \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{i\phi} & -\cos \theta \end{pmatrix}$$
where $\hat{B}(t) = B(t) \hat{n}(t)$, and $\hat{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$. The adiabatic eigenfunctions are

$$|\psi_+ (\hat{n}) \rangle = \begin{pmatrix} u \\ v \end{pmatrix}, \quad |\psi_- (\hat{n}) \rangle = \begin{pmatrix} -\bar{u} \\ \bar{v} \end{pmatrix}$$

where $u = \cos \frac{\theta}{2}$ and $v = \sin \frac{\theta}{2} e^{i\phi}$. Then

$$H(\hat{B}) |\psi_\pm (\hat{n}) \rangle = \pm g\mu_B B$$

The connections are:

$$A_+ (t) = i <\psi_+ | \frac{d}{dt} |\psi_+ \rangle = i (\bar{u} \dot{u} + \bar{v} \dot{v}) = -\frac{1}{2} (1 - \cos \theta) \dot{\phi} = -\frac{1}{2} \dot{\omega}$$

$$A_- (t) = i <\psi_- | \frac{d}{dt} |\psi_- \rangle = i (\bar{v} \dot{v} + u \dot{u}) = +\frac{1}{2} (1 - \cos \theta) \dot{\phi} = +\frac{1}{2} \dot{\omega}$$

where $\text{d}w$ is the differential solid angle subtended by the path $\hat{n}(t)$. Thus $\gamma_\pm (C) = -\frac{1}{2} \omega_c$ is $\pm \frac{1}{2}$ times the solid angle subtended by $\hat{n}_C(t)$ on the Bloch sphere. We stress that $\gamma_\pm (C)$ is dependent only on the path of $\hat{n}$ itself and is time-reparameterization invariant. The components of the connections are then

$$A^\theta_\pm (\hat{n}) = 0, \quad A^\phi_\pm (\hat{n}) = \mp \frac{1}{2} (1 - \cos \theta)$$

and the curvature is

$$\Omega^\phi_\pm (\hat{n}) = \frac{\partial A^\phi_\pm}{\partial \theta} - \frac{\partial A^\theta_\pm}{\partial \phi} = \pm \frac{1}{2} \sin \theta$$
Integrating over the base space \( S^2 \),

\[
C_{\pm} = \frac{1}{2\pi} \int_{0}^{2\pi} \int_{0}^{\pi} \Omega_{\pm}^{\phi} (\theta, \phi) = \mp 1
\]

For any rank-2 Hamiltonian \( H(\vec{\lambda}) = \Delta(\vec{\lambda}) \hat{\mathbf{n}}(\vec{\lambda}) \cdot \vec{\sigma} + E_0(\vec{\lambda}) \mathbb{1} \) the Chern numbers of the two bands are given by

\[
C_{\pm} = \pm \frac{1}{4\pi} \int_{\mathcal{M}} d^2 \lambda \hat{\mathbf{n}} \cdot \frac{\partial \hat{\mathbf{n}}}{\partial \lambda_1} \times \frac{\partial \hat{\mathbf{n}}}{\partial \lambda_2}
\]

- **Two-band models**

  The base space for 2D lattice tight-binding models is the 2-torus \( T^2 \), coordinatized by \((\theta_1, \theta_2)\), where

  \[
  \vec{\kappa} = \frac{\theta_1}{2\pi} \vec{b}_1 + \frac{\theta_2}{2\pi} \vec{b}_2
  \]

  is the Bloch wavevector labeling the adiabatic eigenstates of the Hamiltonian \( H(\vec{\kappa}) \). We may then write

  \[
  \hat{H}(\vec{\kappa}) = E_0(\vec{\theta}) + \Delta(\vec{\theta}) \hat{\mathbf{n}}(\vec{\theta}) \cdot \vec{\sigma}
  \]

  \[
  \begin{cases}
  E_+ = E_0 + \Delta \\
  E_- = E_0 - \Delta
  \end{cases}
  \]

  and

  \[
  C_{\pm} = \pm \frac{1}{4\pi} \int_{0}^{2\pi} \int_{0}^{2\pi} \hat{\mathbf{n}} \cdot \frac{\partial \hat{\mathbf{n}}}{\partial \theta_1} \times \frac{\partial \hat{\mathbf{n}}}{\partial \theta_2}
  \]

  Note that \( C_+ + C_- = 0 \). For a Hamiltonian

  \[
  H = d_0(\vec{\theta}) + d(\vec{\theta}) \cdot \hat{\mathbf{n}}
  \]
we define
\[ \hat{d}(\vec{\theta}) = (\sin \theta \cos \chi, \sin \theta \sin \chi, \cos \theta) \]
where to avoid confusion with the Bloch phases we have taken \((\theta, \phi) \rightarrow (\theta, X)\). The adiabatic WFs are
\[
|\psi_+\rangle = \begin{pmatrix} \cos \frac{1}{2} X \\ \sin \frac{1}{2} V e^{i \chi} \end{pmatrix},
|\psi_-\rangle = \begin{pmatrix} -\sin \frac{1}{2} \theta e^{-i \chi} \\ \cos \frac{1}{2} V \end{pmatrix}
\]
The singularity occurs when \(\theta = \pi\), where \(X\) is ill-defined. We must then find all points \((\theta_1, \theta_2)\) in the \(B^2\) torus \(T^2\) such that \(\theta_1, \theta_2 = \pi\), and then compute the winding of \(S = -X\) as \(\vec{\theta}\) winds counterclockwise around \(\vec{\theta}_i\). (We have to define \(S = -X\) because the singularity is at the south pole on the Bloch sphere.) Two models:

(i) px+ipy superconductor:
\[
H(\vec{\theta}) = \begin{pmatrix} m - 2t \cos \theta_1 - 2t \cos \theta_2 & \Delta (\sin \theta_1 - i \sin \theta_2) \\ \Delta (\sin \theta_1 + i \sin \theta_2) & -m + 2t \cos \theta_1 + 2t \cos \theta_2 \end{pmatrix}
\]
\[ \hat{d}(\vec{\theta}) = (\Delta \sin \theta_1, \Delta \sin \theta_2, m - 2t \cos \theta_1 - 2t \cos \theta_2) \]
\[ = |\hat{d}| (\sin \theta \cos \chi, \sin \theta \sin \chi, \cos \theta) \quad \text{and} \quad d_0(\vec{\theta}) = 0 \]

(ii) Haldane honeycomb lattice model:
\[
d_0(\vec{\theta}) = -2t_2 \left[ \cos \theta_1 + \cos \theta_2 + \cos (\theta_1 + \theta_2) \right] \cos \phi \\
d_1(\vec{\theta}) = -t_1 \left( 1 + \cos \theta_1 + \cos \theta_2 \right) \\
d_2(\vec{\theta}) = t_1 (\sin \theta_1 - \sin \theta_2) \\
d_3(\vec{\theta}) = m - 2t_2 \left[ \sin \theta_1 + \sin \theta_2 - \sin (\theta_1 + \theta_2) \right] \sin \phi
Analysis: see lecture notes §4.4.6

Results:

\[
(i) \quad C_{\pm}(m) = \begin{cases} 
0 & \text{if } m < -4t \\
\pm 1 & \text{if } m \in [-4t, 0] \\
\mp 1 & \text{if } m \in [0, 4t] \\
0 & \text{if } m > 4t
\end{cases}
\]

(ii) phase diagram:

- Chern numbers for Hofstadter's model (square lattice):

\[
H = -t \begin{pmatrix}
2\cos \theta_2 & 1 & 0 & \cdots & e^{i\theta_1} \\
1 & 2\cos(\theta_2 + \frac{2\pi p}{t}) & 1 & \cdots & 0 \\
0 & 1 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
e^{-i\theta_1} & 0 & \cdots & 1 & 2\cos(\theta_2 + \frac{2\pi(4-p)\rho}{t})
\end{pmatrix}
\]

Recall Hofstadter's butterfly, showing the spectra vs. plaquette flux for the square lattice.
Each energy gap is associated with a Chern number. Plotting the Chern numbers in color yields the Avron-Hofstadter butterfly.
Honeycomb lattice Avron-Hofstadter butterfly:

- **Semiclassical dynamics of Bloch wavepackets**

The Hamiltonian is

\[ H(t) = \frac{1}{2m} \left( \vec{p} + \frac{e}{c} \vec{A}(\vec{r}, t) \right)^2 + V(\vec{r}) \]

with \( \vec{E} = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t} \) and \( \vec{B} = \vec{\nabla} \times \vec{A} \). We choose a gauge in which there is no scalar potential: \( \phi = 0 \). We are interested in describing the semiclassical evolution of Bloch wave packets.