(1) For the Hamiltonian
$$\hat{H}(t) = \hat{H}_0 - \sum_i \hat{Q}_i \phi_i(t),$$
the response to second order may be written
$$\langle \Psi(t) | \hat{Q}_i | \Psi(t) \rangle = \int_{-\infty}^{\infty} dt' \chi_{ij}(t, t') \phi_j(t') + \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dt'' \chi_{ijk}^{(2)}(t, t', t'') \phi_j(t') \phi_k(t'') + O(\phi^3).$$

Find an expression for the nonlinear response tensor $\chi_{ijk}^{(2)}(t, t', t'')$ in terms of the spectral properties of $\hat{H}_0$. Some hints:

- From the above expression for $\langle Q_i(t) \rangle$ you can assume $\chi_{ijk}^{(2)}(t, t', t'') = \chi_{ikj}^{(2)}(t, t'', t')$. Your final expression should honor this symmetry.

- To obtain the linear response tensor $\chi_{ij}(t, t')$, we computed the first functional variation,
$$\frac{\delta \langle \hat{Q}_i(t) \rangle}{\delta \phi_j(t')} = \left\{ -\frac{i}{\hbar} \langle \Psi(t_0) | \hat{U}^\dagger(t', t_0) \hat{Q}_j \hat{U}(t, t_0) | \Psi(t_0) \rangle \right\} \times \Theta(t - t') \Theta(t' - t_0) ,$$
with $t_0 \to -\infty$, and then set $\phi = 0$. To obtain the nonlinear response $\chi_{ijk}^{(2)}(t, t', t'')$, we must first functionally differentiate with respect to $\phi_k(t'')$. Since there are three appearances of $\hat{U}$ or $\hat{U}^\dagger$ in each of the above matrix elements, you should get six terms in all.

(2) Sketch the spread of particle-hole excitation frequencies, depicted for a $d = 3$ Fermi gas in Fig. 9.3 of the lecture notes, in dimensions $d = 2$ and $d = 1$.

(3) We previously saw how the static density susceptibility of the electron gas could be written as
$$\chi(q) = \frac{\tilde{\chi}(q)}{1 + \frac{4\pi e^2}{q^2} \tilde{\chi}(q)},$$
where $\tilde{\chi}(q)$ is the polarization function. We can extend this expression to dynamical response, viz.
$$\tilde{\chi}(q, \omega) = \frac{\tilde{\chi}(q, \omega)}{1 + \frac{4\pi e^2}{q^2} \tilde{\chi}(q, \omega)} .$$
Formally this may be taken as a definition of the dynamic polarization $\tilde{\chi}(q, \omega)$. In the random phase approximation (RPA), we replace $\tilde{\chi}(q, \omega) \to \chi^0(q, \omega)$, the noninteracting dynamic
susceptibility, i.e.

\[ \chi^0(q, t) = \frac{i}{\hbar V} \langle [\hat{n}(q, t), \hat{n}(-q, 0)] \rangle \Theta(t) \]

\[ \chi^0(q, \omega) = 2 \int \frac{d^3k}{(2\pi)^3} \frac{f_{k+q} - f_k}{\hbar \omega - \varepsilon(k + q) + \varepsilon(k) + i\epsilon} . \]

Using the RPA, you are invited to determine the plasmon dispersion for the two-dimensional electron gas with interactions \( u(r) = e^2/r \) at \( T = 0 \). Some hints:

– Find the 2D Fourier transform of the interaction potential, \( \hat{u}(q) \).

– Expand \( \chi^0(q, \omega) \) in a series in \( \omega^{-2} \).

– Locate the pole in the RPA response function, and thereby obtain the solution \( \omega(q) \) to order \( q^2 \).