PHU-HUST Lectures (2020-2021)

1. 'Modeling Dynamical Systems' for MFE Core? after Biophysics
   Applications

2. a useful (I hope) methods
   mini-course (mathematical knowledge)

3. Emphasis on non-linear dynamics
   space-time scales

4. Topics:
   \[ \Rightarrow \text{MFE} \]
   \[
   \begin{align*}
   \circ \text{i. Feedback loops} & \quad \text{predator-prey} \\
   \circ \text{ii. Transition evolution and propagation} & \quad \text{Fisher, FW} \\
   \circ \text{iii. Stress flow} & \\
   \circ \text{iv. Avalanches, kink, flux} (\text{conjectures} \rightarrow \text{analyses}) \\
   \circ \text{v. Non-linear diffusion and spreading} \\
   \circ \text{vi. Nematode viscosity and Cahn-Hilliard eqn}
   \end{align*}
   \]
Model reduction and renormalization — a look at the theory

History of Nonlinear Plasma Theory
(1-2 lectures)

EROS II
F.D. Frisch, Y. Amour

Sources:
- J.D. Murray, "Mathematical Biology"
- R. May, "Stability and Complexity in Model Ecosystems"
- F. Morrison, "The Art of Modelling Dynamic Systems"
- More coming
- Selected papers.
PhD-HUST lectures I

1) Feedback loops and Predator - Prey

Plan:

- OV and Motivation
  - Drift Wave + Shear Flow

system

- Constructing the System
  - Solving

- Lessons from Ecology (e.g., May)
  - General structure & constraint
  - Aside: Bigger Models, Stability & Complexity
    - General structure
    - Kolmogorov Theorem
      - Fixed pts. vs. cycles
      - Feedback
    - Time Delays -> Cycles

- Terrible Structure
1. Natural candidate for predator-prey model
2. L → H transition
3. Initial waves and Basin other way
4. Fundamental problem(s)
5. Logistic growth and K
6. Amplifying example
7. Fluctuating logistic, a simple
8. Fluctuating Environments and Physiological Approach
9. Fluctuating Environments and Physiological Approach
10. Fluctuating
11. Fronts
animal

- what?

\[
\frac{dH}{dt} = H \cdot F(H, P) \quad \text{equation}
\]

\[
\frac{dp}{dt} = p \cdot G(H, P)
\]

\[
P(e), H(e)
\]

can extend to 2m x 2m.

Lothe - Volterra

- what does it mean?

\[
H \rightarrow \mathbb{E} - \text{Fluctuation intensity, early}
\]

\[
\mathbb{E} \rightarrow \mathbb{U} = \mathbb{E}^2 - \text{mean square } \mathbb{E}
\]

\[
\mathbb{U} \rightarrow \mathbb{E}^2 \rightarrow \langle \mathbb{E} \rangle \quad \{ \text{avg. over thin layer} \}
\]

\[
\langle \mathbb{E} \rangle \rightarrow \langle \mathbb{U} \rangle \quad \{ \text{ignores optical evolution} \}
\]

- why?

2 populations

\rightarrow \text{self-regulation} \quad \text{(feedback)}

\rightarrow \text{symmetry constraints} \quad \Delta \text{ constructed}
1) self-regulation
   (Shear Flow)
   b) Turbulence

   a) shear stabilizes controls turbulence
      - everyone knows...
   b) turbulence drives flow via Reynolds others

\[ \frac{\partial}{\partial t} \langle \mathbf{u} \rangle = - \nabla \cdot \mathbf{J} \]
\[ \mathbf{J} = \frac{\partial}{\partial t} \mathbf{u} + \mathbf{u} \times \mathbf{\Omega} + \mathbf{f} \]
\[ \mathbf{D} \cdot \mathbf{J} = 0 \]

N.B. more precisely:
From vorticity eqn. \( \mathbf{D} \cdot \mathbf{J} = 0 \)
\[ \frac{\partial}{\partial t} \left( \frac{\partial^2}{\partial t^2} \langle \phi \rangle \right) = - \nabla \cdot \left( \mathbf{u} \nabla \langle \phi \rangle \right) + \ldots \]

\[ \text{net polarization change} \]
\[ \text{design} \]
\[ \text{Rosenbluth} \]
\[ \text{Hinton 97} \]
\[ \text{Ex B shear flow} \]

\[ \text{notar} \]
Reynolds force 1 0 y m.

\[ \langle \mathbf{u} \cdot \mathbf{D} \mathbf{u} \rangle = \Delta n \langle \mathbf{u} \cdot \mathbf{u} \rangle \]

HW: show this!

(G.I.) Taylor identity (100 y old)

Upshot:

- Reynolds stress, force \rightarrow polarisation charge flux

* - Thm: McIntyre & Wood: R. Wood

"PV mixing" [i.e. polarisation charge flux] \rightarrow direction of symmetry

\[ \Rightarrow B \cdot E \text{ generation.} \]

"How reconcile? \rightarrow feedback loop"
2) Symmetry (already encountered)

Why is $\mathcal{E} \mathcal{F}$ special? (P.O., I$^2$H '05)

- $\mathcal{E} \mathcal{F}$ is made of:
  - minimal inertia: easily excited
  - $k_{zz} \rightarrow 0$
  - $H = M \
onumber$
  - minimal transport (none)

- $n = 0 \Rightarrow \nabla n = 0$
  - $M = 0$
  - Can't relax DN, DT, etc.

- minimal damping: (no Landau damping - RH, 47)

Symmetry $\rightarrow P - P$
- Symmetry constraints + consideration of self-regulation

- Z. F. can grow / be excited only via 'feeding off' of turbulence, yet shear turbulence, thus regulating
  ct.

- Natural problem for predator-prey formulation!

- What does it look like? (Cf. PD, et. al., '94) Mixing length process
  - Fluidity
  - \( \frac{1}{2} \frac{dE}{dt} = \gamma_{DE} E - \chi_{E} E^{2} - \chi_{2, UE} \)
  - Growth turbulence
  - Self-stabilization
  - Coupling key
  - \( \frac{1}{2} \frac{dU}{dt} = -\mu U + \alpha_{2, UE} \)
  - Flow damping
  - (5) 2
→ N.B. - simplest

→ An infinity of extensions:
  most use.

\[ \delta_0 \rightarrow \delta_3 ( \delta_0 ) \rightarrow \delta_3 ( a / L_0 - a / ( \sqrt{\pi} \mu_0 ) ) \]

\[ \Delta \xi ( \sin \phi ) + \Delta \xi ( \sin \phi ) = n \rho = n ( \phi + \phi ) \]

\[ \Delta \xi ( \sin \phi ) + \Delta \xi ( \sin \phi ) = n \rho = n ( \phi + \phi ) \]

Molloy P.D. 2009

[Boxed: V_{m} P.D. 63]

\[ V_E = \frac{\delta P}{\lambda} + V_0 \]  
(Corner, Molloy)

→ Toroidal flow, particles, field

Users guide to Ann - Area extension
B.) How is it constructed?

Key terms: $\mathbb{X}_2U\mathbb{E} \rightarrow$ Reynolds Coupling

Point: Reynolds work transfers energy. Flow $\rightarrow$ Fluctuations

\[ \frac{\partial}{\partial t} \langle U_1 \rangle = - \nabla \cdot \langle \mathbf{u} \cdot \mathbf{V}_1 \rangle \]

\[ \frac{\partial}{\partial t} \int dV \langle \frac{\mathbf{V}_1^2}{2} \rangle = \int \langle \mathbf{u} \cdot \mathbf{V}_1 \rangle \Delta \langle \mathbf{u} \cdot \mathbf{V}_1 \rangle \]

But

\[ \frac{\partial}{\partial t} \mathcal{E}_l + \frac{\partial}{\partial V} \int dV \langle \frac{\mathbf{V}_1^2}{2} \rangle = 0 \]

Fact

So

\[ \frac{\partial}{\partial t} \mathcal{E}_l = - \int dV \langle \mathbf{u} \cdot \mathbf{V}_1 \rangle \Delta \langle \mathbf{u} \cdot \mathbf{V}_1 \rangle \]

Formally

\[ \frac{\partial}{\partial t} \langle \mathcal{E}_l \rangle = - \langle \mathbf{u} \cdot \mathbf{V}_1 \rangle \langle \mathbf{u} \cdot \mathbf{V}_1 \rangle \]
What of stress? : 

\[ \sigma = - \text{Eddy Telling Feedback Loop} \]

\[ \langle \sigma_{ij} \rangle = - \frac{c}{2} \sum_{\text{even } f, g} \langle k_x k_y \rangle \langle \phi_x \phi_y \rangle \]

\[ \text{Correlation} \]

\[ \text{Mode structure} \]

\[ \text{Shear flow can induce correlation!} \]

\[ \text{C.S. Law} \]

\[ \frac{d}{dt} \frac{\partial w}{\partial x} = -\frac{2}{3} (\omega + k \cdot \mathbf{v}) \]

\[ \frac{d}{dt} \frac{\partial \mathbf{v}}{\partial x} = -\kappa_0 \langle \mathbf{v} \rangle \]
\[ k_x = k_x^0 - k_0 \langle v_E \rangle t \]

Shearing coordinate

\[ t \sim \frac{1}{T_0} \]

\[ k_x = k_x^0 - k_0 \langle v_E \rangle T_0 \]

\[ \langle k_x k_y \rangle = \langle k_x^0 k_y^0 \rangle - k_0^2 \langle v_E \rangle T_0 \]

\[ \langle \Gamma_{\nu} \Gamma_{\perp} \rangle = \frac{e^2}{2} \sum \frac{\hbar \omega}{\hbar^2} \Gamma_{\nu} \Gamma_{\perp} \frac{1}{m^2} \frac{1}{2} \]
\[ q_t + \langle \delta E \rangle = - \sum_k \langle \delta u^2 \rangle \frac{\partial \langle v \rangle}{\partial x_k} \langle u \rangle \frac{1}{2} \]

\[ \sim - \left( \frac{\partial}{\partial x} \right) \langle u \rangle \langle v \rangle \]

Note: \( \langle \delta v \rangle^2 \) dependence

- Necessarily, \( \delta u \) \text{ and } \delta v \text{ must have opposite sign to conserve energy}

- \( \langle v \rangle^2 \) is natural \( \mathbf{u} \) variable.

- Basic predator–prey structure

- Flow–Fluctuation kinetic energy exchange

Alternate Perspective:

- Spectral Energy Evolution
- Recall fluctuations respond adiabatically to \( Z \).

\[ \omega \approx \omega^* \quad \text{vs.} \quad \omega \approx \bar{\omega} \]

Then, have for wave action density:

\[ N = \frac{E_k}{\omega} \quad \text{energy density} \]

\[ \frac{\partial}{\partial t} N + (\nabla \cdot \mathbf{v}) \cdot \nabla N - 2 \left( \frac{\partial}{\partial x} \left( \omega + 4 \pi \mathbf{v} \cdot \mathbf{v} \right) \right) \frac{\partial N}{\partial x} = \nabla \cdot \mathbf{r} \]

Relaxation

\[ \frac{\partial}{\partial t} \langle \mathbf{W} \rangle + \frac{2}{\omega} \left( -\frac{\partial}{\partial x} \left( \omega \mathbf{v} \cdot \mathbf{v} \right) \right) \mathbf{W} = \nabla \cdot \mathbf{r} \]

\[ \mathbf{W} \rightarrow \text{show} \]

and usual inelastic transfer, see (O, \( I^2 H \), '05)

\[ \frac{\partial}{\partial t} \langle N \rangle - \frac{\partial}{\partial x} \frac{D_n}{k_m} \frac{\partial}{\partial x} \langle N \rangle = \langle \varepsilon \nu \rangle \]

\[ D_n = \sum \frac{\kappa_n}{2} \frac{N_0^2 x^2}{L^2} \]

Induced diffusion

Random shear
Ray chaos

\[ \omega \sim \sqrt{\frac{m}{E}} \]

Note: \( \sqrt{\frac{m}{E}} \) wave

\[ \omega - i \omega' \approx C \omega - iv \]

\[ \omega \sim \sqrt{\frac{m}{E}} \]

\[ \sum_0^{10} \frac{v^2}{v^2} \frac{1}{\mu^2} \]

as \( \delta \rightarrow 0 \).

then:

\[ \langle \mathcal{E} \rangle = \omega \langle \mathcal{N} \rangle \]

\[ \langle \mathcal{E} \rangle = -\Delta \theta \frac{\partial}{\partial \theta} \frac{\partial}{\partial \varphi} \]

\[ = -\Delta \theta \frac{\partial}{\partial \theta} \frac{\partial}{\partial \varphi} \langle \mathcal{N} \rangle \]

\[ \frac{\partial}{\partial \theta} \langle \mathcal{N} \rangle \]

backward wave

fundamentally

\( \omega \sim \sqrt{\frac{m}{E}} \)

\( \text{Same structure} \)

\[ \omega \sim \sqrt{\frac{m}{E}} \]

HW

work out correspondence
Aside

Lessons From Ecology (cf. May)

What can be said about this type of system?

1 predator - 1 prey

General:

\[ \frac{dH}{dt} = H \cdot F(H, P) \]
\[ \frac{dP}{dt} = P \cdot G(H, P) \]

Lotka-Volterra breeding (Linear)

\[ \frac{dH}{dt} = \alpha H - \alpha HP \]
\[ \frac{dP}{dt} = -dP + \alpha HP \]

Mathies

Linear Osc

Linear oscillation / growth tendency → Show
\[ \frac{1}{2} \frac{dE}{dt} = 8.5 E - \frac{1}{2} a E^2 - d_2 UE \]

\[ \frac{1}{2} \frac{dU}{dt} = -u U + d_2 UE \]

\text{Shear Flow} \rightarrow \text{predator}\]

\[ \text{For } 1 \text{ pred., } 1 \text{ prey:} \]

\[ \frac{dH}{dt} = H \left( F(3, P) \right) \]

\[ \frac{dP}{dt} = P \left( G(3, H) \right) \]

\text{Kolmogorov Thm. (1936), based on Poincaré-Bendixon Theorem.}
Predator-prey systems of form \( \frac{dP}{dt} = \frac{r}{K} P \) have either:

- a stable equilibrium point
- a stable limit cycle

\( P, \theta \) continuous, with continuous first derivative. Characterize non-linear system.

and

\( \frac{dF}{dP} \leq 0 \rightarrow \text{rate of prey increase decrease on pred.} \)

\( \frac{dF}{dP} \geq 0 \rightarrow \text{prey growth decrease with population size} \)

\( \frac{dF}{dP} = 0 \rightarrow \text{rate of change} \)

\( \frac{dP}{dF} \rightarrow \text{predators decrease with population size of pred.} \)
\[ \frac{dH}{dt} + p\frac{dS}{dp} \geq 0 \]

- Predator is carrying function of population size.

- \( F(0, 0) > 0 \)

- Prey have positive growth rate (linear for small population).

- \[ \frac{dH}{dt} = F(H) \]

- \[ F = \frac{AB}{1 + B/C} \]

- \( F(0, A) = 0 \), with \( A > 0 \)

- Can have predator population to stop further prey increase for low levels prey.

- Predation induced death.

- \( F(B, 0) = 0 \), with \( B > 0 \)

- Critical prey population beyond which prey cannot increase even absent predators.
\[
\frac{\partial X}{\partial t} = -X \left[ a + \frac{X}{p} - B X \right]
\]

\[B = \frac{\partial g}{\partial X}\]

\(\text{need} \quad \sigma(C, 0) = 0 \quad \text{with} \quad C > 0\)

There exists a critical prey size \(C\) that stops further increase in predators.

\[C = \frac{n}{d}\]

\(B > C\), \(\text{or system collapses}\).

\(\text{need} \quad \frac{\partial g}{\partial X} > 0, \frac{X}{p} > 1\), \(\text{or flows collapse}\).

\(\sigma_{ij}^{(v)} = (v_{ij}) \quad \text{satisfied} \quad \Rightarrow\quad \text{system will have fixed point on LCO solution structure!}\)
N.B.: 

(1) \( \Rightarrow \) (2): are criteria for sustainable, sensible, system

\[ \Rightarrow \text{some can be } \geq, \leq \]

\[ \Rightarrow \text{criteria for (stationary state) equilibrium} \]

\[ \text{i.e. Fixed point} \]

\[ \Rightarrow \text{fixed, not stable} \]

\[ \Rightarrow \text{LCO} \]

\[ \Rightarrow \text{Fixed pt is unstable.} \]

\[ \Rightarrow \text{but LCO encircles c*} \]

\[ \Rightarrow \text{e.g. "Mexican hat" potential} \]

\[ \Rightarrow \text{if system satisfies hi-Thy, and fixed pt unstable, then LCO} \]
$\text{LCO} \rightarrow \text{Time Delay}$

$\text{Sense of effect} \rightarrow \text{Model variables}$

$\text{Starting}$

$\text{From Lotka - Volterra}$:

$$\frac{dH}{dt} = H \left[ a - \alpha P \right]$$

$\text{all} > 0$

$$- \text{often} \ x = b$$

$$\frac{dP}{dt} = P \left[ c - b + \beta H \right]$$

1) $a \rightarrow n \left( d - H/n \right) \rightarrow \text{Logistic}$

$$\text{Carrying capacity} \ (s = 1 - e^{\text{K}n})$$

2) $\alpha HP \rightarrow \text{Prey removal by predator}$

$$\rightarrow kP \left( 1 - e^{\text{K}n} \right) \rightarrow \text{Destabilizing}$

$$\rightarrow \frac{kHP}{H+0} \rightarrow \text{Saturation of predation at 0}$
Transitions → 1 Thm. Conditions

- back to NO 34 model:

Fixed pts:

- travel: \( E = U_f = 0 \)

- no flow: \( E = \frac{V_f}{A_f}, U_f = 0 \) → \( \text{self-sustained no flow} \)

- Flow: \( E = \frac{U_f}{d^2} \) → turbulence set by flow damping

\( B \times \frac{d}{dt} \frac{M}{d^2} + \frac{M}{d^2} > \frac{U_f}{d^2} \)

→ flow set by turb. growth

Superficially counterintuitive → signature of pred–prey systems

→ observed, e.g., Gh simulations

LIMIT, et. al., 78

\( E \sim V \)
Clearly:

- Transition for:
  \[ B_0 \rightarrow d_1, u_1 \]

- \( \Phi = 0 \rightarrow B \) sets critical power, flux

- Can linearize about 2 fixed points (non-trivial)

\[ \Phi_0 \]  

New Flow:

\[ (\Phi_0/d_1, 0) \]

Observe:

- \( E \) mode always heavily damped.

- \( L \) mode [soft] near transition

\[ \Phi \rightarrow 0 \]
\[ E_u \to 0 \text{ at threshold} \]

\[ T \text{ transition} \to \infty \] (Cite a bifurcation)

\[ \text{2nd order transition, soft mode signature.} \]

Now, since \( |\chi| \gg |\psi| \),

- \( E \) will relax to equilibrium/fixed point much faster than \( \psi \) (inside) flows to flow, i.e.

\[ 2 \frac{dE}{dt} = \left( \sigma_0 - x_1 E - x_2 U \right) E \]

\[ E = \left( \sigma_0 - x_2 U \right) / \sigma_1 \]

\[ \frac{1}{2} \frac{dU}{dt} = -M U + x_2 U E \]

\[ = -M U + x_2 U \left( \sigma_0 - x_2 U \right) / \sigma_1 \]
A system described by Flow:

\[ \frac{1}{2} \frac{dU}{dt} = \left( \frac{x_2}{x_1} \frac{dx_2}{dx_1} - 1 \right) U_1 - \frac{x_2^2}{x_1} U_1^2 \]

Logistic Eqn.

Logistic eqn. with growth threshold + Diffn. \[ \text{Fisher Eqn.} \]

\[ \frac{dU_1}{dt} = \left( \frac{x_2}{x_1} \frac{dx_2}{dx_1} - 1 \right) U_1 - \frac{x_2^2}{x_1} U_1^2 \]

\[ \frac{dU_2}{dt} = \frac{1}{2} \frac{dU_1}{dt} \]

TOGL \rightarrow 2nd order transition
and can add external torque/drive:

\[
\frac{d}{dt} \langle V \rangle = \left( \frac{a_2 x_2}{x_1} - y \right) \langle V' \rangle - \frac{a_2^2}{a_1} \langle V \rangle^3 + T_{ext}
\]

- Bias? (ala J-TEXT experiments)

- Allan response of paramagnet near criticality away from criticality

- Topic for serious further work...

- Though transition \( B \to C \)
Further:

- LCO's - time delays
  next class.

- Model developments.

Near criticality — noise

\[
\frac{du}{dt} = \left( \frac{x^2}{2} - uy \right) u - \frac{x^2}{2} \quad 0 = \bar{Q} + \tilde{Q}
\]

\[\delta_c \sim \delta_c^0 \text{ or } \delta_c \sim \delta_c^0 \left( \frac{\bar{Q}}{\dot{Q}} \right)\]

\[\sim \delta + \tilde{\delta}\]

as \[Q = \bar{Q} + \tilde{Q}\]

\[\pm \text{ bursts \& noise, i.e. pdf edge heat flux}\]

mem heat flux
Becomes the question of how deal with noise?

Stochastic model $\Rightarrow$ Pf

Fokker-Planck Eqn.
\[ D_v = \frac{3}{8} \frac{V_t}{m} \sim \frac{\beta}{m} \frac{P_r}{m^2} \]

- **Multiplicative Noise (Simple)**

Consider logistic Eqn. \( \Rightarrow \) Population

\[ \frac{dN}{dt} = N \left( k - N \right) \]

\( k \) is saturation \( \Rightarrow \) competition by competition

\( N = \# \) of population

\( N \sim N^2 \)

Growth (exponential)

\[ x_m = x_m (1 - x_m) \]

**Logistic Map**

\( N = 0, N = k \) are fixed pts

Now, could consider variability in \( k \) and treat as stochastic variable

\[ \frac{dN}{dt} = N \left( k_0 + \delta \left( x - N \right) \right) + \xi(t) \]

\( \xi(t) \) variability in resources

\( \sim \) multiplicative

\( \Rightarrow \) additive noise.
\[ \text{additive:} \]
\[ \text{noise on top} \quad = \quad \text{deterministic base} \]

\[ \text{Multiplicative:} \quad \text{multiplied by} \quad \text{random quantity} \]
\[ F(N\text{,}+1) \rightarrow \text{population pdf} \]

How treat? \[ \rightarrow \text{Fokker-Planck Equation} \]
\[ \rightarrow \text{hence} \quad \langle \xi(t) \xi(\tau) \rangle = \text{delta correlated for simplicity} \]

\[ \text{N.B. This is a "textbook model";} \quad \]
\[ \rightarrow \text{additive, as usual} \]
\[ \langle x^2 \rangle = 0 \]

Then:
\[ \frac{dW}{dt} = N (k \omega + \delta \nabla \cdot \mathbf{A}) - N \] + \(\Theta\)

\[ f(N + A) = -\frac{1}{2N} \left[ (k_0 N - N^2) \frac{\partial \mathbf{F}}{\partial N} + \right. \\
- \frac{2}{N} \left[ D \mathbf{F} \right] \]

For \(\mathbf{D}\):

\[ \langle \Delta N \Delta N \rangle = \int_0^t \int_0^t \langle \delta (\mathbf{x}) \delta (\mathbf{x}) \rangle N^2 \\
+ \int_0^t \int_0^t \langle \delta (\mathbf{x}) \delta (\mathbf{x}) \rangle \]

\[ = k_0 T a_c \frac{N^2}{T} + \frac{N^2}{T} a_c \]

Nonlinearity in \(\mathbf{D}\)

\[ \rightarrow \text{one trademark feature of multiplicative noise} \]
Note: \( N \to \infty \Rightarrow D \to 0 \)

Rate variation \( \Rightarrow \) Adf spread on proportion to population.

- Additive correction significant at low \( N \).

Now, ignoring additive correction,

\[
\frac{dt}{dt} = -\frac{\partial}{\partial N} \left\{ \left( k_0 N - N^2 \right) f (CN) \right\} \equiv -\frac{\partial}{\partial N} \left( \frac{1}{2} \sigma_0^2 \tau + N^2 f (CN) \right)
\]

is Fokker-Planck Equation and stationarity:

\[
N \left( k_0 - N \right) f (CN) = \frac{\partial}{\partial N} \left( \frac{1}{2} \sigma_0^2 \tau \right)
\]
\[ \Phi_{CM} = C \cdot N \left[ \frac{2(\lambda/n)}{1 - 2} \right] e^{-2N/V^2} \]

\[ V^2 = \frac{8\omega^2 \rho \alpha}{4 \pi} \]

\[ \text{Power, exponential tail} \]

Need \( k_0^2 > \left( \frac{\omega^2}{V^2} \right)^2 \)

\[ \Rightarrow \quad f > \frac{\omega^2}{V^2} \]

\[ \Rightarrow \quad k_0 > \frac{8\omega^2 \rho \alpha}{2} \]

due to avoiding log singularity.

Physics of \( k_0 > \frac{8\omega^2 \rho \alpha}{2} \)

Convenient to linearize around fixed point:

\[ \frac{dN}{dt} = \left( \lambda^2 + f - N \right)N \]

\[ N = \lambda + \tilde{N} \]

\[ \frac{d\tilde{N}}{dt} = \lambda \left( \frac{\lambda}{2} + \phi - \frac{1}{2} - \tilde{N} \right) \]

\[ \approx \lambda^2 \phi - \lambda \lambda \tilde{N} + O(\tilde{N}^2) \]
\[ \Delta \psi(n) = -\frac{3}{\pi^2} \left[ -k_n \Delta \psi(n) - \frac{3}{\pi^2} \sum_{n} \frac{k_n^2}{\pi^2} \Delta \psi(n) \right] \]

linearize obt fixed pt.

\[ = -\frac{3}{\pi^2} \left[ -k_n \Delta \psi(n) - \frac{3}{\pi^2} \left( \frac{k_n^2}{\pi^2} \Delta \psi(n) \right) \right] \]

\[ \Rightarrow \text{zero flux / stationarity:} \]

\[ f(n) = \text{c exp} \left[ -\frac{n^2}{k_n} \right] \]

Valid for: \[ \langle (\tilde{\gamma}/k_n)^2 \rangle = \langle (\tilde{\gamma}/k_0)^2 \rangle < 1 \]

Now \[ \langle \tilde{r}^2 \rangle = \frac{3}{2} \frac{k_0}{\delta_0} \]

\[ \delta_0 \langle (\tilde{\gamma}/k_0)^2 \rangle < 1 \Rightarrow \frac{5}{2k_0} < 1 \]
\[ \forall k \text{ s.t. } 0^2 < 2k \]

N.B.: Fluctuations small compared to growth.

- Can determine time evolution
- Can get moments
- Spatio-temporal dynamics.