Invariant Measure and Turbulent Pinch in Tokamaks

M. B. Isichenko
Fusion Research Center, The University of Texas at Austin, Austin, Texas 78712

A. V. Gruzinov and P. H. Diamond
Department of Physics, University of California at San Diego, La Jolla, California 92039-0319
(Received 26 October 1994)

It is shown that electron transport due to a generic low-frequency electrostatic turbulence in tokamak geometry results in a peak, self-sustained plasma density profile \( n_0(r) \), rather than to a diffusion-induced flat distribution. The relaxed density profile depends on the magnetic geometry and the distribution of turbulence. The associated inward pinch velocity \( V_x = D \nabla \ln n_0 \) results from the competition of the turbulent diffusion of trapped electrons over the poloidal magnetic flux coordinate \( \psi \) and the collisional relaxation toward a Maxwellian distribution function.

PACS numbers: 52.25.Fi, 52.35.Ra, 52.55.Fa

Experiments show that particle and energy transport in tokamaks cannot be described in terms of diffusion alone. The necessity of introducing nondiffusive, e.g., convective, fluxes is most transparent from the fact that plasma is confined in the absence of core particle sources and in the presence of a negative radial density gradient \( dn/dr \). A standard phenomenology describes the steady-state particle flux \( \Gamma \) as a sum of a diffusive term and a convective term: \( \Gamma = -D dn/dr + n V_r \), where \( D \) is the diffusivity and \( V_r \) is the radial pinch velocity. Dynamic experiments [1] show the same behavior.

The mechanism of the particle pinch has long been a challenge for theory. The neoclassical pinch [2], which is the radial drift of trapped particles with the velocity \( V_r = -c E_e/B_\phi \), is insufficient to explain the profile resilience, because the Ohmic toroidal electric field \( E_e \) is very small (of order 1 V per loop) in a high-temperature plasma. The tokamak pinch is therefore of an anomalous (turbulent) nature. Indeed, the typical potential fluctuation amplitude is tens of \( V \), which might rescale the Ward-Galeev pinch accordingly. The apparent difficulty of such an interpretation, namely, that the zero-average turbulent fluctuations yield no average pinch velocity but only a diffusion, is resolved by noting that the diffusion is not in the radial coordinate \( r \) but rather in \( \psi \), the poloidal magnetic flux coordinate. If, as a result of such a diffusion, the particles were distributed uniformly over \( \psi \), their Cartesian-space density would be

\[
n_0(\psi) \propto \left[ V(\psi) \right]^{-1} = \left( \int_0^{2\pi} L(\psi, \theta) d\theta / B(\psi, \theta) \right)^{-1}.
\]

(1)

where \( V(\psi) \) is the volume inside the flux surface, \( \theta \) is the poloidal angle, \( L(\psi, \theta) = q RB_\psi / B_\phi \) is the connection length such that the distance along the magnetic field line is \( d\ell = L d\theta \), \( R(\psi, \theta) \) is the major radius, \( q(\psi) \) is the safety factor, and \( B(\psi, \theta) \) and \( B_\psi(\psi, \theta) \) are the total and the toroidal magnetic fields, respectively. A flux-uniform distribution similar to (1) was previously described by Hasegawa, Chen, and Mauel [3] for a magnetospheric-type dipole magnetic field. In the limit of a large aspect ratio tokamak, Eq. (1) reduces to \( n_0(\psi)q(\psi) = \text{const} \), as found by Yankov [4], who introduced the term turbulent equipartition (TEP) for the result of chaotic mixing under fixed adiabatic invariants instead of energy.

Although the peaked density profile (1) is qualitatively consistent with experimental data, it does not quantitatively survive the effect of even rare Coulomb collisions, and the final result [Eq. (12)] turns out more complicated. We derive the electron transport equations based on a nonlinear, parallel-motion-average electron response to a generic low-frequency (less than the electron bounce frequency) electrostatic turbulence. The calculation is performed in a general shaped geometry. The properties of plasma turbulence are not studied; instead, an analysis is given of how these properties enter (or do not) the transport equations. One of the main conclusions is that the tokamak pinch is explained by the turbulent transport of trapped electrons due to the fluctuating toroidal electric fields.

Before discussing specific tokamak issues, it is useful to consider the abstract model of passive turbulent advection with the velocity

\[
dx/dt = v(x,t) = u(x,t)/\lambda(x), \quad \nabla \cdot u = 0,
\]

(2)

representing a short-scale, incompressible, zero-average turbulent field \( u(x,t) \) modulated by a time-independent large-scale amplitude \( 1/\lambda(x) \). Let the density \( n(x,t) \) of a tracer evolve according to the continuity equation \( \partial n/\partial t + \nabla \cdot (nv) = 0 \). Then, for \( \eta(x,t) = n(x,t)/\lambda(x) \), we have \( \partial \eta/\partial t + (1/\lambda)u \cdot \nabla \eta = 0 \), and we see that the distribution \( \eta = \text{const} \), or

\[
n = n_0(x) \propto \lambda(x)
\]

(3)

will not evolve. In general, according to Fick’s law, an average gradient of \( \eta \) will result in the average flux \( \langle \eta(u/\lambda) \rangle = \langle nv \rangle / \lambda = -D(x) : \nabla \eta \), where \( D \) is a turbulent diffusion tensor depending on the properties of
We thus infer the transport (Fokker-Planck) equation for a passive tracer in a compressible turbulence:

\[ \frac{\partial n}{\partial t} = \nabla \cdot (\mathbf{D} \cdot \nabla n) - \mathbf{V}(n), \quad \mathbf{V} = \mathbf{D} \cdot \nabla \ln \lambda, \] (4)

where \( \mathbf{V} \) is the average (“pinch”) velocity.

Thus, even in a zero-average, at a given \( \mathbf{x} \), turbulent field (2), the average particle velocity in (4) is not zero, if the field is compressible. This effect is somewhat similar to the familiar ponderomotive force experienced by a particle oscillating in a high-frequency force field.

From the dynamical standpoint, we note that the time-dependent dynamical system (2) possesses the time-independent invariant measure \( \lambda(x) \) which is conserved for a volume of points evolving according to (2). The much stronger property of the relaxation of the system’s probability distribution function to this measure is equivalent to the ergodic hypothesis or the closely related diffusion approximation (4). A necessary condition for this property is the absence of integrals of motion in the system. Thus, in order to make the invariant measure \( \lambda(x) \) an attracting solution for the diffusion function, we need to eliminate the integrals by effectively reducing the dimension of the phase space using the conserved quantities as new variables. The invariant measure density is then multiplied by the Jacobian of the transformation [5].

The simplest example of a nontrivial invariant measure is given by the two-dimensional chaotic \( \mathbf{E} \times \mathbf{B} \) particle motion in an inhomogeneous magnetic field \( \mathbf{B} = \hat{z} \mathbf{B}(x, y) : \dot{x} = -(c/B)\nabla \phi(x, y, t) \times \hat{z}, \) where \( \phi \) is the electrostatic potential of a low-frequency \( \omega \ll \omega_B = eb/(mc) \) turbulence. The \( \mathbf{E} \times \mathbf{B} \) drift velocity is an incompressible field divided by \( B \), hence, according to (3), the equilibrium density profile is \( n_0(x, y) \propto B(x, y). \) This result can be also verified by a change of variables from the standard phase space \( (x, y, v_x, v_y) \) to the guiding-center variables involving the conserved magnetic moment \( \mu = mv^2/2B \) and the gyrophase \( \alpha \). Since the Liouville theorem implies \( \lambda(x, v) = 1 \), the guiding-center invariant measure is \( \lambda(x, y, \alpha) = \delta(x, y, v_x, v_y) / \delta(x, y, \mu, \alpha) = B(x, y)/m, \) as expected. A pinch velocity similar to the one resulting from this invariant measure, \( \mathbf{V} = \mathbf{D} \cdot \nabla \ln B, \) was obtained by Smolyakov, Callen, and Hirose [6].

For a three-dimensional (3D) geometry the parallel motion and the associated invariants must be included in our theory. The simplest possible model is the motion of deeply trapped particles in a large aspect ratio tokamak. According to Ref. [2] and also to Eq. (5) below, the bounce-average motion of the banana center in the tokamak midplane \( (x, y) \) can be written as \( \dot{x} = (c/B_0)\nabla \phi^*(x, y, t) \times \hat{z}, \) where \( \phi^* = \phi + (\mu/e)B \) is the effective electrostatic potential. Thus the corresponding equilibrium density is proportional to the poloidal magnetic field \( B_0 \), or the reciprocal safety factor \( q. \)

Below we show that, in a general magnetic geometry and for an arbitrary trapping depth, the bounce average turbulent drift of a banana particle can be written in the canonical form

\[ \{ \psi, \phi_0 \} = (2\pi c/e)\{ \partial_{\varphi_0}, \partial_{\varphi} \} e(\psi, \phi_0, J, \mu, t), \] (5)

where \( \psi \) is the poloidal flux labeling a magnetic flux surface, \( \phi_0 = \varphi - q(\psi)\theta \) labels a magnetic field line in the surface, \( \varphi \) and \( \theta \) are the toroidal and the poloidal angles, respectively, \( J = J(\psi, \phi_0, e, \mu, t) = \oint_{\mu} d\mu \) is the conserved longitudinal adiabatic invariant of a trapped particle, and \( e(\psi, \phi_0, J, \mu, t) \) is the total particle energy expressed in terms of \( J \) and \( \mu. \)

Thus, incompressible in the \((\psi, \phi_0)\) plane, Eq. (5) describes a turbulent relaxation to the \((\psi, \phi_0)\) uniform particle distribution \( f_0(\psi, \phi_0, J, \mu) = F(J, \mu). \) The corresponding Cartesian-space particle density is nonuniform and given by Eq. (1).

The turbulent mechanism of the particle pinch can be established by deriving the parallel-motion-average equations of the drift particle motion. Consider the general magnetic field \( \mathbf{B} \) possessing flux surfaces \( \psi(x) = \text{const}. \)

\[ B = [\nabla \chi(\psi) \times \nabla \theta - \nabla \psi \times \nabla \varphi]/2\pi, \] (6)

where \( \chi \) and \( \varphi \) are the poloidal and the toroidal magnetic fluxes and \( q(\psi) = d\chi/d\psi \) is the safety factor.

Suppose that the parallel particle motion occurs much faster than the evolution of the electrostatic potential \( \phi(x, t). \) This approximation holds sufficiently well for electrons and makes it possible to average the electron drift equations,

\[ \dot{x} = v_{\parallel}[b + (v_{\parallel}/\omega_B)c] - (c/B)\nabla \phi^* \times \mathbf{b}, \]
\[ \dot{v}_{\parallel} = -(e/m)\nabla \phi^* \cdot [b + (v_{\parallel}/\omega_B)c], \] (7)

over their fast parallel motion. Here \( v_{\parallel} \) is the parallel velocity, \( \mathbf{b} = \mathbf{B}/B \) is the magnetic unit vector, and \( c = \mathbf{b} \times (\mathbf{b} \cdot \nabla) \mathbf{b} \) is the magnetic curvature. The result of the averaging depends on the topology of the particle orbit.

**Trapped particles.**—Although the separation of the fast parallel motion from the slow cross-field drift and the averaging of the latter over the former are straightforward in the magnetic flux coordinates, we will use the well-known result, which applies to any magnetic field \( \mathbf{B} = \nabla \alpha \times \nabla \psi \) written in the Clebsch coordinates \( \alpha \) and \( \psi. \) [Equation (6) implies \( \alpha = \phi_0/2\pi. \)] In such a field, the bounce-average drift of a trapped particle takes the form

\[ \dot{\alpha} = -(c/e)\partial_\psi e, \quad \dot{\psi} = (c/e)\partial_\alpha e. \] These equations are equivalent to (5) and, according to the above discussion, imply a diffusive random walk of each banana particle over the \( \psi \) coordinate. The quasilinear diffusion over \( \psi \) was calculated in Ref. [8] but not connected with the pinch effect.

More explicitly, the bounce-average equation for \( \psi, \)

\[ \dot{\psi} = 2\pi c(\partial_\varphi \phi(\psi, \theta, \varphi, t)), \] (9)
shows that the effect is proportional to the orbit-average toroidal electric field. The poloidal electric field cancels with the $\nabla B$ drift upon the bounce averaging and does not enter the mean radial displacement of the trapped electrons. For a toroidally symmetrical multivalued $(\text{mod}V_{\text{loop}})\phi$, Eq. (9) reduces to the Ware-Galeev pinch in a general geometry [9], $\psi = -eV_{\text{loop}}$, which is always inward for an Ohmic loop voltage $V_{\text{loop}}$. For general 3D quasistationary potential fluctuations, Eq. (9) represents the instantaneous banana drift velocity, which can have either sign and consequently lead to the trapped particle diffusion over $\psi$. Note that, locally, the Clebsch representation is also valid for a stochastic magnetic field, and we analogously infer that the transport of trapped electrons due to slowly evolving magnetic fluctuations is also a diffusion over (unperturbed) $\psi$.

Passing particles.—The parallel motion of a circulating particle is conditionally (double) periodic, and we need the technique of conditional-periodic averaging, cf. [10]. The time average of a function $A(\psi, \theta, \varphi)$ for a passing particle is defined as

$$\langle A \rangle = \int_0^{2\pi} d\varphi \int_0^{2\pi} d\theta A(\psi, \theta, \varphi) J(\psi, \theta, \varphi) / v_j(\psi, \theta, \varphi, \varphi) .$$

Upon applying this rule to the drift over $\psi$ derivable from Eq. (7), we find $\psi = -eV_{\text{loop}}(B_0^2/B^2)$, where an orbit average is implied. This equation contains no turbulent fluctuations and is just the $\mathbf{E} \times \mathbf{B}$ drift of passing particles due to the Ohmic electric field, an effect even smaller than the typically negligible neo-classical pinch of the trapped particles.

Thus, to first order in $\mathbf{E} \times \mathbf{B}$ and magnetic drifts in quasistationary electrostatic fluctuations, there is neither turbulent diffusion nor pinch of passing electrons. This can be equivalently expressed as the conservation of the suitably defined longitudinal adiabatic invariant $J$ for a passing particle:

$$\langle J(\psi, \theta, \varphi) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} d\varphi \int_0^{2\pi} m v_j(\psi, \theta, \varphi, e, t) L(\theta) d\theta ,$$

with the coefficient chosen so that $J$ is continuous at the boundary between the trapped and the passing modes.

According to Eq. (5) for trapped particles and the conservation of $\psi$ for passing particles, the collisionless electron transport in a turbulent tokamak can be formulated in terms of the diffusion of the distribution function $f(\psi, J, \mu, t)$ (such as the total number of particles is $\int f d\psi dJ d\mu$) over the poloidal flux coordinate $\psi$. The turbulent diffusivity $D^{\psi\psi}(\psi, J, \mu)$ is different from zero only in the trapped region $J < J_e(\psi, \mu)$, where it is given by the time integral of the correlation function of the random velocity (9). Thus, as in the neoclassical theory [11], the trapped particles play the central role in the turbulent transport. In the quasistatic turbulence, the radial motion (9) depends only on the extent, but not on the frequency of the particle bouncing, so the turbulent diffusivity $D^{\phi\phi}$ depends only on $\psi$ and the pitch-angle variable

$$j = J / \sqrt{m \mu} = \int_{-\pi}^{\pi} \Theta(u - B) \sqrt{2(u - B)} L d\theta ,$$

where $\Theta$ is the step function and $u = u(\psi, j) = e/\mu$.

The fully relaxed flux-uniform distribution (TEP) $f_0(\psi, J, \mu) = f(J, \mu)$ caused by the diffusion over $\psi$, however, is incompatible with the flux-surface-local Maxwellian distribution function

$$f_M(\psi, J, \mu) = n(\psi)T^{-3/2}(\psi) \exp[-e(\psi, J, \mu)/T(\psi)],$$

because at no choice of the functions $n(\psi)$ and $T(\psi)$ can $f_M$ be made independent of $\psi$ in the trapped region. Since the collision time is much shorter than the diffusion time through the tokamak minor radius, the turbulent diffusion over $\psi$ must be considered a small perturbation to the collisional transport of electrons over the $J$ and $\mu$ coordinates. Given the trend towards the flux-uniform distribution (1), the direction of this perturbation is clearly toward a peaked density profile.

In the absence of particle sources, the equilibrium profile of $n(\psi)$ is determined by the requirement of the vanishing radial particle flux, $\int dJ d\mu D^{\psi\phi}\delta \phi f_M = 0$. We thus obtain

$$\ln \frac{n_0(\psi)}{n_0(0)} = \int_0^\psi d\psi \int_0^{j_e} d\psi \hat{D}(\psi, j) \delta \phi \ln s/j_e(\psi, j) ,$$

where $\hat{D} = D^{\psi\phi} \mu^{-3/2}(\int_0^{j_e} dj D^{\psi\phi} \mu^{-3/2})^{-1}$ is the normalized turbulent diffusivity and $j_e(\psi)$ corresponds to the pitch angle of marginal trapping. Equilibrium (12) is characterized by an electron convection such that the deeply trapped particles flow inward and the barely trapped ones flow outward.

For illustrative purposes, assume that $D^{\psi\phi}$ is independent of $j$ in the trapped region. Then Eq. (12) yields the following density profile in the large aspect ratio limit $r/R \ll 1$:

$$\ln \frac{n_0(\psi)}{n_0(0)} = 1 - \frac{1}{R} \int_0^r dr \left( \frac{1}{2} \frac{4}{5} \frac{d(\ln q)}{d\ln r} + O\left( \frac{r^2}{R^2} \right) \right) .$$

We see that, leading order in $r/R$, this profile is flat, and peaked only to first order in the inverse aspect ratio. Note that sharp peak at the magnetic axis, $n_0(0) = 0$. This singularity will be regularized by the so far neglected (neo)classical diffusion, because the effective turbulent diffusivity $D(r)$ and the pinch velocity $V_e(r)$ both vanish at the axis in proportion to the fraction of trapped particles, and the classical electron transport takes over.

As shown in Fig. 1, the density and the safety factor profiles in Tokamak Fusion Test Reactor (TFTR) show a reasonably good agreement with formula (13). On the other hand, the data deviate systematically from the TEP profile $nq = \text{const}$.
In conclusion, we have shown that the conservation of adiabatic invariants in a low-frequency electrostatic turbulence induces an invariant measure, to which the particle distribution tends to relax on a turbulent transport time scale. When both invariants $\mu$ and $J$ are conserved, the collisionless electron transport materializes itself as a turbulent diffusion, due to the toroidal turbulent electric field, of trapped particles over the poloidal flux coordinate $\psi$ and a much smaller transport of passing particles. In a tokamak geometry, the relaxation to the flux-uniform profile (1) is competing with the collisionally established flux-surface-local Maxwellian distribution, which results in the peaked density profile (12) depending on both the magnetic geometry and the distribution of turbulent fluctuations. The particle flux $\Gamma = -D_n \nabla (n/n_0)$ consists of diffusive and convective (pinch) parts, the latter involving the gradients of magnetic geometry, normalized turbulence distribution, but not electron temperature, because $n_0$ is independent of the temperature profile. It is thereby demonstrated that turbulent plasma transport is not reducible to the thermodynamic cross fluxes described by the Onsager symmetry characteristic of collisional transport [13]. In general, turbulent particle and energy fluxes are possible even without density and temperature gradients [14].

Although transport in stellarators is also anomalous, the complicated magnetic geometry creates multiple separatrix crossings, of which effectively destroys the longitudinal adiabatic invariant [15]. This should significantly reduce the particle pinch, as confirmed by typically flat density profiles in stellarators [16].

The ion transport is a more delicate problem, because the ion parallel motion frequency is comparable with or less than the mean turbulence frequency. Nevertheless, in a plasma with few impurities, the quasineutrality constraint makes it possible to describe the particle transport in terms of the electron transport alone. The kinetic formalism developed in this work predicts density profiles in good agreement with experiment. A natural extension of this formalism also predicts the electron energy pinch [17]. These and related issues will be discussed in future publications.

The authors wish to thank V. V. Yankov for communicating the result of his work prior to its publication and for stimulating discussions. We are also grateful to A. J. Wootton, B. B. Kadomtsev, and R. Z. Sagdeev for discussions and to L. Chen and C. S. Haas for pointing out relevant references. This work was supported by U.S. DOE under Grants No. DE-FG05-80ET53266 and No. DE-FG03-94ER-54241.