(1) Show that for time scales sufficiently greater than $\gamma^{-1}$ that the solution $x(t)$ to the Langevin equation $\ddot{x} + \gamma \dot{x} = \eta(t)$ describes a Markov process. You will have to construct the matrix $M$ defined in eqns. 2.213 and 2.214 of the lecture notes. You should assume that the random force $\eta(t)$ is distributed as a Gaussian, with $\langle \eta(s) \rangle = 0$ and $\langle \eta(s) \eta(s') \rangle = \Gamma \delta(s-s')$.

Solution:

The probability distribution is

$$P(x_1, t_1; \ldots; x_N, t_N) = \text{det}^{-1/2}(2\pi M) \exp \left\{ -\frac{1}{2} \sum_{j,j'=1}^{N} M_{jj'}^{-1} x_j x_{j'} \right\},$$

where

$$M(t, t') = \int_{0}^{t} \int_{0}^{t'} ds \int_{0}^{s} ds' G(s-s') K(t-s) K(t'-s'),$$

and $K(s) = (1 - e^{-\gamma s})/\gamma$. Thus,

$$M(t, t') = \frac{\Gamma}{\gamma^2} \int_{0}^{t_{\text{min}}} ds (1 - e^{-\gamma(t-s)})(1 - e^{-\gamma(t'-s)}).$$

In the limit where $t$, $t'$, and $|t-t'|$ are all large compared to $\gamma^{-1}$, we have $M(t, t') = 2D \min(t, t')$, where the diffusions constant is $D = \Gamma/2\gamma^2$. Thus,

$$M = 2D \begin{pmatrix} t_1 & t_2 & t_3 & t_4 & t_5 & \cdots & t_N \\ t_2 & t_3 & t_4 & t_5 & \cdots & t_N \\ t_3 & t_4 & t_5 & \cdots & t_N \\ t_4 & t_5 & \cdots & t_N \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ t_N & t_N & t_N & t_N & \cdots & t_N \end{pmatrix},$$

since $t_1 \geq t_2 \geq \cdots \geq t_N$. To find the determinant of $M$, subtract row 2 from row 1, then subtract row 3 from row 2, etc. The result is

$$\tilde{M} = 2D \begin{pmatrix} t_1 - t_2 & 0 & 0 & 0 & 0 & \cdots & 0 \\ t_2 - t_3 & t_2 - t_3 & 0 & 0 & 0 & \cdots & 0 \\ t_3 - t_4 & t_3 - t_4 & t_3 - t_4 & 0 & 0 & \cdots & 0 \\ t_4 - t_5 & t_4 - t_5 & t_4 - t_5 & t_4 - t_5 & 0 & \cdots & 0 \\ t_5 - t_6 & t_5 - t_6 & t_5 - t_6 & t_5 - t_6 & t_5 - t_6 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ t_N & t_N & t_N & t_N & t_N & \cdots & t_N \end{pmatrix}.$$
Note that the last row is unchanged, since there is no row $N+1$ to subtract from it. Since $\tilde{M}$ is obtained from $M$ by consecutive row additions, we have
\[
\det M = \det \tilde{M} = (2D)^N(t_1 - t_2)(t_2 - t_3) \cdots (t_{N-1} - t_N) t_N.
\]
The inverse is
\[
M^{-1} = \frac{1}{2D} \begin{pmatrix}
\frac{1}{t_1-t_2} & \frac{1}{t_1-t_2} & 0 & \cdots & 0 \\
0 & \frac{1}{t_1-t_2} & \frac{1}{t_1-t_2} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \frac{1}{t_{n-1}-t_n} & \frac{1}{t_{n-1}-t_n} \\
0 & \cdots & \cdots & \cdots & \frac{1}{t_{N-1}-t_N} & \frac{1}{t_{N-1}-t_N}
\end{pmatrix}.
\]
This yields the general result
\[
\sum_{j,j'=1}^{N} M^{-1}_{jj'}(t_1, \ldots, t_N) x_j x_{j'} = \sum_{j=1}^{N} \left( \frac{1}{t_{j-1} - t_j} + \frac{1}{t_j - t_{j+1}} \right) x_j^2 - \frac{2}{t_{j-1} - t_{j+1}} x_j x_{j+1},
\]
where $t_0 \equiv \infty$ and $t_{N+1} \equiv 0$. Now consider the conditional probability density
\[
P(x_1, t_1 \mid x_2, t_2; \ldots; x_N, t_N) = \frac{P(x_1, t_1; \ldots; x_N, t_N)}{P(x_2, t_2; \ldots; x_N, t_N)} \frac{\det^{1/2} 2\pi M(t_2, \ldots, t_N)}{\det^{1/2} 2\pi M(t_1, \ldots, t_N)} \exp \left\{ -\frac{1}{2} \sum_{j,j'=1}^{N} M^{-1}_{jj'}(t_1, \ldots, t_N) x_j x_{j'} \right\}
\]
\[
= \frac{\det^{1/2} 2\pi M(t_1, \ldots, t_N)}{\det^{1/2} 2\pi M(t_2, \ldots, t_N)} \exp \left\{ -\frac{1}{2} \sum_{k,k'=2}^{N} M^{-1}_{kk'}(t_2, \ldots, t_N) x_k x_{k'} \right\}
\]
We have
\[
\sum_{j,j'=1}^{N} M^{-1}_{jj'}(t_1, \ldots, t_N) x_j x_{j'} = \left( \frac{1}{t_0 - t_1} + \frac{1}{t_1 - t_2} \right) x_1^2 - \frac{2}{t_1 - t_2} x_1 x_2 + \left( \frac{1}{t_1 - t_2} + \frac{1}{t_2 - t_3} \right) x_2^2 + \ldots
\]
\[
\sum_{k,k'=2}^{N} M^{-1}_{kk'}(t_2, \ldots, t_N) x_k x_{k'} = \left( \frac{1}{t_0 - t_2} + \frac{1}{t_2 - t_3} \right) x_2^2 + \ldots
\]
Subtracting, and evaluating the ratio to get the conditional probability density, we find
\[
P(x_1, t_1 \mid x_2, t_2; \ldots; x_N, t_N) = \frac{1}{\sqrt{4\pi D(t_1 - t_2)}} e^{-(x_1-x_2)^2/4D(t_1-t_2)},
\]

which depends only on \( \{x_1, t_1, x_2, t_2\} \), i.e. on the current and most recent data, and not on any data before the time \( t_2 \). Note the normalization:

\[
\int_{-\infty}^{\infty} dx_1 \ P(x_1, t_1 \mid x_2, t_2; \ldots; x_N, t_N) = 1.
\]

(2) Provide the missing steps in the solution of the Ornstein-Uhlenbeck process described in §2.4.3 of the lecture notes. Show that applying the method of characteristics (see appendix IV) to Eqn. 2.61 leads to the solution in Eqn. 2.62.

Solution:

We solve

\[
\frac{\partial \hat{P}}{\partial t} + \beta k \frac{\partial \hat{P}}{\partial k} = -D k^2 \hat{P}
\]

using the method of characteristics, writing \( t = t_\zeta(s) \) and \( k = k_\zeta(s) \), where \( s \) parameterizes the curve \( (t_\zeta(s), k_\zeta(s)) \), and \( \zeta \) parameterizes the initial conditions, which are \( t(s = 0) = 0 \) and \( k(s = 0) = \zeta \). The above PDE in two variables is then equivalent to the coupled system

\[
\frac{dt}{ds} = 1, \quad \frac{dk}{ds} = \beta k, \quad \frac{d\hat{P}}{ds} = -D k^2 \hat{P}.
\]

Solving, we have

\[
t_\zeta = s, \quad k_\zeta = \zeta e^{\beta s}, \quad \frac{d\hat{P}}{ds} = -D \zeta^2 e^{2\beta s} \hat{P},
\]

and therefore

\[
\hat{P}(s, \zeta) = f(\zeta) \exp \left\{ -\frac{D \zeta^2}{2\beta} (e^{2\beta s} - 1) \right\}.
\]

We now identify \( f(\zeta) = \hat{P}(k e^{-\beta t}, t = 0) \), hence

\[
\hat{P}(k, t) = \exp \left\{ -\frac{D}{2\beta} (1 - e^{-2\beta t}) k^2 \right\} \hat{P}(k, 0).
\]

(3) Consider a discrete one-dimensional random walk where the probability to take a step of length 1 in either direction is \( \frac{1}{2}p \) and the probability to take a step of length 2 in either direction is \( \frac{1}{2}(1 - p) \). Define the generating function

\[
\hat{P}(k, t) = \sum_{n=-\infty}^{\infty} P_n(t) e^{-ikn},
\]

where \( P_n(t) \) is the probability to be at position \( n \) at time \( t \), with \( P_n(0) = \delta_{n,0} \). Solve for \( \hat{P}(k, t) \) and provide an expression for \( P_n(t) \). Evaluate \( \sum_n n^2 P_n(t) \).
Solution:

We have the master equation

\[
\frac{dP_n}{dt} = \frac{1}{2} (1 - p) P_{n+2} + \frac{1}{2} p P_{n+1} + \frac{1}{2} P_{n-1} + \frac{1}{2} (1 - p) P_{n-2} - P_n .
\]

Upon Fourier transforming,

\[
\frac{d\hat{P}(k,t)}{dt} = \left[ (1 - p) \cos(2k) + p \cos(k) - 1 \right] \hat{P}(k,t) ,
\]

with the solution

\[
\hat{P}(k,t) = e^{-\lambda(k)t} \hat{P}(k,0) ,
\]

where

\[
\lambda(k) = 1 - p \cos(k) - (1 - p) \cos(2k) .
\]

One then has

\[
P_n(t) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{ikn} \hat{P}(k,t) .
\]

The average of \(n^2\) is given by

\[
\langle n^2 \rangle_t = -\frac{\partial^2 \hat{P}(k,t)}{\partial k^2} \bigg|_{k=0} = \left[ \lambda''(0) t - \lambda'(0)^2 t^2 \right] = (4 - 3p) t .
\]

Note that \(\hat{P}(0,t) = 1\) for all \(t\) by normalization.

(4) Numerically simulate the one-dimensional Wiener and Cauchy processes discussed in §2.6.1 of the lecture notes, and produce a figure similar to Fig. 2.3.

**Hint:** To generate normal (Gaussian) deviates with a distribution \(p(y) = (4\pi D\varepsilon)^{-1/2} \exp(-y^2/4D\varepsilon)\), we must invert the relation

\[
x(y) = \frac{1}{\sqrt{4\pi D\varepsilon}} \int_{-\infty}^{y} ds \, e^{-s^2/4D\varepsilon} = \frac{1}{2} + \frac{1}{2} \text{erf} \left( \frac{y}{\sqrt{4D\varepsilon}} \right) .
\]

This is somewhat unpleasant. A slicker approach is to use the Box-Muller method, which you can read about on Wikipedia.

Solution:

Most computing languages come with a random number generating function which produces uniform deviates on the interval \(x \in [0, 1]\). Suppose we have a prescribed function \(y(x)\). If \(x\) is distributed uniformly on \([0, 1]\), how is \(y\) distributed? Clearly

\[
|p(y)\,dy| = |p(x)\,dx| \quad \Rightarrow \quad p(y) = \left| \frac{dx}{dy} \right| p(x) ,
\]
where for the uniform distribution on the unit interval we have \( p(x) = \Theta(x) \Theta(1-x) \). For example, if \( y = -\ln x \), then \( y \in [0, \infty] \) and \( p(y) = e^{-y} \), which is to say \( y \) is exponentially distributed. Now suppose we want to specify \( p(y) \). We have

\[
\frac{dx}{dy} = p(y) \implies x = F(y) = \int_{y_0}^{y} ds \, p(s)
\]

where \( y_0 \) is the minimum value that \( y \) takes. Therefore, \( y = F^{-1}(x) \), where \( F^{-1} \) is the inverse function.

To generate normal (Gaussian) deviates with a distribution \( p(y) = (4\pi D\varepsilon)^{-1/2} \exp(-y^2/4D\varepsilon) \), we have

\[
x = F(y) = \frac{1}{\sqrt{4\pi D\varepsilon}} \int_{-\infty}^{y} ds \, e^{-s^2/4D\varepsilon} = \frac{1}{2} + \frac{1}{2} \text{erf}\left( \frac{y}{\sqrt{4D\varepsilon}} \right)
\]

We now have to invert the error function, which is slightly unpleasant.

A slicker approach is to use the Box-Muller method, which used a two-dimensional version of the above transformation,

\[
p(y_1, y_2) = p(x_1, x_2) \left| \frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} \right|
\]

This has an obvious generalization to higher dimensions. The transformation factor is the Jacobian determinant. Now let \( x_1 \) and \( x_2 \) each be uniformly distributed on \([0, 1]\), and let

\[
x_1 = \exp\left(-\frac{y_1^2 + y_2^2}{4D\varepsilon}\right) \quad y_1 = \sqrt{-4D\varepsilon \ln x_1} \cos(2\pi x_2)
\]

\[
x_2 = \frac{1}{2\pi} \tan^{-1}(y_2/y_1) \quad y_2 = \sqrt{-4D\varepsilon \ln x_1} \sin(2\pi x_2)
\]

Then

\[
\frac{\partial x_1}{\partial y_1} = \frac{y_1 x_1}{2D\varepsilon} \quad \frac{\partial x_2}{\partial y_1} = -\frac{1}{2\pi} \frac{y_2}{y_1^2 + y_2^2}
\]

\[
\frac{\partial x_1}{\partial y_2} = -\frac{y_2 x_1}{2D\varepsilon} \quad \frac{\partial x_2}{\partial y_2} = \frac{1}{2\pi} \frac{y_1}{y_1^2 + y_2^2}
\]

and therefore the Jacobian determinant is

\[
J = \left| \frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} \right| = \frac{1}{4\pi D\varepsilon} e^{-(y_1^2+y_2^2)/4D\varepsilon} = \frac{e^{-y_1^2/4D\varepsilon}}{\sqrt{4\pi D\varepsilon}} \cdot \frac{e^{-y_2^2/4D\varepsilon}}{\sqrt{4\pi D\varepsilon}}
\]

which says that \( y_1 \) and \( y_2 \) are each independently distributed according to the normal distribution \( p(y) = (4\pi D\varepsilon)^{-1/2} \exp(-y^2/4D\varepsilon) \). Nifty!
Figure 1: (a) Wiener process sample path $W(t)$. (b) Cauchy process sample path $C(t)$. From K. Jacobs and D. A. Steck, New J. Phys. 13, 013016 (2011).

For the Cauchy distribution, with

$$p(y) = \frac{1}{\pi} \cdot \frac{\varepsilon}{y^2 + \varepsilon^2},$$

we have

$$F(y) = \frac{1}{\pi} \int_{-\infty}^{y} ds \frac{\varepsilon}{s^2 + \varepsilon^2} = \frac{1}{2} + \frac{1}{\pi} \tan^{-1}(y/\varepsilon),$$

and therefore

$$y = F^{-1}(x) = \varepsilon \tan \left( \pi x - \frac{\pi}{2} \right).$$

(5) Due to quantum coherence effects in the backscattering from impurities, one-dimensional wires don’t obey Ohm’s law in the limit where the ‘inelastic mean free path’ is greater than the sample dimensions, which you may assume here. Rather, let $\mathcal{R}(L) = e^2 R(L)/\hbar$ be the dimensionless resistance of a quantum wire of length $L$, in units of $h/e^2 = 25.813 \, \text{k}\Omega$. The dimensionless resistance of a quantum wire of length $L + \delta L$ is then given by

$$\mathcal{R}(L + \delta L) = \mathcal{R}(L) + \mathcal{R}(\delta L) + 2 \mathcal{R}(L) \mathcal{R}(\delta L) + 2 \cos \alpha \sqrt{\mathcal{R}(L) \left[ 1 + \mathcal{R}(L) \right] \mathcal{R}(\delta L) \left[ 1 + \mathcal{R}(\delta L) \right]},$$

where $\alpha$ is a random phase uniformly distributed over the interval $[0, 2\pi)$. Here,

$$\mathcal{R}(\delta L) = \frac{\delta L}{2\ell},$$

is the dimensionless resistance of a small segment of wire, of length $\delta L \ll \ell$, where $\ell$ is the ‘elastic mean free path’.
(a) Show that the distribution function \( P(\mathcal{R}, L) \) for resistances of a quantum wire obeys the equation
\[
\frac{\partial P}{\partial L} = \frac{1}{2\ell} \frac{\partial}{\partial \mathcal{R}} \left\{ \mathcal{R} \left(1 + \mathcal{R}\right) \frac{\partial P}{\partial \mathcal{R}} \right\}.
\]

(b) Show that this equation may be solved in the limits \( \mathcal{R} \ll 1 \) and \( \mathcal{R} \gg 1 \), with
\[
P(\mathcal{R}, z) = \frac{1}{z} e^{-\mathcal{R}/z}
\]
for \( \mathcal{R} \ll 1 \), and
\[
P(\mathcal{R}, z) = (4\pi z)^{-1/2} \frac{1}{\mathcal{R}} e^{-(\ln \mathcal{R}-z)^2/4z}
\]
for \( \mathcal{R} \gg 1 \), where \( z = L/2\ell \) is the dimensionless length of the wire. Compute \( \langle \mathcal{R} \rangle \) in the former case, and \( \langle \ln \mathcal{R} \rangle \) in the latter case.

Solution:

(a) From the composition rule for series quantum resistances, we derive the phase averages
\[
\langle \delta \mathcal{R} \rangle = \left(1 + 2 \mathcal{R}(L)\right) \frac{\delta L}{2\ell}
\]
\[
\langle (\delta \mathcal{R})^2 \rangle = \left(1 + 2 \mathcal{R}(L)\right)^2 \left(\frac{\delta L}{2\ell}\right)^2 + 2 \mathcal{R}(L) \left(1 + \mathcal{R}(L)\right) \frac{\delta L}{2\ell} \left(1 + \frac{\delta L}{2\ell}\right)
\]
\[
= 2 \mathcal{R}(L) \left(1 + \mathcal{R}(L)\right) \frac{\delta L}{2\ell} + O((\delta L)^2),
\]
whence we obtain the drift and diffusion terms
\[
F_1(\mathcal{R}) = \frac{2\mathcal{R} + 1}{2\ell}, \quad F_2(\mathcal{R}) = \frac{2\mathcal{R}(1 + \mathcal{R})}{2\ell}.
\]
Note that \( 2F_1(\mathcal{R}) = dF_2/d\mathcal{R} \), which allows us to write the Fokker-Planck equation as
\[
\frac{\partial P}{\partial L} = \frac{\partial}{\partial \mathcal{R}} \left\{ \mathcal{R} \left(1 + \mathcal{R}\right) \frac{\partial P}{\partial \mathcal{R}} \right\}.
\]

(b) Defining the dimensionless length \( z = L/2\ell \), we have
\[
\frac{\partial P}{\partial z} = \frac{\partial}{\partial \mathcal{R}} \left\{ \mathcal{R} \left(1 + \mathcal{R}\right) \frac{\partial P}{\partial \mathcal{R}} \right\}.
\]
In the limit \( \mathcal{R} \ll 1 \), this reduces to
\[
\frac{\partial P}{\partial z} = \mathcal{R} \frac{\partial^2 P}{\partial \mathcal{R}^2} + \frac{\partial P}{\partial \mathcal{R}},
\]
which is satisfied by \( P(\mathcal{R}, z) = z^{-1} \exp(-\mathcal{R}/z) \). For this distribution one has \( \langle \mathcal{R} \rangle = z \).
In the opposite limit, $\mathcal{R} \gg 1$, we have
\[
\frac{\partial P}{\partial z} = \frac{\partial}{\partial \mathcal{R}} \left( \mathcal{R}^2 \frac{\partial}{\partial \mathcal{R}} \right) = \frac{\partial^2 P}{\partial \nu^2} + \frac{\partial P}{\partial \nu},
\]
where $\nu = \ln \mathcal{R}$. This is solved by the log-normal distribution,
\[
P(\mathcal{R}, z) = (4\pi z)^{-1/2} e^{-(\nu+z)^2/4z}.
\]
Note that
\[
P(\mathcal{R}, z) \, d\mathcal{R} = \tilde{P}(\nu, z) \, d\nu = \frac{1}{\sqrt{4\pi z}} e^{-(\nu-z)^2/4z} \, d\nu,
\]
One then obtains $\langle \nu \rangle = \langle \ln \mathcal{R} \rangle = z$. Furthermore,
\[
\langle \mathcal{R}^n \rangle = \langle e^{n\nu} \rangle = \frac{1}{\sqrt{4\pi z}} \int_{-\infty}^{\infty} d\nu \, e^{-(\nu-z)^2/4z} e^{n\nu} = e^{k(k+1)z}
\]
Note then that $\langle \mathcal{R} \rangle = \exp(2z)$, so the mean resistance grows exponentially with length. However, note also that $\langle \mathcal{R}^2 \rangle = \exp(6z)$, so
\[
\langle (\Delta \mathcal{R})^2 \rangle = \langle \mathcal{R}^2 \rangle - \langle \mathcal{R} \rangle = e^{6z} - e^{4z},
\]
and so the standard deviation grows as $\sqrt{\langle \mathcal{R}^2 \rangle} \sim \exp(3z)$ which grows faster than $\langle \mathcal{R} \rangle$. In other words, the resistance $\mathcal{R}$ itself is not a self-averaging quantity, meaning the ratio of its standard deviation to its mean doesn’t vanish in the thermodynamic limit – indeed it diverges. However, $\nu = \ln \mathcal{R}$ is a self-averaging quantity, with $\langle \nu \rangle = z$ and $\sqrt{\langle \nu^2 \rangle} = \sqrt{2z}$. 