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Possibility of an inverse cascade of magnetic helicity in magnetohydrodynamic turbulence

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Some of the consequences of the conservation of magnetic helicity

$$\int \mathbf{a}.\mathbf{b}\,d^3\mathbf{r} \quad (\mathbf{a} = \text{vector potential of magnetic field } \mathbf{b})$$

for incompressible three-dimensional turbulent MHD flows are investigated. Absolute equilibrium spectra for inviscid infinitely conducting flows truncated at lower and upper wavenumbers $k_{\min}$ and $k_{\max}$ are obtained. When the total magnetic helicity approaches an upper limit given by the total energy (kinetic plus magnetic) divided by $k_{\min}$, the spectra of magnetic energy and helicity are strongly peaked near $k_{\min}$; in addition, when the cross-correlations between the velocity and magnetic fields are small, the magnetic energy density near $k_{\min}$ greatly exceeds the kinetic energy density. Several arguments are presented in favour of the existence of inverse cascades of magnetic helicity towards small wavenumbers leading to the generation of large-scale magnetic energy.

1. Introduction

It is known that turbulent flows which are not statistically invariant under plane reflexions may be important for the generation of magnetic fields; for example, Steenbeck, Krause & Rädler (1966) have shown within the framework of the kinematic dynamo problem (prescribed velocity fields) that in helical flows, i.e. flows in which the velocity and vorticity are statistically correlated, a mean magnetic field may be amplified by the so-called α-effect. It has been recognized for quite some time (e.g. Batchelor 1950; Moffatt 1970, 1973) that, owing to the back reaction of the Lorentz force on the velocity field, the ultimate fate of the magnetic field and the magnetic energy can only be studied within the framework of the full MHD equations. These equations are

$$(\partial/\partial t - \nu\nabla^2)\,\mathbf{v} = -(\mathbf{v}.\nabla)\,\mathbf{v} + (\mathbf{b}.\nabla)\,\mathbf{b} - \nabla p + \mathbf{f}, \tag{1}$$

$$(\partial/\partial t - \lambda\nabla^2)\,\mathbf{b} = -(\mathbf{v}.\nabla)\,\mathbf{b} + (\mathbf{b}.\nabla)\,\mathbf{v}, \tag{2}$$

$$\nabla.\mathbf{v} = 0, \quad \nabla.\mathbf{b} = 0, \tag{3}$$

where $\mathbf{v}(\mathbf{r},t)$ and $(\mu\rho)^{\frac{1}{2}}\mathbf{b}(\mathbf{r},t)$ are the velocity and magnetic fields (SI units), ν is the kinematic viscosity, λ the magnetic diffusivity, μ the magnetic susceptibility, ρ the fluid density, ρp the total pressure, and $\mathbf{f}(\mathbf{r},t)$ a given solenoidal driving term which maintains statistical stationarity in the dissipative system. The ordinary Navier–Stokes equations resulting from (1)–(3) with $\mathbf{b} = 0$ will be called the *non-magnetic* equations.

The present paper is essentially based on the fact that for inviscid, unforced, infinitely conducting, differentiable flows vanishing at infinity, the MHD equations possess three quadratic invariants: one scalar, the total energy

$$E_T = \tfrac{1}{2}\int (v^2+b^2)\,d^3\mathbf{r}, \tag{4}$$

and two pseudo-scalars, the *magnetic helicity* (Elsässer 1956)

$$H_M = \tfrac{1}{2}\int \mathbf{a}\,.\,\mathbf{b}\,d^3\mathbf{r} \tag{5}$$

(where $\mathbf{a}$ = vector potential of magnetic field) and the *cross-helicity* (Woltjer 1958)

$$H_C = \tfrac{1}{2}\int \mathbf{v}\,.\,\mathbf{b}\,d^3\mathbf{r}. \tag{6}$$

An interpretation of the invariants (5) and (6) has been provided by Moffatt (1969). Note that the invariance of H_M follows from (2) and (3) alone (with $\lambda = 0$) and therefore puts a constraint even on the kinematic dynamo theories. For statistically homogeneous turbulence, the above integrals are divergent and should be replaced by their mean values per unit volume, denoted by $\langle E_T\rangle$, $\langle H_M\rangle$ and $\langle H_C\rangle$.

If we assume isotropy but not invariance under plane reflexions the simultaneous second-order moments may be written in the Fourier representation (within constant factors)

$$2\langle v_i(\mathbf{k})\,v_j^*(\mathbf{k})\rangle = P_{ij}(\mathbf{k})\,U_K(k) - i\epsilon_{ijf}k_f\tilde{U}_K(k), \tag{7}$$

$$2\langle v_i(\mathbf{k})\,b_j^*(\mathbf{k})\rangle = P_{ij}(\mathbf{k})\,\tilde{U}_C(k) - i\epsilon_{ijf}k_f U_C(k), \tag{8}$$

$$2\langle b_i(\mathbf{k})\,b_j^*(\mathbf{k})\rangle = P_{ij}(\mathbf{k})\,U_M(k) - i\epsilon_{ijf}k_f\tilde{U}_M(k), \tag{9}$$

where
$$k = |\mathbf{k}|, \quad P_{ij}(\mathbf{k}) = \delta_{ij} - k_ik_j/k^2 \tag{10}$$

and ϵ_{ijf} is the fundamental antisymmetric tensor. For non-helical turbulence all pseudo-scalars $\tilde{U}(k)$ vanish. The kinetic, magnetic and total energy spectra are given by

$$E_K(k) = 2\pi k^2 U_K(k), \quad E_M(k) = 2\pi k^2 U_M(k), \tag{11}$$

$$E_T(k) = E_K(k) + E_M(k), \tag{12}$$

and the kinetic, magnetic and cross-helicity spectra by

$$\left.\begin{aligned} H_K(k) &= 2\pi k^4\tilde{U}_K(k), \quad H_M(k) = 2\pi k^2\tilde{U}_M(k),\\ H_C(k) &= 2\pi k^2\tilde{U}_C(k). \end{aligned}\right\} \tag{13}$$

A consequence of the positive definiteness of the spectral tensors is that the three helicity spectra satisfy realizability conditions:

$$|H_K(k)| \leqslant kE_K(k), \tag{14}$$

$$|H_M(k)| \leqslant E_M(k)/k \leqslant E_T(k)/k, \tag{15}$$

$$|H_C(k)| \leqslant \{E_K(k)\, E_M(k)\}^{\frac{1}{2}} \leqslant \tfrac{1}{2}E_T(k). \tag{16}$$

When (14) is an equality for some k, we say that the kinetic helicity is *maximal* at that wavenumber (similarly for the magnetic and cross-helicity). When this holds for arbitrary k, we say that there is a state of maximal helicity. As noticed by Kraichnan (1973), maximal kinetic helicity cannot persist under evolution determined by the equations of motion.

Note that $\langle E_T\rangle$, $\langle H_M\rangle$ and $\langle H_C\rangle$ are related to their spectra by

$$\langle E_T\rangle = \int_0^\infty E_T(k)\,dk, \quad \langle H_M\rangle = \int_0^\infty H_M(k)\,dk, \quad \langle H_C\rangle = \int_0^\infty H_C(k)\,dk. \quad (17)\text{–}(19)$$

It must be stressed that $H_K(k)$, $H_M(k)$ and $H_C(k)$ are *not* positive definite; therefore, for example, the mean magnetic helicity $\langle H_M\rangle$ may be zero with

$$H_M(k) \not\equiv 0.$$

In the *non-magnetic* case the corresponding invariants are kinetic energy and kinetic helicity (Betchov 1961;† Moffatt 1969). Curiously, in the transition to MHD the energetic invariant becomes the total energy whereas kinetic helicity is no longer invariant; magnetic helicity does not even have the same dimensions as kinetic helicity. The effect of kinetic helicity on non-magnetic turbulent flows has been considered in recent papers (Brissaud *et al.* 1973; Kraichnan 1973; Frisch, Lesieur & Brissaud 1974). The main conclusions of these papers are now briefly summarized for later use.

(i) When kinetic helicity is injected at a constant rate η into a turbulent fully developed isotropic flow with an energy spectrum $E_K(k)$, this kinetic helicity is carried along by the energy cascade, giving rise to a simultaneous helicity cascade of the form

$$H_K(k) \sim (\eta/\epsilon)\, E_K(k), \tag{20}$$

where ϵ is the energy transfer rate. Since $H_K(k)$ is linear in η, this cascade is conveniently called *linear*. This result is supported by phenomenological arguments (Brissaud *et al.* 1973) and numerical calculations on a soluble stochastic model, which indeed show that, in the inertial range, $(\epsilon/\eta)\; H_K(k)/E_K(k)$ is a purely numerical constant (Lesieur 1973). The possibility of an inverse helicity cascade, considered by Brissaud *et al.* (1973), seems now to be ruled out both by the results of numerical experiments and by an argument of Kraichnan (1973). He argued that, owing to the absence of absolute equilibrium states with energy peaked at low wavenumbers (as for two-dimensional turbulence), a simultaneous cascade of energy from intermediate wavenumbers to low wavenumbers and of helicity from intermediate wavenumbers to high wavenumbers is unlikely.

† This reference contains certain minor errors, particularly in equations (2) and (8).

(ii) The presence of kinetic helicity, especially maximal helicity, inhibits energy transfer to large wavenumbers. This was first demonstrated by calculations for two interacting helical waves (Kraichnan 1973) and by numerical resolution of the direct-interaction approximation equation (J. Herring, private communication). It is also supported by direct numerical integration of the Navier–Stokes equations (S. Patterson, private communication) and by a simple argument due to S. Patterson and W. V. R. Malkus: a non-zero helicity $\int \mathbf{v}.\boldsymbol{\omega} d^3\mathbf{r}$ imposes certain phase relations between the velocity $\mathbf{v}$ and the vorticity $\boldsymbol{\omega}$, e.g. $H_K > 0$ means that $\mathbf{v}$ and $\boldsymbol{\omega}$ have a tendency to align; this obviously reduces the nonlinear term $\mathbf{v}\times\boldsymbol{\omega}$ in the Navier–Stokes equations, which means that transfer is inhibited.

(iii) Application of Gibbs statistical mechanics to the inviscid non-magnetic equations truncated in Fourier space produces *absolute-equilibrium* solutions with a density in phase space proportional to $\exp\{-\alpha E_K - \beta H_K\}$, where α and β are expressible in terms of the mean kinetic energy and helicity. The corresponding energy spectrum is

$$E_K(k) = 4\pi\alpha k^2/(\alpha^2-\beta^2k^2) \quad (k_{\min} < k < k_{\max}), \tag{21}$$

with the positivity conditions α, $\alpha^2-\beta^2k^2 > 0$. The presence of helicity ($\beta \neq 0$) may strongly enhance the excitation found at large wavenumbers near $k_{\max}$ (Kraichnan 1973).

2. Absolute-equilibrium helical MHD ensembles

Following Kraichnan (1967, 1973), we consider the absolute statistical equilibrium ($\nu = \lambda = 0$, $\mathbf{f} = 0$) of the truncated MHD equations. They are obtained by considering solutions which, being periodic in space, are represented by Fourier series and then retaining only those Fourier components $\mathbf{v}(\mathbf{k})$ and $\mathbf{b}(\mathbf{k})$ of the velocity and magnetic fields for which $|\mathbf{k}| = k \in I = (k_{\min}, k_{\max})$, and dropping any nonlinear interaction terms involving one or several wavenumbers outside this interval I. The energy invariant

$$E_T = \tfrac{1}{2}\sum_k (v_i(k)\,v_i^*(k) + b_i(k)\,b_i^*(k)) \tag{22}$$

survives under truncation. Indeed, take the *untruncated* equations and assume that, at some time t, $\mathbf{v}(\mathbf{k}) = \mathbf{b}(\mathbf{k}) = 0$ for $k \notin I$. This does not imply that their time derivatives $\dot{\mathbf{v}}(\mathbf{k})$ and $\dot{\mathbf{b}}(\mathbf{k})$ are zero for $k \notin I$. However, such terms will not contribute to dE_T/dt at time t since $v_i(\mathbf{k})\,\dot{v}_i^*(\mathbf{k})$, $\dot{v}_i(\mathbf{k})\,v_i^*(\mathbf{k})$, $b_i(\mathbf{k})\,\dot{b}_i^*(\mathbf{k})$ and $\dot{b}_i(\mathbf{k}) \times b_i^*(\mathbf{k})$ are all zero for $k \notin I$. It follows that, at time t, dE_T/dt has the same value, zero, for the untruncated and truncated equations (R. H. Kraichnan, private communication). The same proof holds for any quadratic invariant which is a sum of contributions from individual Fourier components, e.g. the magnetic and cross-helicity

$$H_M = \tfrac{1}{2}\sum_k ik^{-2}\epsilon_{rmn}\,b_r(\mathbf{k})\,b_m^*(\mathbf{k})\,k_n, \tag{23}$$

$$H_C = \tfrac{1}{4}\sum_k (v_i(\mathbf{k})\,b_i^*(\mathbf{k}) + v_i^*(\mathbf{k})\,b_i(\mathbf{k})). \tag{24}$$

Notice that in (22)–(24) E_T, H_M and H_C are not the mean values of invariants but the stochastic values corresponding to individual realizations.

The truncated equations of motion possess canonical equilibrium ensembles for which the density in phase space is

$$\rho = Z^{-1} \exp(-\alpha E_T - \beta H_M - \gamma H_C), \tag{25}$$

where Z is a normalizing constant. This density is the exponential of a quadratic form and therefore defines a Gaussian ensemble. To calculate the second-order moments of the components of the velocity and magnetic fields the following lemma is used.

LEMMA. The multivariate Gaussian density

$$\rho(x) = Z^{-1} \exp\{-\tfrac{1}{2} \sum_{i,j} A_{ij} x_i x_j\} \tag{26}$$

has second-order moments given by

$$\langle x_i x_j \rangle = A^{-1}_{ij}, \tag{27}$$

where $\mathbf{A}^{-1}$ is the inverse of $\mathbf{A}$.

This result is obvious when $\mathbf{A}$ is diagonal and the general case is reducible to this special case by a linear orthogonal transformation (see also Lumley 1970, p. 38).

To apply this result we must determine, for each wave vector $\mathbf{k}$, the real and imaginary parts of the two components of $\mathbf{v}(\mathbf{k})$ in a plane perpendicular to $\mathbf{k}$ (since $\mathbf{k}.\mathbf{v}(\mathbf{k}) = 0$) and similarly for $\mathbf{b}(\mathbf{k})$. Clearly the matrix $\mathbf{A}$ separates into 8×8 blocks corresponding to the different wave vectors. Each such block reads

$$\mathbf{A}(k) = \begin{pmatrix} \alpha & 0 & 0 & 0 & \frac{1}{2}\gamma & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & \frac{1}{2}\gamma & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 & \frac{1}{2}\gamma & 0 \\ 0 & 0 & 0 & \alpha & 0 & 0 & 0 & \frac{1}{2}\gamma \\ \frac{1}{2}\gamma & 0 & 0 & 0 & \alpha & 0 & 0 & \beta/k \\ 0 & \frac{1}{2}\gamma & 0 & 0 & 0 & \alpha & -\beta/k & 0 \\ 0 & 0 & \frac{1}{2}\gamma & 0 & 0 & -\beta/k & \alpha & 0 \\ 0 & 0 & 0 & \frac{1}{2}\gamma & \beta/k & 0 & 0 & \alpha \end{pmatrix}. \tag{28}$$

The inverse of this is given by

$$\mathbf{A}^{-1}(k) = \frac{1}{\Delta} \begin{pmatrix} P & 0 & 0 & X & Q & 0 & 0 & Y \\ 0 & P & -X & 0 & 0 & Q & -Y & 0 \\ 0 & -X & P & 0 & 0 & -Y & Q & 0 \\ X & 0 & 0 & P & Y & 0 & 0 & Q \\ Q & 0 & 0 & Y & R & 0 & 0 & W \\ 0 & Q & -Y & 0 & 0 & R & -W & 0 \\ 0 & -Y & Q & 0 & 0 & -W & R & 0 \\ Y & 0 & 0 & Q & W & 0 & 0 & R \end{pmatrix}, \tag{29}$$

where
$$\Delta = (\alpha^2 - \tfrac{1}{4}\gamma^2)^2 - k^{-2}\alpha^2\beta^2,$$

$$W = -\alpha^2\beta/k, \quad X = -\beta\gamma^2/4k, \quad Y = \alpha\beta\gamma/2k,$$

$$P = \frac{\Delta}{\alpha} + \frac{\gamma^2}{4\alpha}\left(\alpha^2 - \frac{\gamma^2}{4}\right), \quad Q = -\tfrac{1}{2}\gamma(\alpha^2 - \tfrac{1}{4}\gamma^2), \quad R = \alpha(\alpha^2 - \tfrac{1}{4}\gamma^2).$$

It can be checked that the second-order moments given by (27) have the general form given by (7)–(9) since we are dealing with an isotropic homogeneous situation. Finally, the various energy and helicity spectra are

$$E_k(k) = \frac{4\pi}{\alpha}\left(1+\frac{tg^2\phi}{D(k)}\right)k^2, \quad H_K(k) = -\frac{4\pi k_c tg^2\phi}{\alpha}\frac{k^2}{D(k)}, \tag{30}$$

$$E_M(k) = \frac{4\pi}{\alpha\cos^2\phi}\frac{k^2}{D(k)}, \quad H_M(k) = -\frac{4\pi k_c}{\alpha\cos^2\phi}\frac{1}{D(k)}, \tag{31}$$

$$H_C(k) = -\frac{4\pi}{\alpha}\frac{\sin\phi}{\cos^2\phi}\frac{k^2}{D(k)}, \tag{32}$$

where $\sin\phi = \gamma/2\alpha$ $(-\frac{1}{2}\pi < \phi < \frac{1}{2}\pi)$, $k_c = \beta/\alpha\cos^2\phi$ and $D(k) = 1-(k_c/k)^2$. For the above calculations to be correct, the density ρ must be normalizable, i.e. the quadratic form $\alpha E_T + \beta H_M + \gamma H_C$ must be positive definite; this is easily seen to require $\alpha > 0$, $|\gamma| < 2\alpha$ and $D(k) > 0$, which implies that $|k_c| < k_{\min}$.

By integrating (30)–(32) over k from $k_{\min}$ to $k_{\max}$, it is possible to express the three fundamental invariants $\langle E_T\rangle$, $\langle H_C\rangle$ and $\langle H_M\rangle$ in terms of α, β, γ, $k_{\min}$ and $k_{\max}$ (see appendix). Conversely, given $k_{\min}$, $k_{\max}$, $\langle E_T\rangle$, $\langle H_C\rangle$ and $\langle H_M\rangle$ satisfying realizability conditions, α, β and γ are uniquely determined but cannot be obtained in closed analytic form. As special cases of interest, notice that $\beta = 0$ is equivalent to $\langle H_M\rangle = 0$ and $\gamma = 0$ is equivalent to $\langle H_C\rangle = 0$.

If an ensemble of inviscid truncated MHD flows has a non-equilibrium initial distribution in phase space, it will relax, assuming the mixing property of the dynamical system (Wightman 1971), to the canonical ensemble (25) with α, β and γ depending only on the initial energy, cross-helicity and magnetic helicity and not on the initial kinetic helicity.

Let us now consider some of the implications of the equilibrium spectra (30)–(32). In the absence of mean magnetic helicity ($\beta = 0$), (30)–(32) give the classical $E(k) \sim k^2$ spectrum obtained by Lee (1952) with equipartition of magnetic and kinetic energy. When $\beta \neq 0$, there is still approximate equipartition ($E(k) \sim k^2$) for $k \gg |k_c|$, a result substantially different from the corresponding non-magnetic one (see § 1 (iii)). However, for values of k near $k_{\min}$ the situation may be quite different if

$$k_{\min} - |k_c| \ll |k_c|. \tag{33}$$

At this point it is convenient to consider separately the cases $\gamma = 0$ and $\gamma \neq 0$. When $\gamma = 0$ and (33) holds, there is near $k_{\min}$ a region of almost maximal helicity with very large values of $E_M(k)$ and $H_M(k)$; the mean total energy $\langle E_T\rangle$ in the interval $(k_{\min}, k_{\max})$ is made up essentially of magnetic energy and is related to the mean magnetic helicity $\langle H_M\rangle$ by

$$k_{\min}|\langle H_M\rangle| \simeq \langle E_T\rangle. \tag{34}$$

Since $k_{\min}|\langle H_M\rangle| \leqslant \langle E_T\rangle$ is a consequence of (15), these states with high magnetic excitation near $k_{\min}$ can also be defined as states of high magnetic helicity. For $\gamma \neq 0$, with (31) still holding, the correlations between **v** and **b** produce also a high magnetic excitation near $k_{\min}$ but the magnetic energy density near $k_{\min}$ still exceeds the kinetic energy density by a factor of order $1/\sin^2\phi$ and the ratio

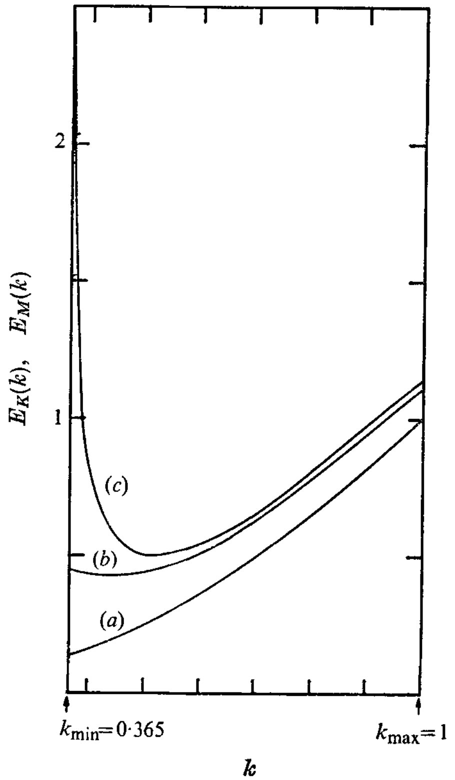

FIGURE 1

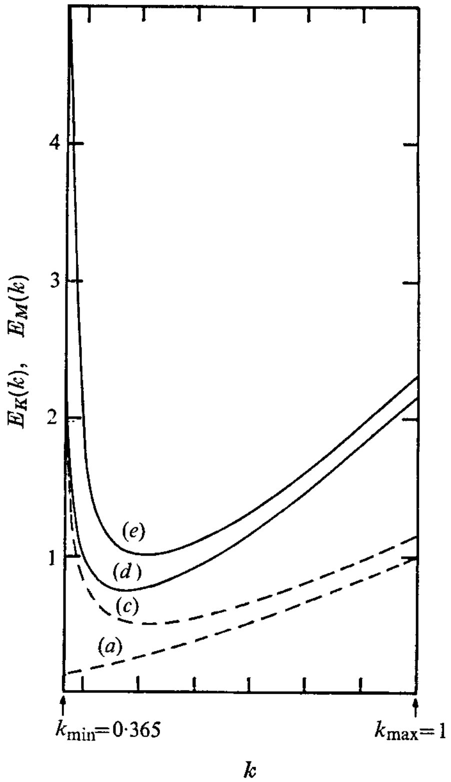

FIGURE 2

FIGURE 1. Energy densities: effect of magnetic helicity when $\langle H_C \rangle = 0$; $k_{min} = 0{\cdot}365$, $k_{max} = 1$, $\alpha = 4\pi$, $\phi = 0$; $\langle E_K \rangle \simeq 0{\cdot}32$. (*a*) Kinetic energy density for $k_c = 0$, 0·32 and 0·355 and magnetic energy density for $k_c = 0$; $\langle H_M \rangle = 0$, $\langle E_M \rangle \simeq 0{\cdot}32$. (*b*) Magnetic energy density for $k_c = 0{\cdot}32$; $\langle H_M \rangle \simeq -0{\cdot}30$, $\langle E_m \rangle \simeq 0{\cdot}41$, $k_{min}|\langle H_M \rangle|/\langle E_T \rangle \simeq 0{\cdot}15$. (*c*) Magnetic energy density for $k_c = 0{\cdot}355$; $\langle H_M \rangle \simeq -0{\cdot}45$, $\langle E_M \rangle \simeq 0{\cdot}47$, $k_{min}|\langle H_M \rangle|/\langle E_T \rangle \simeq 0{\cdot}21$.

FIGURE 2. Energy densities: effect of cross-helicity; $k_{min} = 0{\cdot}365$, $k_{max} = 1$, $\alpha = 4\pi$. (*a*), (*c*) As in figure 1. (*d*) Kinetic energy density and (*e*) magnetic energy density for $k_c = 0{\cdot}355$, $\phi = \frac{1}{4}\pi$; $\langle H_M \rangle \simeq -0{\cdot}9$, $\langle H_C \rangle \simeq -0{\cdot}67$, $\langle E_K \rangle \simeq 0{\cdot}79$, $\langle E_M \rangle \simeq 0{\cdot}95$, $k_{min}|\langle H_M \rangle|/\langle E_T \rangle \simeq 0{\cdot}19$, $2|\langle H_C \rangle|/\langle E_T \rangle \simeq 0{\cdot}77$.

$k_{min}\langle H_M \rangle/\langle E_T \rangle$ cannot exceed $(1+\sin^2\phi)^{-1}$. To illustrate the above results, we have plotted $E_K(k)$ and $E_M(k)$ for different values of the ratio $k_{min}\langle H_M \rangle/\langle E_T \rangle$ for $\gamma = 0$ in figure 1 and for two different values of γ in figure 2.

The situation described here is reminiscent of two-dimensional non-magnetic absolute equilibria (Kraichnan 1967). In both cases, there is the possibility of high excitation at low wavenumbers. The essential differences are that in the two-dimensional case the equilibria with high excitation near k_{min} have negative α, i.e. negative temperatures, and are low enstrophy states.

3. Magnetic helicity cascades in fully developed isotropic MHD turbulence

For MHD turbulent flows with statistical invariance under plane reflexions (no helicity) it is generally accepted that energy (kinetic + magnetic) cascades towards large wavenumbers. Following Kraichnan (1965), Kraichnan & Nagarajan (1967) and Orszag & Kruskal (1968), let us assume that there is a (local or semi-local) inertial range with an energy spectrum $E_T(k)$ and an energy transfer rate ϵ. What happens if magnetic helicity is injected at a constant rate η at some wavenumber in the inertial range? The case when cross-helicity is also injected will not be considered in this paper. Since magnetic helicity is conserved by the nonlinear interactions, a cascade process is expected. Contrary to the non-magnetic case (cf. § 1 (i)), we must now rule out an upward linear cascade of magnetic helicity to increasing wavenumbers of the form

$$H_M(k) \sim (\eta/\epsilon)\, E_T(k), \tag{35}$$

because such a cascade cannot be continued beyond the wavenumber $k \sim \epsilon/\eta$ where maximal magnetic helicity is reached. It is thus difficult to escape the conclusion that the presence of magnetic helicity must produce some important change in fully developed MHD turbulence.

Instead of an upward cascade, a possibility which must be seriously envisaged is that of an *inverse cascade* of magnetic helicity to small wavenumbers, which cannot be a linear cascade because it is not carried by an energy cascade. Furthermore, all the triple correlations appearing in the equation for the evolution of magnetic helicity involve the velocity field, and thus a purely magnetic and local cascade is impossible (R. H. Kraichnan, private communication). In fact, large-scale velocity fields will necessarily be generated by the Lorentz-force term in the Navier–Stokes equations; the inverse cascade then proceeds through the interaction of the velocity fields with the large-scale magnetic fields.

The results of § 2, showing the possibility of high magnetic excitation at small wavenumbers, indicate that an inverse cascade of magnetic helicity accompanied by magnetic energy may indeed exist. It is true that the absolute-equilibrium ensembles are very far from the actual non-equilibrium state of turbulence. Nevertheless (Kraichnan 1973) their value may be in showing the directions in which the actual states may plausibly be expected to transfer excitation, since within any given wavenumber band the characteristic time for reaching absolute equilibrium is the same as the eddy turnover time. For two-dimensional non-magnetic turbulence, the prediction by the absolute-equilibrium ensembles of an inverse energy cascade has, in fact, been verified by numerical experiment (Lilly 1969). As an additional argument in favour of an inverse cascade (cf. Fjørtoft 1953), note that simultaneous up-transfer of total energy and magnetic helicity is impossible. Suppose that an initial state of maximal helicity is confined to two wavenumbers **p** and **q** ($p < q$), and let this excitation be entirely transferred to the wavenumber **k**. From the conservation of total energy and magnetic helicity we have

$$E_T(k) = E_T(p) + E_T(q), \tag{36}$$

$$H_M(k) = H_M(p) + H_M(q) = E_T(p)/p + E_T(q)/q. \quad (37)$$

This obviously violates (15) if $k > q$.

The invariance of magnetic helicity holds only under the assumption that $\mathbf{b}(\mathbf{r}, t)$ vanishes at infinity or (in the statistically homogeneous case) that the mean magnetic field vanishes. In the presence of a prescribed uniform magnetic field $\mathbf{b}_0$, the invariance of magnetic helicity disappears and so probably do the corresponding cascades. This should not be surprising, since it is known that large-scale magnetic fields differ very much from large-scale velocity fields in their effects: they cannot be suppressed by a Galilean transformation.

4. Concluding remarks

We have found new evidence that the absence of statistical symmetry in isotropic turbulent flows may be of interest in the MHD turbulence problem, possibly leading to the growth of magnetic energy and the appearance of large-scale magnetic fields, a question of considerable astrophysical interest.

It is tempting to compare our present approach with the approach of Steenbeck *et al.* (1966). There are two important differences: (i) Steenbeck *et al.* make use only of the linear Ohm's law, which incorporates the magnetic helicity invariant but no energy invariant, whereas our approach is based on the invariants of the full nonlinear equations; (ii) in the analysis of Steenbeck *et al.*, only kinetic helicity appears. As kinetic helicity is not conserved when nonlinear effects are included, the magnitude of the α-effect is modified (as analysed for example, by Moffatt 1970). Magnetic helicity is however conserved, and our theory includes explicit consideration of this constraint.

We are grateful to Dr R. H. Kraichnan for a very illuminating discussion and to the National Center of Atmospheric Research (Boulder, Colorado), where part of this work was done.

Appendix

The expressions for $\langle E_K \rangle$, $\langle E_M \rangle$, $\langle H_M \rangle$ and $\langle H_C \rangle$ in terms of α, β, γ, $k_{\max}$ and $k_{\min}$ are [from (30)–(32)]

$$\langle E_K \rangle = \frac{4\pi}{\alpha \cos^2 \phi}\left(\frac{k_{\max}^3 - k_{\min}^3}{3} + k_c^2 I \sin^2 \phi\right),$$

$$\langle E_M \rangle = \frac{4\pi}{\alpha \cos^2 \phi}\left(\frac{k_{\max}^3 - k_{\min}^3}{3} + k_c^2 I\right),$$

$$\langle H_M \rangle = -\frac{4\pi}{\alpha \cos^2 \phi} k_c I,$$

$$\langle H_C \rangle = -\frac{4\pi \sin \phi}{\alpha \cos^2 \phi}\left(\frac{k_{\max}^3 - k_{\min}^3}{3} + k_c^2 I\right),$$

where $$I = \int_{k_{\min}}^{k_{\max}} \frac{dk}{D(k)} = k_{\max} - k_{\min} + \frac{k_c}{2}\log\frac{(k_{\max} - k_c)(k_{\min} + k_c)}{(k_{\max} + k_c)(k_{\min} - k_c)}.$$

Note added in proof. The prediction of an inverse cascade of magnetic helicity is now supported by (i) a direct numerical simulation of the MHD equations with 32^3 Fourier modes due to S. Patterson and A. Pouquet, which shows a transfer of magnetic excitation towards small wavenumbers, and (ii) the numerical integration of the spectral equations obtained from the eddy damped quasi-normal approximation, which yields an indefinite inverse cascade.

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† This and subsequent papers appear in translation in Roberts & Stix (1971).