Taylor Relaxation and its Dynamics

- Taylor Relaxation
- Buried bodies on Taylor Hypothesis
  - Relation to stochastic fields, turbulence
- KFP
- Mean Field Theory
- Selective Decay
so... \[ k_1 = \Phi_1 \Phi_2 \Rightarrow \text{product of fluxes} \]

Similarly \[ k_2 = \Phi_1 \Phi_2 \]

.: \[ k = 2\Phi_1 \Phi_2 \]

If \( n \) winding \[ k = k_1 + k_2 = \pm 2n\Phi_1 \Phi_2 \]

\[ \Rightarrow \text{helicity is measure of self-linkage of magnetic configuration.} \]

Topological constraint.

Why care \( \Rightarrow \text{Taylor Conjecture} \) \( (1974) \) \( \text{(J.B. Taylor)} \)

- in magnetic confinement of great interest to determine how field, currents self-agranize \( \nabla \times B = 0 \)

- RFP \[ \Rightarrow \text{toroidal} \]
  \[ \Rightarrow \text{toroidal current} \]

well fit by \[ B_2 = B_0 \frac{\nabla}{\nabla} (x \cdot \n) \]
\[ B_0 = B_0 \frac{\nabla}{\nabla} (x \cdot \n) \]
\[ \nabla \times B = 0 \]
\[ \nabla B = 0 \]

Force free

\[ \Rightarrow \text{why so robust?} \]
especially since RFP so turbulent
Taylor Relaxation

- Transition to quiescent period
- Relaxation $\rightarrow$ turbulent regime
- Magnetic energy minimized
  (fast only and $\beta << 1$)
- What constraint $\Omega$

$\text{in \underline{cylindrical \ N\,orma}}$

$\int d^3x \mathbf{A} \cdot \mathbf{B}$ conserved for all

$\int d^3x$ c.i.e any tube, around line

$\int_{\text{tube}} \int d^3x \mathbf{A} \cdot \mathbf{B} = \text{const.}$

$\mathbf{A} \times \mathbf{B} = \partial \mathbf{A} \times \partial \mathbf{B}$
\[ \nabla \times B = \nabla \times (\gamma g \beta) B \quad \text{force-free case} \]

\[ \gamma \cdot \nabla \times \gamma = 0 \]

\[ \nabla \times B = \nabla \times (\gamma g \beta) B \nabla \cdot B = 0 \]

force-free or micro-tube only line

But \[ \nabla \times (\gamma g \beta) \neq \nabla \times (\gamma g \beta') \]

\( \Rightarrow \) Each tube line defines conserved helicity

\( \Rightarrow \infty \) of invariants due freezing on.

\( \Rightarrow \) Relaxation occurs in resistive turbulent plasma, \( T_R \sim \frac{1}{\tau_{A \mu}} \)

\( \Rightarrow \) Small tubes are destroyed by reconnection

\( \Rightarrow \text{as } T_R \to 0 \) only very largest tube survives \( \Rightarrow \) global helicity is asymptotic survivor

---

Could also view from static lines \( \Rightarrow 1 \) line
\[ V = \frac{U}{\sqrt{Rm}} \approx \frac{\| \mathbf{A} \|}{L} \]

\[ \frac{L}{R_{L}} \sim \frac{L}{L^{3/2}} \Rightarrow \text{smaller scales reconnect faster,} \]

\[ \Rightarrow \text{smaller tubes destroy first.} \]

- Arguments for conjecture of global helicity as neglected invariant:
  - enhanced dissipation (above) \( \rightarrow \) largest scales reconnect most slowly
  - Stochasticity \( \rightarrow \) if field lines stochastic then (at Fermi \( \rightarrow \) MNR)
  - 1 field line \( \rightarrow \) 1 tube \( \rightarrow \) conserved helicity \( \rightarrow \) global helicity is only on vortex
  - \( \Rightarrow \) RFP has only 1 field line.
- Selective decay

- Magnetic helicity
  (inverse cascades) on
  3D MHD

- Global
  Close scale
  Helicity accumulated

- N-B component
energy

**Heuristic**

\[ W = -2 \sqrt{\langle B^2 \rangle} \quad (\text{if } \mathbf{m} = 0) \]

\[ \frac{\mathbf{L} = \int \mathbf{B} \times \mathbf{A} \cdot \mathbf{B} \geq 0}{\text{left}} \]

\[ W \sim -2 \mathbf{m} \frac{\langle B^2 \rangle}{L^2} \quad \text{left} \]

\[ L \sim -2 \mathbf{m} \frac{\langle B^2 \rangle}{L^2} \quad \text{left} \]

\[ \mathbf{F} \quad \text{left} \sim \Delta \sim L \sqrt{Rm} \sim M^{1/2} \]
\[ \omega \rightarrow \eta \rightarrow \text{finite} \rightarrow \text{infinite dissipation} \quad \omega \rightarrow \text{finite} \rightarrow \text{infinite} \quad \eta \rightarrow 0 \]

\[ \omega \rightarrow \eta = \text{constant} \rightarrow \text{constant} \]

\[ \nabla \times B = \mu \nabla B \]

\[ \nabla \cdot B = \text{constant} = \eta \]

\[ \nabla \cdot B \bigg/ B^2 \rightarrow \text{constant} = \eta \]

\[ \text{homogeneous} \]

\[ \int B \times A \text{ related to volt-second in } \Phi = \text{plasma, } U/\Omega \text{ transformer.} \]
Taylor theory predicts $E \rightarrow 0$ curve well.

\[ \Theta = \mu q/2 = 2 I/a B_0 \]

need $\mu q > 2.4$ created externally.

$\Theta > 1.2$

$F = B_{z \text{wall}} / <B>$

pretty good
N.B. An unpleasant reality:
- Relaxation $\rightarrow$ stock/tub.
- stock/tub $\rightarrow$ loss.

\[ L = \int \frac{4x^3 uJ^2}{v_2 n} \]

\[ Q = \varphi \text{ de. } \sim \varphi^2 \text{ for } \gamma \]

\[ \frac{b u}{D m} \frac{J}{L} \]

- Heat flux driven dynamo...
  
  Confinement bed.
- Taylor conjectured conservation of magnetic helicity constrains relaxation to force-free state.

Key Point: helicity conserved in flux tubes to
- toroidal plasma → many small tubes
- etc.

Recall Sweet-Parker model: magnetic reconnection/negligible dissipation effective on small scales.

$\Rightarrow$ Taylor Conjecture: At finite $M$, helicity of small tubes dissipated but $\rightarrow$ Global helicity conserved.

$c.e.: \quad \int_A \cdot B \, d^3x = \Phi_0 \rightarrow \circ \text{ conserved.}$

Taylor conjectured that actual magnetic configuration could be explained by minimum principle.
- Taylor conjectured conservation of magnetic helicity constrains relaxation to a free-free state.

key point - helicity conserved in flux tubes to
- toroidal plasma → many small tubes \( \frac{\gamma A \sim L^{3/2}}{L} \)
- etc. \( \frac{V}{L} = \frac{V_{L} / \kappa_{m}}{L^{3/2}} \)
- recall Sweet-Parker model: magnetic reconnection / resistive dissipation effective on small scales.

⇒ Taylor Conjecture: At finite \( M \), helicity of small tubes dissipated but \( H_{0} \) (global) helicity conserved.

c.e. \( \int A \cdot B \, d^{3}x = H_{0} \rightarrow \odot \) conserved.

⇒ Plasma volume

Taylor conjectured that actual magnetic configuration could be explained by minimum principle.
\[
\delta \left[ \int \frac{d^3 x}{8\pi} B^2 + \lambda \int d^3 A \cdot B \right] = 0
\]

d.e. minimize magnetic energy subject to constraint of conserved global helicity,

Comments:

→ it works! — indeed amazingly well — for RFPs, spheromaks, etc. Departures only recently being discovered.

→ inspired idea of helicity injection as way to maintain configurations.

→ it is a conjecture — no proof.

Hypothesis: Selective Decay

Dust Cascade

- Relevance to driven system? c.e. in real RFP, transformer on $T \sim L^{3/2}$
→ dynamics?

→ how does relaxation occur

→ more in discussion of limits.

\[
\int \left[ \int \frac{\partial^2}{\partial t^2} + \lambda A \cdot B \right] dV =
\]

\[
\frac{\partial E}{\partial t} + \lambda A \cdot DB = 0
\]

\[
\frac{\partial A}{\partial t} + \lambda A = 0
\]

\[
V = UB
\]

\[
\nabla \times B = \mu B
\]

\[
\nabla \times \nabla \times B = \mu B
\]

\[
\nabla \cdot J = 0 \rightarrow \text{Parallel current; homogenized}
\]
II) Dynamics of Taylor Relaxation

1. How represent dynamics of relaxation?
   How does system evolve to Taylor state?
   (general)

2. How does RHP drive poloidal currents
   which produce reversed toroidal field
   (specific)

3. How relate to more general concepts of
   relaxation, dynamo?
   - Self-organized
   - Criticality...

4-6. Mean Field Electrodynamics
   - i.e. how calculate \( \langle \mathbf{\Omega} \times B \rangle \)
   - goal to turbulence driven EMF
   - akin \( \langle E \circ \mathbf{f} \rangle \) in QLT
   - issues: structure, symmetry
     - origin of irreversibility
     - conservation properties

- topic is fundamental to subject of dynamo
  theory
  - flow counterpart: zonal flow generation
    (Monday lecture)
Good resource:

www.cgf.edu.A/KB/HKM

items 28, 46

Keith Moffatt picks

§ Structural/Symmetry Argument

Approach I (Boozer '86)

Write Ohm's Law in form:
(Mean Field)

\[ \langle E \rangle + \langle v \times B \rangle = \langle S \rangle + n \langle J \rangle \]

Hereafter, ignore un-resolved EMF, "something"

What is \(<S>\)?

Taylor \(
\begin{align*}
\text{i}) & : S & \text{must not dissipate } H \text{M} \\
\text{ii}) & : S & \text{must dissipate } E \text{M}
\end{align*}
\)

Now

\[ 2 \int d^3x \langle \mathbf{E} \times \mathbf{B} \rangle = 2 \int d^3x \left[ \mathbf{A} \times \mathbf{E} \times \mathbf{B} \right] \]

\[ = -2c \int d^3x \left[ \langle \mathbf{E} + \nabla \phi \rangle \cdot \mathbf{B} \right] \]

\[ = -2c \int d^3x \left[ \langle E \rangle \cdot \mathbf{B} \right] \]

\[ \mathbf{S} \cdot \mathbf{B} = 0 \]

\[ \text{to } S \text{.} \]

Now

\[ = -2c \int d^3x \left[ \mathbf{S} \cdot \mathbf{B} + n \mathbf{S} \cdot \nabla \phi \right] \]
Now, to conserve $H, M$, 2nd term must integrate to $S.T., \delta$:

$$\left\langle S \right\rangle = \frac{B \cdot \nabla \cdot \mathbf{H}}{B^2} \implies \text{Flux driving helicity evolution}$$

For form $\mathbf{H}$, consider energy:

$$\int d^3x \frac{B^2}{8\pi} = \int d^3x \frac{\mathbf{B} \cdot \mathbf{d} \mathbf{B}}{4\pi}$$

$$= -\int d^3x \frac{\mathbf{B} \cdot \nabla \times \mathbf{E}}{4\pi}$$

$$= -\int d^3x \mathbf{E} \cdot \mathbf{J}$$

$$= -\int d^3x \left[ \frac{n J^2 + (\mathbf{J} \cdot \mathbf{B}) \nabla \cdot \mathbf{H}}{B^2} \right]$$

$$= -\int d^3x \left[ \nabla \cdot \mathbf{J} - \mathbf{H} \cdot \nabla \left( \frac{\mathbf{J} \cdot \mathbf{B}}{B^2} \right) \right]$$

\[\text{flux force}\]

i.e., \[\frac{dS}{dt} = -\left( \nabla \cdot \mathbf{F} \right) = \propto \mathbf{B} \cdot \mathbf{v}^2, \text{ general form}\] (entropy)
\[ \partial_t E_M = \int d^3x \, \left[ \mathbf{H} \cdot \nabla \left( \frac{\mathbf{J}_H}{B} \right) \right] \]

so \[ \mathbf{H} = -x \nabla \left( \frac{\mathbf{J}_H}{B} \right) \]

\[ \partial_t E_M = -\int d^3x \, \lambda \left[ \nabla \left( \frac{\mathbf{J}_H}{B} \right) \right]^2 \]

and:

\[ \langle E \rangle = m \langle J \rangle = \frac{e B}{B^2} \mathbf{V} \cdot \left[ + x \nabla \left( \frac{\mathbf{J}_H}{B^2} \right) \right] \]

simplified form:

\[ \langle E \rangle = m \mathbf{J}_H - \frac{e}{2} \mathbf{V} \times \mathbf{J}_H \]

\( \lambda \) = 'hyper-resistivity', 'electron viscosity'

structurally:

\[ \lambda = \frac{c^2 Q_0}{\omega^2 R_0} \]

as \[ \mathbf{M} = \frac{c^2 \mathbf{V}}{\omega^2} \]

\( \lambda \equiv \mathbf{M} \)

\( D_J \to \mathbf{M} \mathbf{H} \mathbf{D} \)

\( \to \) multi-fluid

in extended stochastic field argument
Exercises:

5-p reconnection, with $E_{11} = -u \frac{V^2}{c} J_{11}$

$V_1/V_A = \frac{1}{(S_A)^{1/4}}$  \hspace{1cm} $S_A = \frac{VA L^3}{m}$

1/5 $\rightarrow m/V_A L^3$

To derive structure of $\Phi$

for ensemble stochastic fields

(i.e. shifted electron Maxwellian $\Rightarrow$

$J_{\mu}(x)$ ...)

Compare $\Phi$ to $X_0$ for various turbulence models.

In MHD:

--as seeks $\langle E_{11} \rangle$ and centered with

locally strong field

\[
\left( \frac{E + V \times B}{c} = mJ \right) \cdot \frac{B}{|B|}
\]

Stevens

\[
-\frac{1}{c} \partial_t A_{11} - n \cdot \nabla \Phi - \nabla_{\perp} \times \vec{n} \cdot \vec{\nabla} \Phi = mJ_{11}
\]

Here $R = \frac{B}{|B|}$

Then for mean field:
\[-\frac{1}{c} \varepsilon_{\alpha} \left< \mathbf{A}_\alpha \right> + \frac{1}{2} \varepsilon_{\alpha \beta \gamma} \left[ \partial_\gamma \mathbf{A}_\alpha \right] = \mathbf{m} \left< \mathbf{J}_\parallel \right> \]

Flat n. induced EMF.

Note naturally on Flux Form.

\[
\left< \mathbf{D} \cdot \mathbf{\nabla} \mathbf{A}_\parallel \right> = \left< \mathbf{D} \cdot \mathbf{\nabla} \mathbf{A}_\parallel \right> + \left< \mathbf{A}_\parallel \cdot \mathbf{\nabla} \mathbf{D} \right>
\]

1. \( \text{Ohm's Law} \)
2. \( \text{Vorticity} \) \( \propto \mathbf{n} \)

\[e_i \partial_i \mathbf{A}_\parallel + \mathbf{A}_\parallel \cdot \mathbf{\nabla} \mathbf{A}_\parallel = \partial_\parallel \mathbf{A}_\parallel - \nu k^2 \mathbf{A}_\parallel \]

\( \text{turbulent mixing, bending, resistive dissipation.} \)

\[
\left< \mathbf{D} \cdot \mathbf{\nabla} \mathbf{A}_\parallel \right> = \sum_k \left< k \cdot k \right> \left( \mathbf{D} \partial_\parallel - \nu k^2 \mathbf{A}_\parallel \right)
\]

In pure QLT, irreversibility from resistive diffusion, only \( \mathbf{E} \) can be large unless \( k^2 \) large.

\( \partial_\parallel \text{if undist norm-izations} \)

\[
\left< \mathbf{D} \cdot \mathbf{\nabla} \mathbf{A}_\parallel \right> = \alpha \left< \mathbf{B} \right> \rightarrow \text{alpha effect}
\]

\( \alpha \) = above formula.

\( k \cdot k \) \( \text{motion has handedness} \)
\[ x \rightarrow -x \quad \Rightarrow \quad x \rightarrow -x \]

\[ c_{II}k = \frac{k_{II}^2 x}{L_0} \]

\[ \frac{\partial}{\partial t} <A|W> = \frac{\partial}{\partial t} <B> \]

\[ \frac{\partial}{\partial t} <B> = \frac{\partial}{\partial t} <J> \]

i.e. how generate a field parallel/antiparallel parallel to a current? (Parker)

![Diagram of a current and vector fields]

Make By Far Bg.

\[ \partial \rightarrow \partial \rightarrow \rightarrow J \]

\[ \rightarrow J \quad \rightarrow J \]

need \( \langle \nabla \cdot \mathbf{A} \rangle \neq 0 \) \quad \Rightarrow \quad \text{fluctuations have net helicity}

Here \( \langle \mathbf{A} \times \mathbf{B} \rangle \) is magnetized analogue of handedness.
\[ \Theta = - \langle \nabla A_{\parallel} \cdot \delta \phi \rangle \]

Vorticity \\begin{align*}
\omega + & \nabla^2 \phi + \nabla \times \nabla \times \phi \\
= & \left( \frac{\partial \phi}{\partial t} \right) + \nabla \cdot \mathbf{J} + \mathbf{B} \cdot \nabla \times \mathbf{F} + \omega \nabla^2 \phi \\
\end{align*}

\[ \phi = \frac{\partial \mathbf{A}}{\partial t} - \nabla \times \mathbf{E} \]

\[ \Theta = - \sum_{\text{type}} k_i k_{ii} |A_{\parallel}|^2 \left( \Delta A_{\parallel} + u_k k_{ii}^2 \right) \]

- Magnetic \times effect
- Opposite sign to

\[ \Box \]
\[ \mathcal{O}_B = \sum_{u} \left| \mathcal{A}_{uu} \right|^2 \left( \omega u_{u} + u u_{u} \right) \frac{\partial \langle J_{uu} \rangle}{\partial \gamma} \]

Clearly comes from hyper-M

e.g.

\[ -\frac{1}{c} \frac{\partial \langle A_{uu} \rangle}{\partial t} + \partial_{\gamma} \langle (\mathcal{Q} \Phi \tilde{A}_{uu}) \rangle = n \langle J_{uu} \rangle \]

\[ \langle (\mathcal{Q} \Phi \tilde{A}_{uu} \rangle = \sum_{k} \sum_{m} \frac{x_{k}}{x_{m}} \left| L_{uu} \right|^{2} \left| L_{uu} \right|^{2} \frac{x_{k}}{x_{m}} \]

\[ L_{uu} = \frac{(\omega u_{u} + u u_{u})}{\omega^2 + (\omega u_{u} + u u_{u})^2} \]

\[ + \sum_{u} \left| \frac{\tilde{B}_{uu}}{B_{u}} \right|^2 \left| L_{uu} \right|^2 \frac{x_{k}}{x_{m}} \frac{x_{m}}{x_{k}} \]

\[ \text{hyper-reactivity} \]

N.B. - \( \omega \)'s both come from bending
- \( \omega_u, \omega_m \) opposite sign
- \( \omega \)'s from MHD exterior

\[ \tilde{A}_{uu} \rightarrow \frac{\tilde{A}_{uu}}{\tilde{A}_{uu}} \text{ where} \]
- hyper- \( n \) from resonance i.e. where vorticity driven
  \( \Rightarrow \) reconnection process site

- hyper- \( n \) tied to basic tearing drive
- \( \Delta u + \text{hyper-} \) cancel in exterior
  \( \nu \) survives in exterior. vanish near Res. surf
- note total EMF encompasses more than hyper- \( n \)

6. RFP

\[ \langle B_2 \rangle \]

exterior

\[ \theta \]

\[ 2 = \frac{1}{m} \text{ resonances} \]

\[ z < 1 \quad \Rightarrow \quad n = 5 \]

\[ z' < 0 \] unstable

\[ m = 1 \text{ para-die (global tearing turbulence)} \]

\[ \text{to compute induced EMF, seek} \]

\[ \langle \Omega \times \mathbf{B} \rangle \]

\( \forall \) in exterior.

\[ \mathbf{v} = \alpha \frac{\hat{z}}{L} \]

\( \Rightarrow \) displacement
\[ \vec{B} = n \times \sum \langle \vec{E} \rangle \]
\[ = -\vec{E} \cdot \vec{D} \langle \vec{B} \rangle + \langle \vec{B} \rangle \cdot \vec{D} \vec{E} - \langle \vec{B} \rangle \, d \sigma / d \theta \]

field advection
irrelevant.

\[ \vec{E} = \langle \vec{B} \rangle \cdot \nabla \vec{E} \]

\[ \langle \vec{v} \times \vec{B} \rangle = \sum \nabla \langle \vec{E}_x \cdot \vec{B}_y \rangle \]
\[ = \sum \nabla \langle \vec{E}_x \rangle \times i \nu_{im} \langle \vec{B}_y \rangle \vec{E}_n \]

\[ \to \text{field primarily poloidal near } B_z\]

\[ \nabla \cdot \vec{E} = 0 \Rightarrow \frac{\partial \vec{E}_x + c \theta \vec{E}_y}{i \nu} = \vec{E}_n \]

then

\[ \langle \vec{v} \times \vec{B} \rangle = \sum \nabla \langle \vec{E}_x \rangle \nu_{im} \langle \vec{B}_y \rangle \left[ \vec{E}_2 \vec{E}_x - \vec{E}_x \vec{E}_2 \right] \]
\[ = \sum \nabla \langle \vec{E}_x \rangle \nu_{im} \langle \vec{B}_y \rangle \left( M \right) \]
\[ M = \left( \overline{\alpha} \overline{\alpha}_r^* - i \alpha \sigma_0 \overline{\alpha}_r \right) \overline{\alpha}_r \]

\[ + \overline{\alpha}_r \left( \overline{\alpha} \sigma_0 + i \alpha \sigma_0 \overline{\alpha}_r \right) \]

\[ M = \overline{\alpha}_r \left( \overline{\alpha} \sigma_0 + i \alpha \sigma_0 \overline{\alpha}_r \right) \]

but \[ \lim_{\alpha \to 0} \overline{\alpha} = 0 \]

\[ \alpha \to 0 \Rightarrow \partial \alpha \gg \alpha \]

\[ \left< \sigma \times \vec{b}_0 \right> = \sum_{\beta} \frac{k_{\beta \beta}}{k_2} \left< B_0 \right> \partial \alpha \overline{\alpha}_r \]

\[ \Rightarrow k_{\beta \beta} \overline{k}_2 = \left( \frac{m}{r} B_0 - \frac{n}{r} B_0 \right) \] / \( B_0 \)

\[ \Rightarrow \frac{k_{\beta \beta}}{k_2} = \frac{\frac{m}{r} - \frac{n}{r}}{B_0} \]

\[ = \frac{1}{2} (4\pi n - n 2\pi r) \]

\[ k_2 = \frac{n}{r} \]

\[ \Rightarrow k_{\beta \beta} = (4\pi n) \left( \frac{m}{r} \right) - \frac{r}{r} 2\pi r \]

\[ = (4\pi n) (Z_{\text{ref}} - Z(r)) \]
\[ \langle \nabla \times \mathbf{B} \rangle = -\sum_{n} \frac{1}{n} \frac{R}{n} (Z_{\text{rev}} - Z(r)) \langle B_{0} \rangle d\nu \frac{\bar{\epsilon}_{n}}{\varepsilon_{n}} \]

\[ \Rightarrow 2\pi \varepsilon_{n} \frac{\bar{\epsilon}_{n}}{\varepsilon_{n}} < 0 \]

\[ \Rightarrow \varepsilon_{n} \rightarrow \text{irreversibility} \]

\[ \Rightarrow Z_{\text{rev}} - Z(r) \Rightarrow \begin{array}{l} < 0 \quad \text{on axis} \\ > 0 \quad \text{at } Z_{\text{rev}} \end{array} \]

\[ \Rightarrow \]

\[ \langle E \rangle + \langle \nabla \times B \rangle = n \langle J_{0} \rangle \]

\[ \Rightarrow \langle J_{0} \rangle = \frac{1}{\varepsilon_{n}} \langle \nabla \times B \rangle \]

\[ \Rightarrow \langle B_{2} \rangle < 0 \rightarrow \text{neither drive reversed} \]

But what about irreversibility and/or locking in $\theta$?
S - T - F - R

\[ \frac{1}{n} \rightarrow \frac{2}{2n+1} \rightarrow 1 \]

\[ \frac{1}{n}, e^t \rightarrow t/n \]

\[ m = 0 \text{ delete } \Rightarrow \text{ reconnect } \rightarrow \text{ look in} \]
(a) 4/5 Law - See Lecture 1.

(iii) Cascade and Relaxation

⇒ Selection Decay

Recall: \[
\text{Taylor Relaxation} \begin{cases} \text{3D} \\ 2D = \text{"Taylor in Flatland"} \end{cases}
\]

Argued: \[ \int d^3 x \frac{B^2}{8 \pi} \text{ minimized} \]

subject to constraint of \[ \int d^3 x \mathbf{A} \cdot \mathbf{B} \text{ conserved} \]

⇒ \[ J_n = J \frac{\mathbf{B}}{\mathbf{B}^2} \rightarrow \text{const.} \]

(2D \( J/A \rightarrow \text{const.} \)).

Arguments heuristic \( \begin{cases} \text{Power counting (4)} \\ \text{tech. fields} \end{cases} \)

Now, dissipation at small scale \( \mathcal{M}_V \)

→ expect energy transfer to small scale.
Inverses cascades of magnetic helicity would set up a "selective decay" scenario.

Magnetic energy scattered to small scales and dissipated.

\[ \Rightarrow \text{relaxation} \]

Magnetic helicity inverse cascades avoid dissipation. Constraint: some survives.

c.f. Frisch (75), Pouquet et al. (76)

\[ \text{(posted)} \]

See also: Montgomery

Why, where, from?

- Primarily: Statistical Mechanics
  - c.f. Frisch 75, though not transparent.
  - easier → "Taylor in Flatland" Problem.
Recall: Relaxation \( \begin{cases} \text{minimize } \langle \mathbf{B}^2 \rangle \\ \text{conserving } \langle \mathbf{A}^2 \rangle \end{cases} \)

Does this follow from selective decay?

\[ \Rightarrow \text{ Explore Absolute Equilibrium} \]

\[ \text{i.e.} \]

\[ \begin{array}{c}
\begin{array}{c}
\text{finite box}
\end{array}
\end{array} \]

\[ \text{kinin} \quad \text{kina} \]

- remove forcing, dissipation etc.

- input excitation.

For 2D MHD (ignoring cross helicity),

\[ \text{have} \quad A \Rightarrow X^c \]

\[ \Rightarrow \text{Model amplitude} \]

\[ E_m = \sum_{i=1}^{N} k_i^2 x_i^2 \]

\[ H = \sum_{i=1}^{N} x_i^2 \quad - \langle A^2 \rangle \]
\[ \phi \rightarrow y_i \]
\[ E_k = \sum_{i=1}^{\infty} \kappa_i x_i^2 \]

Now, \[ H \rightarrow \frac{1}{\beta} F = \frac{F}{\beta} \rightarrow \beta \rightarrow \text{conserved} \]

conserved, so \[ \frac{\partial F}{\partial \beta} = 0 \] of this closed system is given by micro-canonical ensemble distribution:

\[ P(x_1, x_2, \ldots, x_N) = \frac{c}{\text{norm}} \exp \left[ -\sum_{i=1}^{N} (x_i + \beta \kappa_i^2) x_i^2 + \beta \kappa_i^2 \right] \]

and can integrate out \( y_i \) \((\kappa E)\) part, so:

\[ P(x) = \frac{c}{\text{norm}} \exp \left[ -\sum_{i=1}^{N} (x_i + \beta \kappa_i^2) x_i^2 \right] \]

then:

\[ \langle x^2(k) \rangle = \int dx_i \; x_i^2 \; P(x_i) \]

\[ = \frac{1}{(x + \beta \kappa_i^2)} \]

\[ \langle x^2(k) \rangle = \left[ \frac{k^2}{x + \beta \kappa_i^2} \right] \]
\( \text{observe immediately:} \)

\[ A^2 \]

\[ \text{"} A^2 \text{ wants remain at large scale"} \]

\[ k \] \[ \text{min} \rightarrow 0 \]

\[ B^2 \]

\[ \text{"} B^2 \text{ approaches equipartition"} \]

\[ k \]

- \( A^2 \) distribution most populated at larger scales. Damps at small.

- \( B^2 \) distribution most populated at smaller. Approaches equipartition at small scale.

- suggests \( A^2 \) populates large scales, \( B^2 \) approaches equipartition.

- suggestive of inverse cascade of \( A^2 \) along with forward cascade of energy.
Supports Selective Decay Hypothesis as Foundation for "Taylors in Flatland."

Similarly for Magnetic Helicity, though more laborious.

N.B. For 2D Fluid:

\[ E = \int d^3x \left( \frac{1}{2} \mathbf{B}^2 \right) \] - energy

\[ \mathcal{L} = \int d^3y \left( \mathbf{B} \cdot \mathbf{\nabla} \right)^2 \] - enstrophy

\[ \mathcal{L} = k_i^2 E_i. \]

\[ \mathbf{\nabla} \rightarrow \mathbf{\nabla} \cdot \mathbf{\epsilon} \]

\[ P(\mathbf{x}) = c \exp \left[ - \sum_{i=1}^{N} \left( \mathbf{x} + \delta \mathbf{k}^2 \right) \cdot \mathbf{x}_i \right] \]

So \[ \mathcal{E}_i \left( N^2(\mathbf{x}) \right) = \left( k + \delta k^2 \right) \]

\[ \mathbf{\alpha} = k^2 \left( k + \delta k^2 \right) \]

Similar suggestion of dual cascade and minimum enstrophy state.
→ Is this story true?

→ What does dynamics tell us?

Consider interactions in 2D MHD.

Observation:

- Reduced MHD

\[ \frac{\partial \psi}{\partial t} + \frac{\partial }{\partial x} \left( \psi \cdot \mathbf{B}_0 \right) = \frac{\partial }{\partial x} \left( \psi \mathbf{D}_x \right) \]

- 2D MHD

\[ \frac{\partial \psi}{\partial t} + \frac{\partial }{\partial x} \left( \psi \cdot \mathbf{D}_x \right) = \mu_0 \mathbf{D}_x \times \mathbf{B}_0 \]

so with strong \( B_0 \):

\[ \langle A \times B \rangle \rightarrow \langle \psi \rangle B_0 \]

so mean \( \psi \) in 2D captures magnetic helicity dynamics of strongly magnetized system.

For \( \langle A^3 \rangle \) transfer, consider closure of \( \langle A^3 \rangle \) equation, much akin to wave kinetics, though closure required.

See: Diamond, Hughes, Kim (forthet)
Can write (see DHK):

\[
\frac{1}{2} \left[ 2 \langle A^2 \rangle_T + T(\nu) \right] = -\frac{\Gamma_A(k) A^2}{\nu_x} - \eta \langle B^2 \rangle_T
\]

\[
\text{triplet:} \quad \left< D \cdot \left< V A^2 \right> \right>_T
\]

\[
\frac{f/uv}{A^2} = \left[ \int_0^\infty \langle V^2 \rangle_A \right. \\
- \int_0^\infty \langle V^2 \rangle_B
\]

\[
T_n = \sum \left( \frac{1}{n^2} - \frac{1}{n^4} \right) \sum \left< \phi^2 \right>_B
\]

\[
\text{coherent damping, incoherent emission}
\]

\[
\text{akin to scattering of passive scalar, small scale 'chop-up', conserve } \left< \psi^2 \right> \text{ upon } \Sigma
\]

\[
\sum \phi \to 1.0
\]
(2) coherent damping/growth - from back reaction \( J_2 R \) into von st. Lo
correlation \( \langle A^2 \rangle \) to larger scale. sign \( \hbar^2 \) vs \( H_j^2 \)
\[ \sum \frac{1}{h} \text{ conserve independently.} \]

correspondence to condensation and zero (currents) attracting.

(1 + 2) net effective resistivity sign:
see \( \Gamma_A \), too. - negative resistivity
- adsorption state

\[ E_k > E_m \Rightarrow \langle A^2 \rangle \text{ shuffled to smaller scale.} \]

\[ E_m < E_k \Rightarrow \langle A^2 \rangle \text{ transferred to larger scale.} \]

and transfer need not be global.
\[ \text{In dynamic evolution is complex.} \]

\[ \text{N.B. Recall Flux expulsion:} \]
\[ \frac{\nabla \times \mathbf{B}}{\nabla} \to \mathbf{J} \times \mathbf{B} \text{ disrupts vortex expulsion} \]
\[ \text{\( \nabla \cdot \mathbf{B} = 0 \) implies \( \mathbf{B} \to \mathbf{B} \text{ expelled} \)} \]
\[ \nabla \times \mathbf{B} \to \mathbf{J} \times \mathbf{B} \text{ expels \( \mathbf{B} \)} \]

\[ \Rightarrow \mathbf{B}^2 < \frac{\partial (\mathbf{B}^2)}{\partial t} / \mathcal{R} \]

\[ \text{but } \left< \mathbf{B}^2 \right> \to \mathbf{B}^2 \text{ upon stretching} \]

Zeldovich: \text{weak } \mathbf{B} \text{ is sufficient!} \]

\[ \frac{\partial \mathbf{A}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{A} = -\nu \Delta \mathbf{A} + \nu \mathbf{D} \frac{\partial^2 \mathbf{A}}{\partial x^2} \]

\[ \mathbf{A} \text{ and } \mathbf{B} \to \]

\[ \mathbf{B} \left< \mathbf{B}^2 \right> = \left< \mathbf{J} \times \mathbf{A} \right> \frac{\partial \left< \mathbf{A} \right>}{\partial x} \]

\[ \left< \mathbf{B}^2 \right> = \frac{\mathbf{B}^2}{\eta} \]

\[ \left< \mathbf{B}^2 \right> = \frac{\mathbf{B}^2}{\eta} \]

\[ = \frac{\mathbf{B}^2}{\eta} \quad \text{for } \mathbf{B}^2 \approx \mathcal{R} \text{m} \mathbf{B}^2 \nu \]
\[ \langle B^2 \rangle / R_m > \langle D_{000} \rangle / R_m \]

== Questions still open ==

Taylor conjecture remains a conjecture
iii - how describe global dynamics of relaxation and self-organization

\[ \frac{\Lambda}{\Lambda V R^m_1} \]

S.P.

Usually envision as localized event involving irreversible dissipation etc. at a singularity

From relaxation to relaxation

Preamble
Examples of Self-Organization Principles
Quiet Period is origin of RFP

Reversal

\[ B_I(q) > 0 \]

\[ T_e \sim 1 \text{msec} \]

\[ T_i \sim 150 \text{eV} \]

Reduced fluctuations

Macro-stability

Properties of Quiet Period:

Access to "Quiet Period"

\[ \theta < \theta_{\text{crit}} \]

\[ I_p < I_p^{\text{stabilized pinch}} \]

Stabilized pinch 

Week \[ B_I \]

Susceptibility eliminated

Initial results

Violent macro-instability, short life time

toroidal pinch = vessel + gas + transformer

Prototype of RFP's Zeta

(UK: late 50's - early 60's)

II. Focus | Magnetic Relaxation

(Derek C. Robinson)
Needed: Unifying Principle

steady, albeit modest, improvement in RFP performance, operational space

\[ \int_{x^p} \mathcal{A} \cdot B \]

follows from minimized \( \mathcal{E} \) at conserved \( \mathcal{F} \)

- L. Wolfer (1958): Force-Free Fields at constant \( \mathcal{G} \)

observed to correlate well with observed \( \mathcal{B} \) structure

\[ J^B = J^F \]

\[ B^{\theta}_0 = B^{\mathcal{F}}_0 \quad (\alpha\mathcal{I}) \]

- Resistive Interchange

... Kink-Tearing \( \Rightarrow \) tend toward force-free state

- Turbulence

Further Developments
Works amazingly well for recovering BFM, with reversals for heterogeneous form free force.

Taylor state is:

\[ \theta < \frac{\partial \mu}{\partial I} \]

\[ \frac{\partial B}{\partial t} = \frac{B}{\parallel J} \times \Delta \Rightarrow \]

\[ \text{const} = \frac{\frac{\partial B}{\partial t}}{B \cdot J} \Rightarrow B' = B \times \Delta \]

\[ 0 = \left[ B \cdot \nabla x_3 p \int \chi + \frac{\kappa}{2 B} x_3 p \int \right] \]

\[ \text{in profile follows from:} \]

Global magnetic helicity hypothesis that relaxed state minimises magnetic energy subject to constant

Theory of Turbulent Relaxation (J.B. Taylor, 1974)
Central Issue: Origin of Irreversibility

What does the pinch say about dynamics?

Theory predicts end state — what can be said about dynamics?

Why only global magnetic helicity as constraint?

What is magnetic helicity and what does it mean?

Questions:

\[ \langle B \rangle / \eta_{\text{m}} B = \beta \]

\[ \frac{\partial B^0}{\partial \tau} = \tau / \eta \eta = \theta \]

Result:
Thus larger tubes persist longer. Global flux tube most robust.

\[ \frac{3}{2} \text{ for } S \text{-P reconnection} \]

in turbulent resistive plasma. Reconnection occurs on all scales, but:

- line \( l \) tube \( \Leftrightarrow \) only global helicity meaningful.

- if turbulenceIGGERfeld lines stochastic, then \( l \) field line, field pinch.

\[ (g', \gamma) \neq (g, \gamma) \]

\( \text{i.e.} \)

- turbulent mixing eradicates identity of individual flux tubes, lines!

\[ \mathbf{B}(g', \gamma) \neq \mathbf{B}(g, \gamma) \]

\( \text{i.e.} \)

- in ideal plasma, helicity conserved for each line, tube

\textbf{Why Global Helicity Only?}
- But, no dynamical insight
  - Taylor's theory, simple and successful

Bottom Line

- Confinement penalty: No free lunch!
  
Most plausible argument for $H^W$ is stochasticization of field lines -- forces

(impact (apologies to some present...)

Many attempts to expand/supplement the Taylor conjecture have had little lasting

unproven in any rigorous sense

Taylor's conjecture that global helicity is most rugged invariant remains a conjecture

Comments and Caveats
In the sense of perturbation theory, the conjecture is not systematic. Note this is ad-hoc, forcing $\langle S \rangle$ to

$$\langle B \times a \rangle \leftrightarrow \text{something related to} \quad \langle S \rangle + \langle f \rangle u = \langle E \rangle$$

Structural Approach (Boozer): Plasma frame

MFT is often very useful, but often fails miserably.

Caveat: MFT assumes fluctuations are small and quasi-Gaussian. They are often not.

How relate to relaxation?

Mean Field Theory \leftrightarrow How represent

(of others - see D. Hughes, Thursday lecture)

The question of dynamics brings us to mean field theory (C. Moffat, 78, and an infinity)

Dynamics 1:
Homogenized current: \( 0 \leftarrow (B / \parallel f) \Delta \)

Relaxed state:

\[ \langle B \rangle \Delta = \gamma \]

Turbulent hyper-resistivity:

Simple form consistent with Taylor hypothesis:

\[ \omega \frac{\partial}{\partial t} \]

To dissipate:

\[ (B / \parallel f) \Delta \gamma = H \frac{1}{\omega} \]

So

\[ \left[ \frac{\partial}{\partial t} \Delta - \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \right) \right] x_p \int_{-} = \frac{\mu_0}{\varepsilon B} x_p \int_{-} \]

Conservation (Helicity flux):

\[ H \frac{1}{\omega} \Delta \frac{\partial}{\partial B} = \langle S \rangle \]

\[ \langle B \cdot S \rangle x_p \int_{-} - \langle B \cdot J \rangle x_p \int_{-} \omega x_p - \omega H = \omega H \]

Now
\[ \langle \theta b \rangle (\frac{s}{R})^4 \frac{\partial}{\partial \eta} |_{\eta=0} \int_{\Omega} \approx \langle B \times A \rangle \]

Approach: QL \langle B^z \times a \rangle in MHD exterior - exercise: derive!

Approach: QL \langle B \times a \rangle in MHD exterior - exercise: derive!

Issue: What drives reversal near boundary?

Tearing modes resonant in core \( B^z \) global structure

Point: Dominant fluctuations controlling relaxation are \( m=1 \)

Aspects of hyper-resistivity do enter but so do other effects

Boozer model not based on fluctuation structure, dynamics

Dynamics II: The Pinch's Perspective
Bottom Line: How Pinch Taylor-Israeli remains unclear in detail

region broadens

driven c'nn't shear at

\( \text{driven current sheet at } \mathcal{J}_{P} \)

driven c'nn't shear at

\( \text{driven current sheet at } \mathcal{J}_{P} \)

\( n=1 \quad m=0,1 \)

\( n=1 \quad m=0,1 \)

\( 2n+1 \quad m=2,1 \)

\( 2n+1 \quad m=2,1 \)

beat

sum

difference beat

difference beat

driving scattering relaxation front

\( \text{driving scattering relaxation front} \)

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<td><strong>process:</strong> stochastization of fields, turbulent reconnection</td>
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<td><strong>constraint released:</strong> local helicity</td>
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<td><strong>players:</strong> tearing modes</td>
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