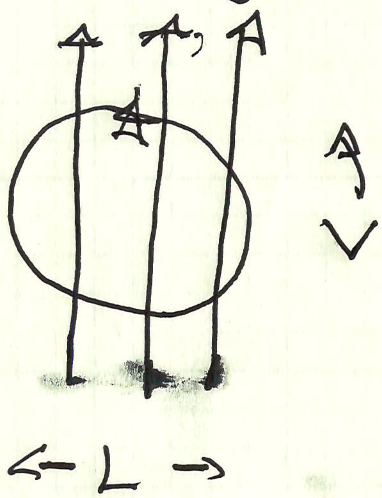
Flux Expulsion — "The Simplest Problem"

- Flux expulsion is simplest dynamic problem in non-ideal MHD

- closely related to problem of "homogenization".



- consider an eddy rotating (differentially) at speed V

- $Rm \sim VL/\eta \gg 1$

What happens?

- eddy winds-up the field

- tends to wind up and fold over the field

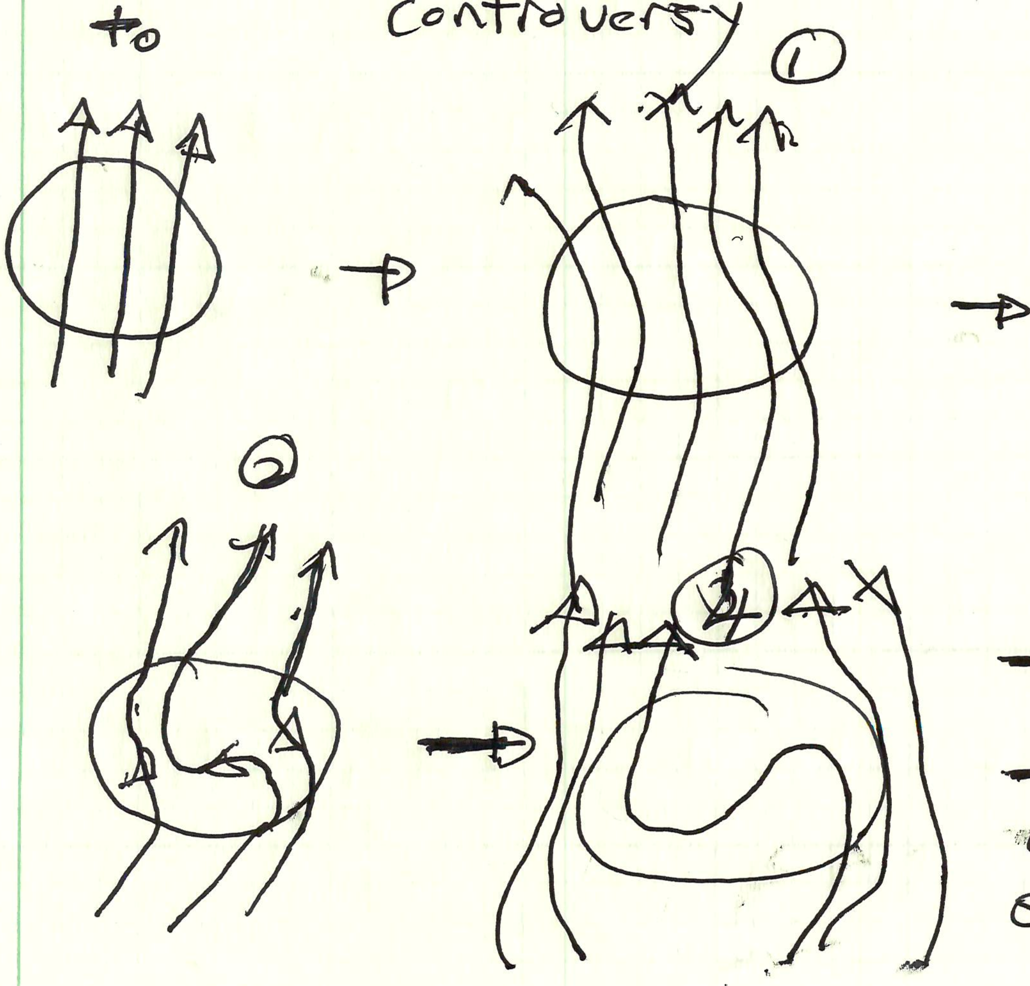
- field inside the eddy drops, but expelled to boundary layer on rim of the eddy

– hence ⇒ "flux expulsion"

"homogenization": $B \to 0$, $A \to$ const within

– Questions: – time scale
– layer width
– boundary of kinematic regime

– N.B.: Problem has history of controversy



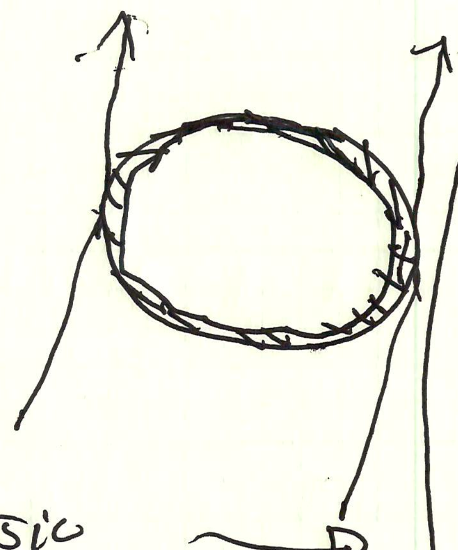
In ④:
– field expelled
– as freezing in, up to small η

⇒ – field "expelled" from eddy

but

– Large Rm ⇒ field concentrated in boundary layer on edge of eddy:

i.e.



begs the questions:

→ how thick a layer?

→ how strong the field in surface layer?

→ characteristic time scale to expell?

basic physics →

Obviously → – larger Rm, greater expulsion (weaker field in interior).

– thinner the BL.

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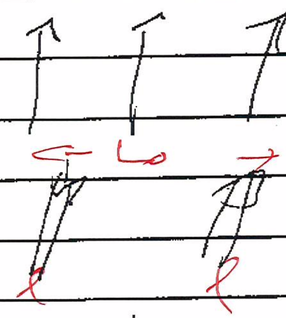
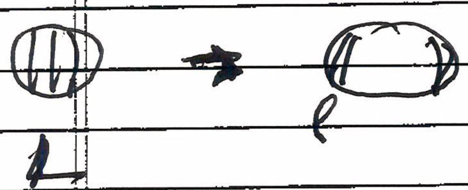
- Field stretched / compressed in n-words

$n\ell = L_0$

$\ell = L_0/n$



- Flux (vertical) conserved, so



$n\ell = L_0$

$L_0 B_0 \sim \ell b$

and

$$\frac{V}{L_0} B_0 \sim \frac{\eta}{\ell^2} \underset{\text{stretched}}{b}$$

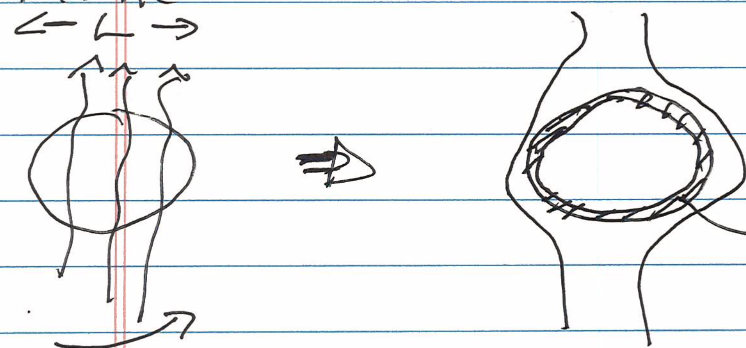
$$\frac{V}{L_0} B_0 \sim \frac{\eta}{\ell^3} B_0 L_0$$

$$\frac{\ell^3}{L_0^3} \sim \frac{\eta}{V L_0}$$

$$\frac{\ell}{L_0} \sim (R_m)^{-1/3}$$

Recall:

Considered problem of flux expulsion
→ i.e. dynamical counterpart of Sweet-Parker:



→ flux expelled to B.L.

→ BL thickness:

$B_0 L = b\delta$ (flux conservation)

$\frac{V_0}{L} B_0 = \frac{\eta}{\delta^2} b$ (rate balance)

b → stretched field

⇒

$$\delta \sim L_0 (R_m)^{-1/3}$$

Now further:

$$b \sim \frac{B_0 L}{\delta} \approx B_0 n$$

n → $\#$ turns / windings.

so $\delta = \frac{L_0}{n}$

$\therefore \quad n \sim (R_m)^{1/3}$

and $\tau_{expul} \cong \tau_c (R_m)^{1/3} \simeq \left(\frac{L_0}{V_0}\right) R_m^{1/3}$

$\Downarrow$

characteristic time for expulsion

$\Rightarrow$

What is the physics of flux expulsion?

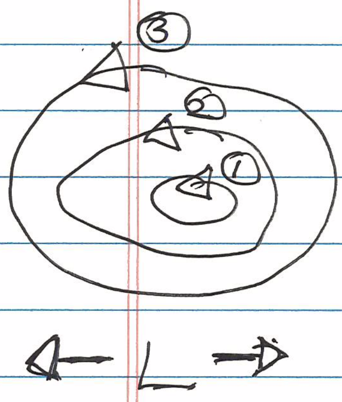
$\Rightarrow$ combination of $\begin{cases} \text{wind-up} \\ \text{diffusion} \end{cases}$

Key contrast: $\eta^{-1/2} \rightarrow$ neglects stretching

$\eta^{-1/3} \rightarrow$ stretching + diffusion

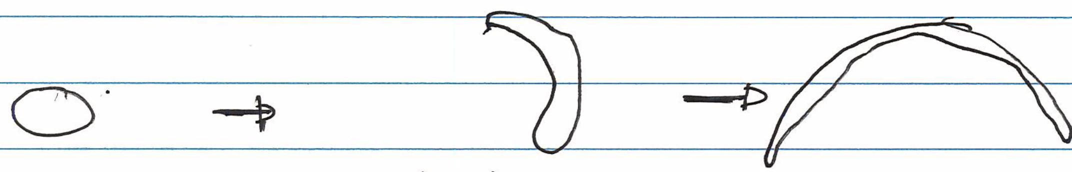
$\Rightarrow$ shear dispersion (G. I. Taylor, ...)

i.e. consider differentially rotating sheared flow



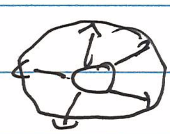
$V_3 > V_2 > V_1$

so



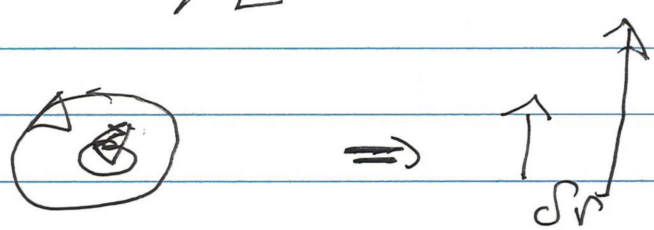
stretched, extended to finer scale till diffusion smears.

i.e. (a) diffusion only (includes solid body)



$1/\tau \sim D/L^2$

(b)



$$\frac{d\delta r}{dt} = \tilde{v} \qquad \langle \tilde{v}(t')\tilde{v}(t'')\rangle = \overline{\tilde{v}^2}\tau_{ac}\delta(t'-t'')$$

random walk

$$\langle \delta r(t)\delta r(t')\rangle = \overline{\tilde{v}^2}\tau_{ac}\, t$$
$$\equiv D t$$

Now, $r\theta = y$

$$\frac{dy}{dt} = V_{0y}(r)$$

for excursions

$$\frac{d}{dt}\delta y = \left(\frac{\partial V_{0y}}{\partial r}\right)\delta r$$

$$\delta y = \int \left(\frac{\partial V_{0y}}{\partial r}\right)\delta r\, dt$$

$$\langle \delta y^2 \rangle \sim \left(\frac{\partial V_{0y}}{\partial r}\right)^2 \langle \delta r^2 \rangle\, t^2$$

$$\langle \delta r^2 \rangle \sim D_r t$$

$$\langle \delta y^2 \rangle \sim \left(\frac{\partial V_{0y}}{\partial r}\right)^2 D_r\, t^3$$

→ shear dispersion

→ scatter rate
i.e. hybrid of
radial diffusion
and differential
rotation

Now, for decorrelation time, make $\langle \delta y^2 \rangle \sim l_0^2$

l_0 → relevant scale comparison

Here, $l_0 \sim L_0$ so

$$1/\tau_{mix} \sim \left(\left(\frac{\partial V_y}{\partial r} \right)^2 \frac{D}{L_0^2} \right)^{1/3}$$

$$\sim \left(\left(\frac{V_0}{L_0} \right)^2 \frac{D}{L_0^2} \right)^{1/3}$$

$$\sim \left(\frac{V_0^2}{L_0^2} \frac{D}{L_0^2} \right)^{1/3}$$

$$\sim \frac{V_0}{L_0} \left(\frac{D}{V_0 L_0} \right)^{1/3}$$

$$\Rightarrow \boxed{\tau_{mix} \sim \tau_{circ} R_m^{1/3}}$$

- mixing, expulsion $\Rightarrow$ hybrid time scale, hence shear dispersion.

- $\tau_c < \tau_{mix} < \tau_{diffn}$
  $\quad\quad\sim R_m^{1/3}, \quad \sim R_m.$

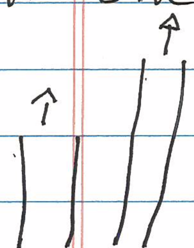
Systematically: shearing coordinates!

N.B.: Shear dispersion is operative thru out the entire time of expulsion.

So, for 2D MHD

$$\frac{\partial A}{\partial t} + \underline{v}\cdot\underline{\nabla} A = \eta \nabla^2 A$$

recall shearing sheet configuration:



$$V_y = V_y(x) = V_{y_0} + x V_y' + \dots$$

so

$$\frac{\partial A}{\partial t} + \cancel{V_{y_0}\partial_y A} + x V_y' \partial_y A - \eta(\partial_x^2 + \partial_y^2) A = 0$$

Now,

$$A = A(t) e^{i\underline{k}(t)\cdot\underline{x}}$$

$\left\{\right.$ i.e. $\underline{k}$ tilts in shearing field

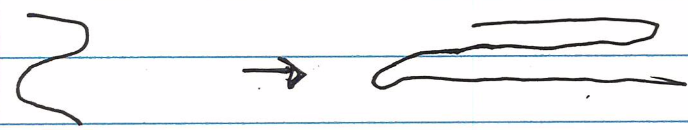
$$\frac{dk}{dt} = -\frac{\partial}{\partial x}\left(\omega + k_y V_y\right)$$

$$\frac{dk_x}{dt} = -k_y V_y'$$

$$\frac{dk_y}{dt} = 0$$

i.e. selects k to eliminate fast shearing evolution

i.e.



$$A = A(t)\, e^{i(k_{x0} - k_y V_y' t)x}\, e^{i k_{y0} y}$$

$\Rightarrow$

$$\frac{\partial A}{\partial t} = -i k_{y0} V_y' x A + \frac{\partial A(t)}{\partial t}$$

$$x V_y' \partial_y A = i k_{y0} V_y' x A$$

$$\eta\left(\partial_x^2 + \partial_y^2\right) = -\eta k_{y0} V_y'^2 t^2 A - k_{y0}^2 A$$

so

$$\cancel{-i k_{y0} V_y' x A} + \frac{\partial A}{\partial t} \cancel{+ i k_{y0} V_y' x A} = -\eta k_{y0}^2 V_y'^2 t^2 A$$

$$-\eta k_{y0}^2 A = 0$$

$$\therefore \quad \frac{\partial A}{\partial t} = -\eta k_{y_0}^2 V_y'^2 t^2 A$$

$$\Rightarrow \quad A = \exp\left[-\eta k_{y_0}^2 \left(t + \tfrac{1}{3} V_y'^2 t^3\right)\right]$$

$$\text{so} \quad \tau_{mix}^{-1} \cong \left(\frac{\eta k_{y_0}^2 V_y'^2}{3}\right)^{1/3}$$

$$= V_y' \left(\frac{\eta k_{y_0}^2}{V_y'}\right)^{1/3}$$

$$\boxed{\tau_{mix} = \tau_{shear} \left(R_m\right)^{1/3}}$$

$$\left.\begin{aligned} \tau_{shear}^{-1} &= V_y' \\ R_m &\sim V_y' / \eta k_{y_0}^2 \end{aligned}\right\}$$

For $\left.\begin{aligned} k_{y_0} L_y &\sim 1 \\ \frac{1}{V_y} V_y' L_y &\sim 1 \end{aligned}\right\} \Rightarrow$ recovers previous.

N.B.: Shearing coordinates $\Leftrightarrow$ Normal Modes!?!

2.3
The time-scale associated with flux expulsion

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A simple model problem is solved in order to show that the time-scale associated with the process of flux expulsion is

$$t_{fe} = R_m^{1/3} t_0$$

where t_0 is a time-scale characterising the flow (for example, the eddy turnover time, or inverse shear rate) and R_m is the magnetic Reynolds number. This estimate is in agreement with that of Weiss (1966) based on numerical experiments. By decomposing the vector potential into a product of a rapidly varying part (in space) and a slowly varying part, it is shown how numerical work can be extended to much higher values of R_m than has been achieved hitherto.

1. Introduction

When a steady two-dimensional motion $\mathbf{u}(\mathbf{x})$ with closed streamlines acts upon a magnetic field in the plane of the motion, it is well known that, if $R_m \gg 1$, the field is eventually expelled from regions of closed streamlines, and is ultimately concentrated in layers of thickness $0(R_m^{-1/2})$ at the boundaries of these regions.

The process is described by the equation for the vector potential $A(x,y,t)\mathbf{k}$ of the magnetic field, viz

$$\partial A/\partial t + \mathbf{u}\cdot\nabla A = \eta\nabla^2 A. \tag{1.1}$$

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During an initial phase, diffusion is negligible, and

$$A(\mathbf{x},t) = A(\mathbf{a},0), \tag{1.2}$$

where $\mathbf{x}(\mathbf{a}, t)$ is the position at time t of the fluid particle initially at $\mathbf{a}$. During this phase, the magnetic field is distorted into a tight double spiral within each eddy, and the magnetic energy increases essentially like t^2. Obviously the field gradient increases during this process, and so after a time, say t_{fe}, diffusion must become important. This is the stage at which closed field loops form (Parker, 1966), and the process of flux expulsion commences. The magnetic energy within any eddy reaches a maximum at $t \approx t_{fe}$ and then falls off, ultimately to a value of order R_m^{-1}.

The computational study of Weiss (1966) suggested that $t_{fe} \sim R_m^{1/3} t_0$, and that in consequence, $B^2_{max} \sim R_m^{2/3}$. The purpose of this note is to provide a simple theoretical explanation for this scaling, and to explain why the alternative scaling $t_{fe} \sim R_m^{1/2} t_0$, $B^2_{\max} \sim R_m$ suggested by Moffatt (1978, §3.8) is in fact incorrect.

2. The action of uniform shear on a space-periodic magnetic field

Flux expulsion from an eddy occurs essentially because, at $t = 0$, $\mathbf{u} \cdot \mathbf{B}$ varies (and indeed changes sign) on each closed streamline within the eddy. A much simpler flow and field configuration, with a similar property, is sketched in Figure 1. We suppose that $\mathbf{u} = (\alpha y, 0, 0)$, and that at $t = 0$,

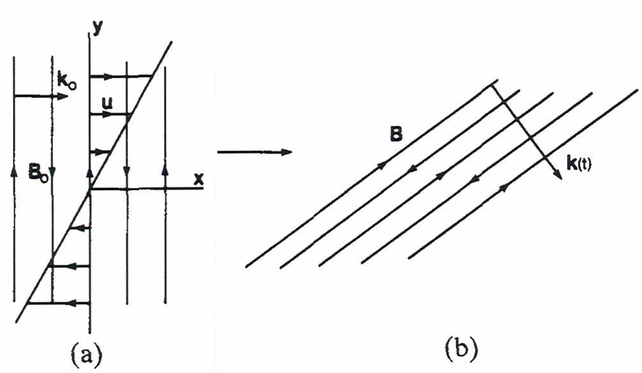


FIGURE 1 Effect of uniform shear on a unidirectional space-periodic magnetic field.

$$\mathbf{B}(\mathbf{x}, 0) = (0, B_0 \cos k_0 x, 0). \tag{2.1}$$

Correspondingly,

$$A(x, y, 0) = -k_0^{-1} B_0 \operatorname{Im}\{e^{ik_0 x}\}, \tag{2.2}$$

and the solution of (1.1) has the form

$$A(x, y, t) = -k_0^{-1} B_0 \operatorname{Im}\{a(t)\, e^{i\mathbf{k}(t)\cdot\mathbf{x}}\}, \tag{2.3}$$

where

$$a(0) = 1, \qquad \mathbf{k}(0) = (k_0, 0, 0). \tag{2.4}$$

It is easily shown that

$$\mathbf{k}(t) = (k_0, -\alpha t k_0, 0), \tag{2.5}$$

so that the wave-fronts of **B** are progressively tilted as indicated in Figure 1b, and that

$$da/dt = -\eta\, \mathbf{k}^2 a,$$

so that

$$a(t) = \exp\{-\textstyle\int \eta \mathbf{k}^2\, dt\} = \exp\{-\eta k_0^2 (t + \tfrac{1}{3}\alpha^2 t^3)\}. \tag{2.6}$$

The effect of the shear is represented in the term $\tfrac{1}{3}\alpha^2 t^3$. If $\alpha = 0$, the time-scale of decay of **B** is the usual diffusion time-scale $t_\eta = (\eta k_0^2)^{-1}$. If $\alpha \neq 0$, and more particularly if $\alpha \gg \eta k_0^2$, then the time-scale of decay is

$$t_{fe} = (\alpha^2 \eta k_0^2)^{-1/3} = \alpha^{-1} R_m^{1/3}, \tag{2.7}$$

where $R_m = \alpha/\eta k_0^2$.

3. The action of non-uniform shear on a space-periodic magnetic field

Suppose now that $\mathbf{u} = (u(y), 0, 0)$, so that

$$\partial A/\partial t + u(y)\,\partial A/\partial x \;=\; \eta\,\nabla^2 A. \tag{3.1}$$

Figure 2a, b shows the effect of such a velocity field on a magnetic field given initially by (2.1). Flux expulsion occurs from the region in which $|du/dy| \gg \eta k_0^2$, the field topology changing through the diffusion process. The solution of (3.1) now has the form

$$A(x, y, t) \;=\; -k_0^{-1}\, B_0 \,\mathrm{Im}\,\{\, a(y, t)\, e^{i\,\mathbf{k}(y,t)\cdot\mathbf{x}} \,\}\,, \tag{3.2}$$

where

$$\mathbf{k}(y, t) \sim (k_0, -k_0\,(du/dy)\,t, 0) \tag{3.3}$$

and

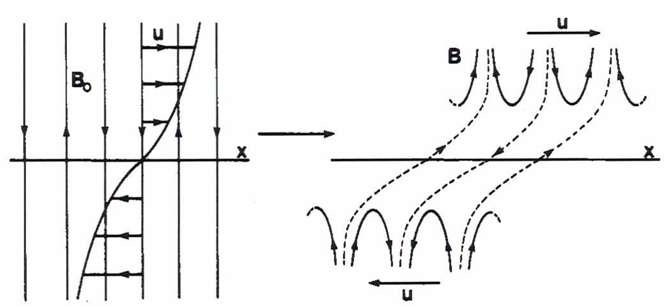
$$a(y, t) \sim \exp\,\{-\eta k_0^2\,(t + \tfrac{1}{3}(du/dy)^2\, t^3)\}\,. \tag{3.4}$$

This solution describes flux expulsion on the time-scale (2.7) where now $\alpha = |du/dy|_{\text{max}}$.

4. Flux expulsion from a single eddy with circular streamlines

Suppose now that

$$\mathbf{u} \;=\; (0, s\omega(s), 0) \tag{4.1}$$



(a) $t=0$ (b) $t \gg t_{fe}$

FIGURE 2 Effect of non-uniform shear on the same initial field as in Figure 1. Accelerated ohmic diffusion in the region of maximum shear leads to apparent flux expulsion in a time t_{fe} of order $R_m^{1/3}\,|du/dy|_{\text{max}}^{-1}$.

in cylindrical polar coordinates (s, θ, z), and that

$$A(s, \theta, 0) = B_0 s \sin \theta, \tag{4.2}$$

corresponding to a uniform field $\mathbf{B}_0$ parallel to $\theta = 0$. Then

$$A(s, \theta, t) = B_0 \operatorname{Im} \{f(s, t) e^{i\theta}\}, \tag{4.3}$$

where

$$\partial f / \partial t + i\omega(s) f = (\partial^2/\partial s^2 + s^{-1}\partial/\partial s - s^{-2}) f, \tag{4.4}$$

with $f(s, 0) = s$.

The results of §§2 and 3 now suggest the best way to proceed. If $\eta = 0$, the solution of (4.4) is

$$f(s, t) = f(s, 0)\, e^{-i\omega t} = s\, e^{-i\omega t}. \tag{4.5}$$

The function $e^{-i\omega t}$ is a rapidly varying function of s, when t is large. When $\eta \neq 0$, let

$$f(s, t) = e^{-i\omega t} g(s, t), \quad g(s, 0) = s, \tag{4.6}$$

so that

$$\partial f / \partial s = (-it\omega' g + \partial g / \partial s)\, e^{-i\omega t}, \tag{4.7}$$

$$\partial^2 f / \partial s^2 = (-t^2 \omega'^2 g - it\omega'' g - 2it\omega' \partial g / \partial s + \partial^2 g / \partial s^2)\, e^{-i\omega t}. \tag{4.8}$$

Substitution in (4.4), and retaining only the term on the right which increases like t^2, we have

$$\partial g / \partial t = -\eta(\omega'^2 t^2 g + \ldots), \tag{4.9}$$

giving

$$g(s, t) \sim s\, e^{-\frac{1}{3}\eta\omega'^2 t^3}, \tag{4.10}$$

which again describes flux expulsion on the time-scale

$$t_{fe} = R_m^{1/3} (s\, \omega')^{-1}, \tag{4.11}$$

in agreement with the results of Weiss (1966).

The asymptotic symbol $\sim$ in (4.10) needs interpretation in terms of a double limiting process

$$|s\omega' t| \gg 1 \text{ and } |\eta t^2 \omega''| \ll 1, \tag{4.12}$$

the latter arising from back-substitution of the solution (4.10) in (4.8) to see at what stage the term $\partial^2 g/\partial s^2$ becomes comparable with $t^2\omega'^2 g$. The solution (4.10) is thus valid for

$$1 \ll \omega_0 t \ll R_m^{1/2}, \tag{4.13}$$

where ω_0 is a typical value of $|s\omega'|$, and it is supposed that $|\omega'/s\omega''| = O(1)$. The flux expulsion time $t_{fe} = \omega_0^{-1} R_m^{1/3}$ is within the range (4.13), so that the description given by (4.10) is self-consistent.

The important point to note is that, for t in the range (4.13), the function $g(s, t)$ defined by (4.6) is slowly varying as a function of s, whereas $f(s, t)$ is rapidly varying. Computer experiments based on the exact equation for $g(s, t)$ have in fact been carried out for values of R_m up to 10^9 (Kamkar, 1981), and the $R_m^{1/3}$ behaviour for t_{fe} (defined as the value of t for which the field perturbation energy is maximal) persists, as expected, to these high values. Computer experiments based on the equation for f fail for $R_m \gtrsim 10^3$ due to inadequate radial resolution of the developing field structure.

5. Discussion

It remains to explain why the argument given by Moffatt (1978, §3.8), though appealing in its simplicity, is in fact incorrect. This argument involved simple evaluation of the diffusion term $\eta\nabla^2 A$ in (1.1) on the basis of the Lagrangian solution (1.2), and the assertion that, when $\eta\nabla^2 A$ becomes of the same order as either term on the left of (1.1), neglect of diffusion is no longer valid. It is this argument that leads to the estimate $t_{fe} \sim R_m^{1/2} t_0$ referred to in the introduction. The reason that diffusion becomes significant at an *earlier stage* ($\sim R_m^{1/3} t_0$) is that, whereas the $\mathbf{u}\cdot\nabla A$ term in (1.1) leads to periodic variation of A at any fixed point (with period of order t_0) the diffusion term $\eta\nabla^2 A$ is cumulative in its effect, which must therefore be estimated *by an integration from zero to t*, rather than simply an evaluation at time t; it is this integration which leads to the crucial t^3 term in (2.6). It is rather interesting that the normal procedure for neglecting a 'small' term in an equation, viz "neglect it, solve the equation, then evaluate the neglected term to see whether it was indeed negligible" is here unreliable and gives a misleading result!

Acknowledgements

We are grateful to Dr Nigel Weiss, whose disbelief in the $R_m^{1/2}$ estimate led to a continuing investigation of the problem, and to Dr John Chapman, who contributed to some of our earlier discussions.

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Gordon & Breach, London, 1983

→ Flux Expulsion and Homogenization – Non-Ideal, cont'd

– so far, have encountered :

→ S-P reconnection ⇒ weak dissipation ($R_m \gg 1$) has strong effect at singularity – BOUNDARY LAYER

→ Taylor Hypothesis ⇒ small flux tubes destroyed by stochasticity, leaving $\int d^3x\, \underline{A} \cdot \underline{B}$ as robust invariant

∴ diffusive dissipation most effective at breaking freezing-in on small scales

Another examples : $\left\{ \text{singular behavior in 2D closed-streamline flow} \right.$

→ $\boxed{\text{Homogenization Theory}}$ → $\left\{ \begin{array}{l} \text{Prandtl, Batchelor} \\ \text{Weiss} \\ \text{Rhines, Young} \end{array} \right.$

recall $\underline{\omega}$ evolution for $\underline{\nabla} \cdot \underline{V} = 0$

$$\frac{\partial}{\partial t} \underline{\omega} + \underline{V} \cdot \underline{\nabla}\, \underline{\omega} = \underline{\omega} \cdot \underline{\nabla}\, \underline{V} + \nu \nabla^2 \underline{\omega}$$

2D → $\underline{\omega} \cdot \nabla V = 0$ $\qquad \underline{\omega} = \omega(\hat{z})$

$\qquad V = \nabla\phi \times \hat{z}$

c.f. [P.A. Davidson 2004] – c.f.

A little viscosity ~~goes a long way~~ makes a global difference.

then $\partial_t \omega + \underline{\nabla}\phi \times \hat{z} \cdot \underline{\nabla}\omega = \nu \nabla^2 \omega$

more generally scalar ξ: $\left\{\begin{array}{l}\text{active} \\ \text{or} \\ \text{passive}\end{array}\right.$

$$\partial_t \xi + \underline{\nabla}\phi \times \hat{z} \cdot \nabla \xi = \nu \nabla^2 \xi$$

Now: $t \to \infty$, $\partial_t \xi \to 0$

$$\underline{\nabla}\phi \times \hat{z} \cdot \nabla \xi = \nu \nabla^2 \xi$$

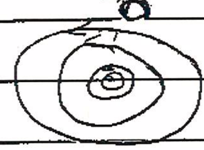
$\nu \to 0$ $\underline{\nabla}\phi \times \hat{z} \cdot \nabla \xi = 0$

i.e. $P_e \sim \frac{VL}{\nu} \to \infty$ ~Re

$$\xi = \xi(\phi)$$

i.e. bounded domain, closed streamline solution

ξ_0

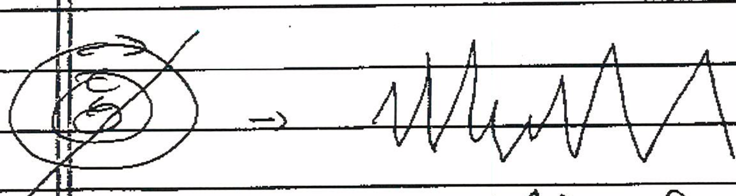


→ $\xi = \xi(\phi(r))$ is arbitrary solution

can develop arbitrarily fine scale $\xi(\phi)$ → closed streamlines perfect memory ⇒

→ ∴ fine scale structure develops, no inter-streamline communication, & persists

i.e.



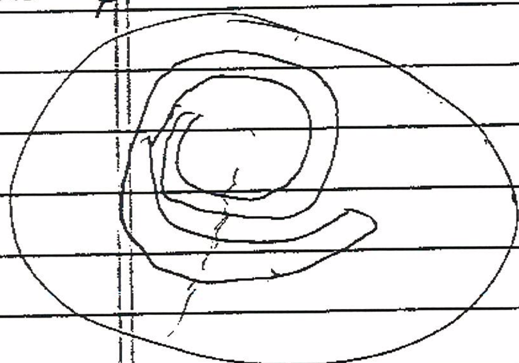
$\left\{\begin{array}{l}\text{i.e. tag each} \\ \text{streamline} \\ \text{arbitrarily}\end{array}\right.$

~ no smoothing of sharp gradients

"Not all solutions of the Navier-Stokes equations are realized in nature."
Landau & Lifshitz

→ i.e. of  ⇒ blob in concentric shear flow

blow-up



⇒ non-diffusive stretching produces arbitrarily fine scale structure!

now, point is that for $\nu \neq 0$
$Re, Pe \gg 1$

instead of arbitrarily fine scale structure

must have: $\omega(\psi) \to \text{const}$ as $t \to \infty$

i.e. small ν ⇒ global behavior

⇒ i.e. finite ν at large Re ⇒ (P)Vorticity homogenization!

$\omega \to \text{const}$ within C_0

⇒ highly singular behavior!

$\nu = 0$ → Euler Eqn. (2D) → $\omega = \omega(\psi)$ solves

$\nu \neq 0$ → large Re 2D Navier-Stokes Eqn. ⇒ $\omega = \text{const}$ solves

note contrast!

Issues:

→ how long to homogenization? (what means asymptotic)

→ where is $\nabla\omega \neq 0$ ⇒ boundary layer thickness?

→ analogy in MHD? — Flux Expulsion

$$\underline{E} + \frac{\underline{v}\times\underline{B}}{c} = \eta \underline{J} \qquad \begin{aligned} \underline{v} &= \nabla\phi\times\hat{z} \\ \underline{B} &= \nabla A\times\hat{z} \end{aligned}$$

$$-\frac{1}{c}\partial_t \underline{A} - \nabla\phi + \frac{(\nabla\phi\times\hat{z})}{c}\times(\nabla A\times\hat{z}) = \eta \underline{J}$$

$\hat{z}\cdot(\quad)$

$$\Rightarrow -\frac{1}{c}\partial_t A_z - \cancel{\nabla\phi\cdot\hat{z}} + \hat{z}\cdot\left[\cancel{(\nabla\phi\times\hat{z})\cdot\hat{z}}\right]\nabla A$$

$$-\frac{(\nabla\phi\times\hat{z})\cdot\nabla A}{c} = \eta \underline{J}$$

$$\therefore \quad \partial_t \underline{A} + \nabla\phi\times\hat{z}\cdot\nabla A = \eta\nabla^2 A$$

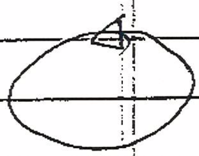
$$\Rightarrow 2D \text{ convection} \begin{cases} \nabla\cdot\underline{v} = 0 \\ \eta \neq 0 \end{cases}$$

$\Rightarrow$ expect $\nabla A = 0$, except boundaries
$t\to\infty$

→ analog of "flux expulsion"

(E.) Prandtl–Batchelor Theorem

* G. Batchelor, JFM 1 177 (1956) (posted)
P.B. Rhines and W.R. Young, JFM 122, 347 '82 (posted)
JFM 133 133 '83
J. Pedlosky, "Ocean Circulation Theory"
see also Springer 1996; esp. 3.8



Prandtl – Batchelor Theorem

Thm: Consider a region of 2D incompressible flow (i.e. vorticity advection) enclosed by closed streamline C_0. Then, if diffusive dissipation,

i.e. $\partial_t \omega + \nabla\phi\times\hat{z}\cdot\nabla\omega = \nabla\cdot(\nu\nabla\omega)$

then vorticity → uniform (homogenization), as $t \to \infty$, within C_0.

N.B.: ① finite ν ⇒ radically different final state

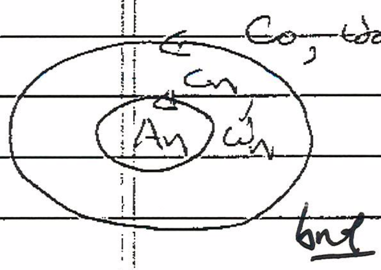
② no comment on "how long"!

$t \to \infty$ before $\nu \to 0$ $\underline{6a}$

$$-\nabla\phi \times \hat{z} \cdot \nabla\omega = \nabla \cdot \nu \nabla \omega$$

for stationarity [note $t \to \infty$ before $\nu \to 0$]

- choose arbitrary closed C_n within C_0.
  Here C_n a streamline



n.b. – assume simply connected region, i.e. no holes
– stationarity ⇒ ω constant along streamlines.
– C_0 → specified on 1 bndry.

∴ $\omega \to \omega_0$ on C_0 (ultimately C_0 sets b.c.)
$\omega \to \omega_n$ on C_n

if A_n is area enclosed by C_n;

$$\int_{A_n} d^2x \, \underline{v} \cdot \underline{\nabla}\, \omega = \int_{A_n} d^2x \, \underline{\nabla} \cdot \left(\nu \, \underline{\nabla} \omega \right)$$

but

$$\int_{A_n} d^2x \, \underline{v} \cdot \underline{\nabla}\, \omega = \int d^2x \, \underline{\nabla} \cdot [\underline{v} \omega]$$

$$= \int_{C_n} dl \, \hat{n}_{C_n} \cdot (\underline{v} \omega)$$

normal

streamline, C_n

Now $\vec{V}$

$\hat{n}_{C_n}$ → normal to streamline C_n

so $$\int_{C_n} dl \, (\hat{n}_{C_n} \cdot \mathbf{V}) \, \omega = 0$$

as V is along streamline

∴ $$0 = \int_{A_n} d^3x \, \nabla \cdot (\nu \nabla \omega)$$

$$= \nu \int_{C_n} dl \, \hat{n}_{C_n} \cdot \nabla \omega$$

now in stationary state must have ω → const along streamline

∴ $\omega = \omega(\phi)$

so $\omega_{C_n} = \omega(\phi_n)$

$$0 = \nu \int_{C_n} dl \, \hat{n}_{C_n} \cdot \nabla \phi_n \, \frac{\partial \omega}{\partial \phi_n}$$

$$0 = \nu \frac{\partial \omega}{\partial \phi_n} \int_{C_n} dl \, \hat{n}_{C_n} \cdot \nabla \phi_n$$

<u>but</u>

$$\Gamma = \int d\underline{l} \cdot \underline{v}$$

$$= \int d\underline{l} \cdot (\underline{\nabla}\phi \times \hat{z})$$

$$= \int (\hat{z} \times \vec{n}) \cdot (\underline{\nabla}\phi \times \hat{z})$$

$$= -\int dl \, (\underline{\nabla}\phi \cdot \vec{n}) = -\int dl \, (\underline{\nabla}\phi \cdot \vec{n})$$

$\therefore$

$$0 = \nu \frac{\partial \omega}{\partial \phi_n} \Gamma_n$$

$\therefore$

$$\partial \omega / \partial \phi_n = 0$$

but ϕ arbitrary $\Rightarrow$ $\boxed{\partial \omega / \partial \phi = 0 \;, \; \underline{\underline{\text{all}}} \; \phi}$

arbitrary $\Rightarrow$ no variation from line to line

$\Rightarrow$ $\boxed{\boxed{\omega \text{ homogenized}}}$!

so, expect $\nabla \omega$ larger at bounding contour C_0.

$\nabla \omega \neq 0$, within $\Rightarrow$ $\nabla \omega$ held at boundary

closed bounded streamline

Some Comments:

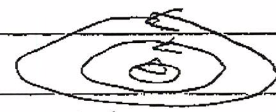
⇒ Homogenization theory looks 'magical' → caveat emptor!

i.e.

* 1.) note assumptions of:

$t \to \infty$ ⇒ time asymptopia

$q = q(\phi)$ ⇒ concentric streamlines

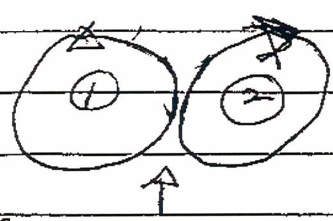


how long to achieve configuration?!

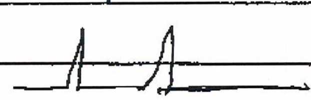
2.) simply connected domain ⇒ annulus? (2F?)

3.) single structure → expulsion from neighbors and possible interaction not addressed

i.e. what happens if? → 'interference of boundary layers?'



or



⇒ straining interaction

⇒ ② 'strips' ①, etc.

4.) Key Assumptions:

⇒ {closed, bounding streamline
  {viscous dissipation

i.e. can envision:

→ [exact streamline, molecular viscosity
or
→ coarse-grained streamline, eddy viscosity

⇒ correspond to homogenization of
→ total vorticity
→ mean/coarse-grained vorticity

⇔ time scales different

→ $\boxed{\dfrac{\tau_{circulation}}{\tau_{diffusion}} \ll 1 \Rightarrow Re \gg 1 \quad \leftarrow L \rightarrow}$

then
- to establish concentric circulation lines
- diffuse accross to homogenize → but slow ??

$$\frac{\tau_c}{\tau_d} = \frac{1}{(V/L)} \frac{D}{L^2} \ll 1 \Rightarrow \frac{D}{VL} \ll 1$$

i.e. $Re \gg 1$

or equivalently $\frac{VL}{D} \gg 1$

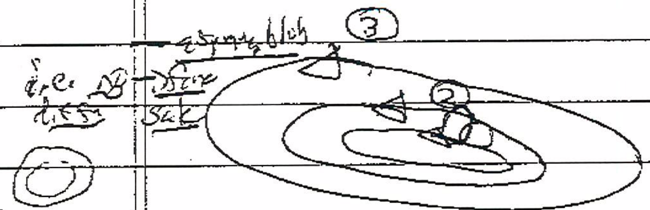
i.e. $(Re)_{\substack{eff \\ cell}} \gg 1$ $Pe \gg 1$

related: – essential idea is that q constant along streamlines established on fast ($\sim \tau_c$) scale few

– dissipation homogenizes on slower ($\sim \tau_D$) time scale (but this is slow...)

→ what are the time scales? – how resolve slow time scale problem

– useful to consider differentially rotating sheared flow within closed pattern



$V_1 \neq V_2 \neq V_3$

need blob with finite l_y

what is the mixing time scale?

finer, finer scale ...

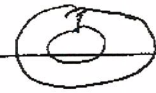
key: synergism between {shear, diffusion}!

$\boxed{\text{shear dispersion}}$

c.f. {H. Biglari, P.H. Diamond, P.W. Terry, Phys Fluids (B2), 1, 1990

(first noted by G.I. Taylor)

mixing Shear Dispersion

i.e. compare  $\leftarrow L \rightarrow$

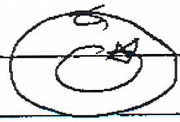
time

(a) pure diffusion $1/\tau \sim D/L^2$ $\boxed{\langle \delta r^2 \rangle \sim D t}$ → usual diffusion

radial diffusion

$D_r \sim \langle V_r^2 \rangle \tau_c$

any radial scattering process (unspecified)

(b) diffusion + shear hybrid with  $V' \sim V/L$ δ

shear

now $\frac{dr}{dt} = \tilde{V}_r$ → random walk

$r\theta = y$

$\frac{dy}{dt} = V_{\theta}(r)$ → streaming

shear

$\frac{d}{dt}\delta y = \left(\frac{\partial V_{\theta}}{\partial r}\right)\delta r$

$\delta y = \int \left(\frac{\partial V_{\theta}}{\partial r}\right)\delta r\, dt$

$\langle \delta y^2 \rangle \sim \left(\frac{\partial V_{\theta}}{\partial r}\right)^2 \boxed{\langle \delta r^2 \rangle} t^2$

$\langle \delta r^2 \rangle \sim D_r t$

$\Rightarrow \boxed{\langle \delta y^2 \rangle \sim \left(\frac{\partial V_{\theta}}{\partial r}\right)^2 D_r t^3}$ → shear dispersion hybrid decorrelation $\langle \delta y^2 \rangle \sim t^3$

scale of comparison

$$\langle \delta y^2 \rangle \sim L_y^2 \Rightarrow \text{arbitrary}$$

$$1/\tau_{mix} = \left(\left(\frac{\partial V_y}{\partial x}\right)^2 \frac{D}{L_y^2} \right)^{1/3}$$

$$\sim \left(\left(\frac{V_0}{L_0}\right)^2 \frac{D}{L_y^2} \right)^{1/3}$$

$$\sim \frac{V_0}{L_0} \left(\frac{D}{V L_0} \right)^{1/3}$$

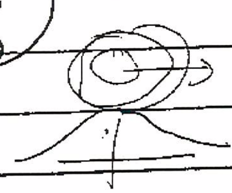
$$\therefore \boxed{1/\tau_{mix} \sim \frac{1}{\tau_0} (Re)^{-1/3}}$$

$Re \gg 1$ by construction $\rightarrow$ consistent ✓

so have:

→ mixing / homogenization on hybrid time scale → time to come to ⊙ symmetric state

$$1/\tau_{mix} = 1/\tau_0 \left(\tau_0 / \tau_D \right)^{1/3}$$



only radial diffusion possible

$$\Rightarrow \frac{1}{\tau_0} > \frac{1}{\tau_{mix}} > \frac{1}{\tau_D}$$

time to uniformize vs τ_D

⇒ PV homogenization most relevant to closed eddys with sheared rotation

Some Points

i.) Time scales

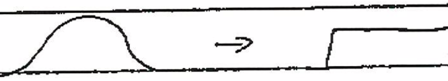
have $Re, \; Pe \gg 1 \quad \Rightarrow \quad \frac{\tau_D}{\tau_c} \gg 1$

but $\quad \tau_{mix} \sim Re^{1/3} \tau_c \sim \frac{\tau_D}{Re^{2/3}}$

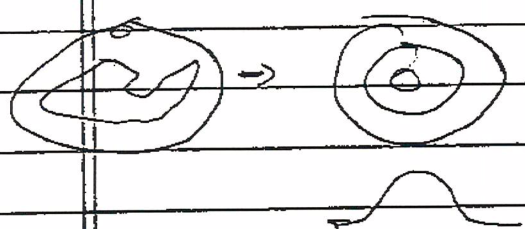
$\tau_{mix} < \tau_D$

↓ → time to homogenize

time to establish ~ azimuthally symmetric state



i.e.



but radially profiled

Compute mixing NS.

ii.) $\boxed{\text{Point of theorem is global impact of small dissipation.}}$

iii.) Interesting to note that P-B theorem applies to both active, passive scalar.

ii) Observe: all that is really required for applicability of theory is:

- → incompressible advection – 2D: $\nabla\phi \times \hat{z} \cdot \nabla$
- → closed streamline → {fine, coarse}
- → diffusive dissipation → {molecular, eddy}

i.e. can apply to magnetic potential, as noted previously, i.e.

2D MHD

$\underline{v} = \nabla\phi \times \hat{z}$

$\underline{B} = \nabla A \times \hat{z}$

$$\frac{\partial A}{\partial t} + \underline{v} \cdot \nabla A = \eta \nabla^2 A$$

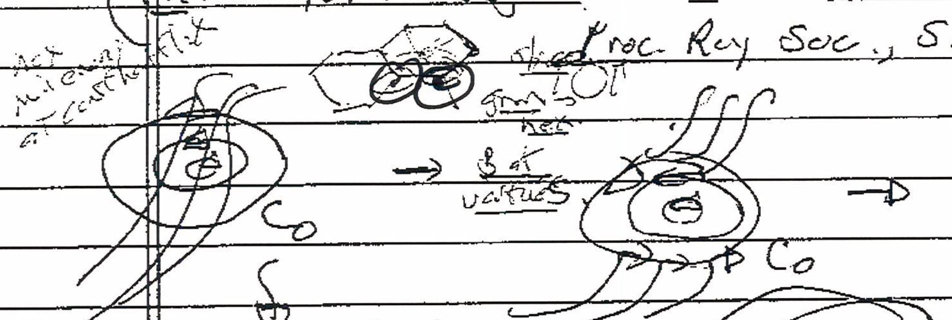
$E + \underline{v} \times B = \eta J$

$-\frac{1}{c}\frac{\partial A}{\partial t} - \cancel{\nabla\phi} + (\nabla\phi \times \hat{z}) \times (\nabla A \times \hat{z}) = \eta \underline{J}$

$\hat{z}\left\{\left(-\frac{1}{c}\frac{\partial A}{\partial t}\right) + \cancel{(\nabla\phi \times \hat{z}) \cdot \hat{z}} - (\nabla\phi \times \hat{z} \cdot \nabla A)\hat{z}\right\} = \eta \underline{J}$

Why? → ~~sun spots~~

→ famous problem of Flux Expulsion (i.e. N. Weiss, Proc. Roy. Soc., Series A 293, 310, 1966)



↓ specified convective cell (has magnetoconvection as aim)

see 15a

~~Outcome~~ of Magneto Convection [illegible] T_B

→ $\nabla A \to 0$ within

→ B expelled to boundary (i.e. $B = \nabla A \times \hat{z}$)

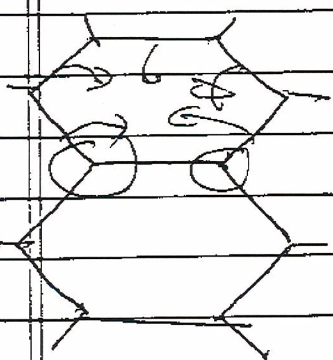
⇒ $B \sim 0$ in cell.

→ obvious that above argument can be recycled, so

$\delta A_n / \delta\phi_n = 0$, for all C_n in C_0.

treat kinematically → why?

i.e. top view → solar granulation



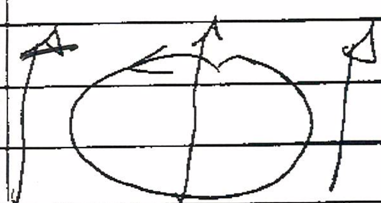
~ hexagonal pattern

~ field strongest at Vertices

result of

~ expulsion

suggests → side view → toy problem of flux expulsion



→ i.e.

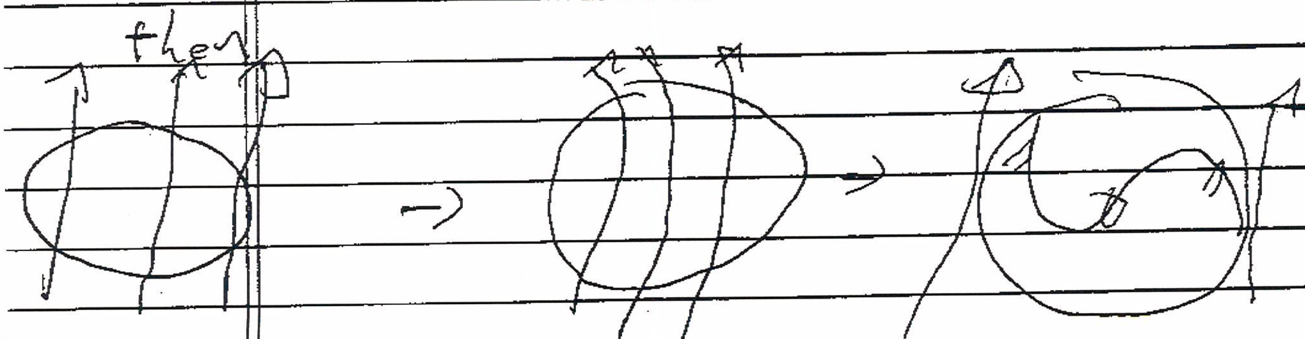
what happens?

→ cell linked by field

cell in uniform field

What happens

then



see Weiss, Proc. Roy Soc. A 293, 310-1966

also Moffatt, 3.7-3.10

→ here requirement is $R_m = \frac{LV}{\eta} \gg 1$

and

$1/\tau_{hang} \sim 1/\tau_{mix} \sim \frac{V_0}{L_0}(R_m)^{-1/3}$

↓

time scale for flux expulsion

→ physically, can see relevant time scale by noting:

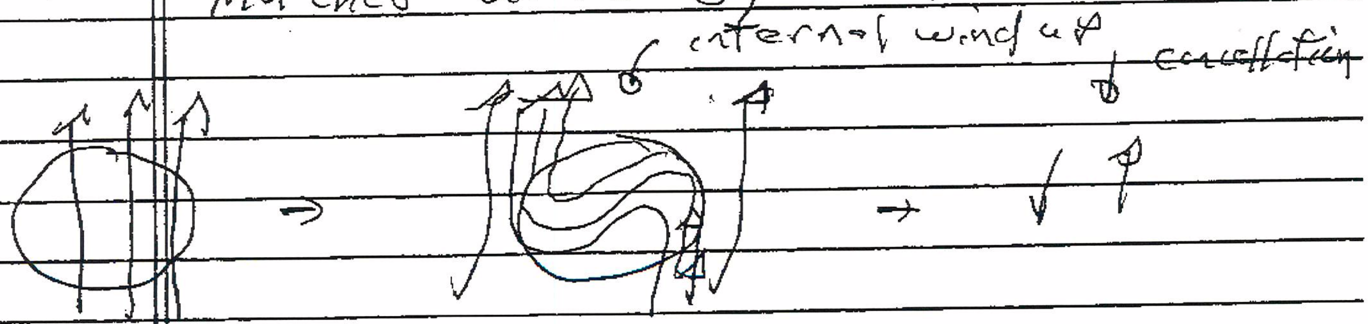
Flux

→ wind up must conserve volume/mass

→ $\boxed{\text{wind up must conserve flux}}$

→ $\boxed{\text{irreversibility sets in when } R_m|_{local} \sim 1}$

⇒ dissipation of field matches drive by wind-up



internal cancellation → { field expulsion
flux homogenization

18. [redacted]

so $B_0 L \sim \ell B$ ($B \uparrow$ upon winding)
flux conservation

⇒ $B \sim n B_0$

wind-up in = dissipation out

⇒ now, expect freezing-in lost when compressed field / amplified

$(R_m)_{eff} \sim 1 \Rightarrow \frac{V B_0}{L_0} \sim \eta \frac{B}{\ell^2} \sim \eta \frac{n B_0}{L_0^2/n^2}$

⇒

$n^3 \sim \left(\frac{V_0 L_0}{\eta}\right) \Rightarrow \boxed{n \sim R_m^{1/3}}$

not $R_m^{1/2}$

[CHKM not realize field amplified]

\# of turns to render boundary diffusive

~~thickness of layer~~

$\frac{V B}{L} \sim \frac{\eta B}{\delta^2}$ $\frac{\delta}{L} \sim 1/\sqrt{R_m}$

but: - diffusion in boundary layer ⇔ homogenization within

so

- $n \sim R_m^{1/3}$ ⇒ # turns for homogenization

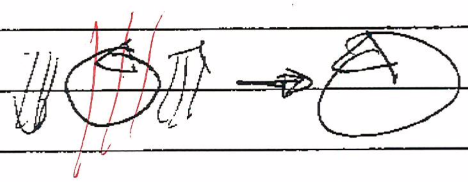
- $\boxed{\tau_{hom} \sim \tau_0 R_m^{1/3}}$ → time ✓

agreed above

⇒ expect flux expelled from closed cell
→ field strongest at boundary → spots

→ possibly explain why strongest cells in 2D

magneto convection often
reach a state independent of B
while neighbors quenched.



Magnetoconvection = convection in B field

Now: ⇒ why care about this?
Why homogenization important
→ simulated [illegible]

① identifies a trend; i.e. in spirit of
Taylor Theory (E_{mag} minimized s/t
$\int \underline{A} \cdot \underline{B}\, d^3x$ conserved), homogenization
theory identifies a trend, i.e.

Relaxation Principle

if f [– conserved locally by 2D flow, $\nabla \cdot v = 0$
– diffused
– enclosed

⇒ f homogenized

Strong B_0
→ ~~[illegible]~~
Alfven

② trend applies to ~~non-passive~~ → vorticity
~~passive~~ → A, C

In particular

③ trend severely constrains form of vorticity flux, flow evolution

i.e. zonal flows → 2D closed streamline flows ....

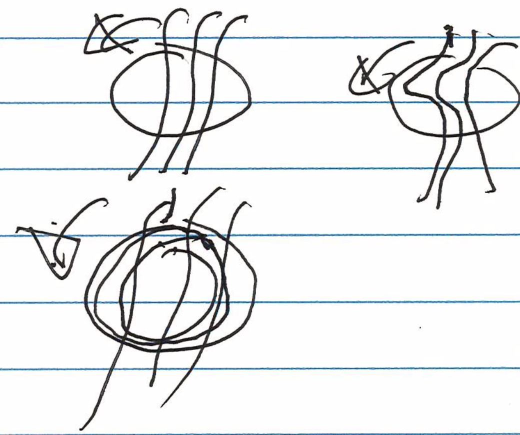
→ Do zonal flows tend homogenize PV?

→ if noise (i.e. emission), what scale selected?

→ When does kinematic expulsion end?

(MGH)

~ qualitatively, when magnetic tension grows to point of competition with vortex evolution.



Now, $b\ell = B_0 L_0$

$$\frac{\eta}{\ell^2} b = \frac{B_0 U_0}{L_0}$$

so $$\eta \frac{b}{\left(\frac{B_0 L_0}{b}\right)^2} = \frac{B_0 U_0}{L_0}$$

$$\Rightarrow \boxed{b \sim B_0 \left(\frac{U_0 L_0}{\eta}\right)^{1/3}} \rightarrow \frac{b}{B_0} \sim \frac{L_0}{\ell} \sim R_m^{1/3}$$

For tension;

stretched field

$$\underline{B}\cdot\underline{\nabla}\underline{B} = -\frac{|B|^2}{r_c}\hat{n} + \frac{d}{ds}\left(\frac{|B|^2}{2}\right)\hat{t}$$

$$|\underline{B}\cdot\underline{\nabla}\underline{B}| \approx \frac{b^2}{L_0} \qquad \left\{ \begin{array}{l} r_c \sim L_0 \\ \frac{d}{ds} \sim L_0^{-1} \end{array} \right.$$

then:

$$\rho \frac{d\omega}{dt} = \left[\nabla \times (B \cdot \nabla B)\right] \cdot \hat{z}$$

picks up short length scale

$$\rho\, u \cdot \nabla \omega \sim \frac{b^2}{\ell L_0}$$

$$\frac{u}{L_0}\omega \sim \frac{b^2}{\rho_0 L_0 \ell}$$

but

$$\frac{u}{L_0}\omega \sim \frac{u^2}{L_0^2} \sim \frac{b^2}{\rho L_0 \ell}$$

$$\sim \frac{B_0^2 L_0^2}{\ell^3 L_0 \rho_0}$$

$\therefore$ for given L_0, critical B_0 to prevent expulsion:

$$\frac{u^2}{L_0^2} \sim \frac{B_0^2 L_0}{\ell^3 \rho_0} \Rightarrow \left(\frac{V_{A_0}}{u}\right)^2 \sim \left(\frac{L_0^3}{\ell^3}\right)^{-1}$$

$$\frac{b}{B_0} \sim \left(\frac{u_0 L_0}{\eta}\right)^{1/3} \sim R_m^{1/3} \qquad \sim R_m^{-1}$$

so $$\boxed{Rm\left(\frac{V_{A0}}{u}\right)^2 \sim 1}$$

is feedback criterion!

→ akin feedback criterion for magnetic flux transport in 2D

→ Can easily have $V_{A0} \ll u$ with $Rm \gg 1$, and still encounter feedback.

⟺ weak field is sufficient.