Consider the one-dimensional Bloch oscillations discussed in §4.5.3 of the Lecture Notes. Setting \( k(0) = 0 \), show that a Taylor expansion of the motion \( x(t) \) agrees with the ballistic result to order \( t^2 \), if the ballistic mass is taken to be the effective mass \( m^* \) at \( k = 0 \).

**Solution:**
According to the Lecture Notes, if \( E(k) = -2\beta \cos(ka) \), and setting \( k(0) = 0 \), the motion of a Bloch wavepacket is given by

\[
x(t) = x(0) - \frac{2\beta}{eE} \left[ \cos \left( \frac{eaEt}{h} \right) - 1 \right] = \frac{\beta e E a^2 t^2}{h^2} \frac{F}{2m^*} t^2,
\]

with \( F = -eE \) the force on the electron. The relation between \( \beta \) and \( m^* \) is obtained from \( E(k) = -2\beta + \beta a^2 k^2 + \mathcal{O}(k^4) \), whence we identify \( \beta = \frac{h^2}{2m^* a^2} \).

[2] Make a sketch of the extended Brillouin zones like in Fig. 5.2.2 of the Lecture Notes, but for the triangular lattice. Then make plots the free electron Fermi surface for valences \( Z = 2 \) and \( Z = 3 \), such as in Fig. 2.3.

**Solution:**
See Fig. 1. For \( Z = 2 \) the free electron Fermi sphere extends into the second Brillouin zone, and for \( Z = 3 \) it extends into the third Brillouin zone!

![Figure 1](image)

**Figure 1:** Left: Brillouin zones for the triangular lattice structure. Right: The concentric circles correspond to electron fillings of \( Z = 1, Z = 2 \), and \( Z = 3 \) per unit cell.

[3] Suppose, in the vicinity of the \( \Gamma \) point for some material, the electron dispersion is of the form \( E(k) = \frac{1}{2}(m^*)^{-\frac{1}{2}} k^\mu k^\nu \), where \( m^* \) is the effective mass tensor. Find an expression for the low-temperature molar heat capacity in terms of density \( n \) and temperature \( T \). Your expression should involve the tensor \( m^* \) in some way.
Solution:
The heat capacity is $C_V = V \gamma T$, where $\gamma = \frac{1}{3} \pi^2 k_B^2 g(\varepsilon_F)$. The density of states at the Fermi energy is $g(\varepsilon_F) = m k_F / \pi^2 h^2$ for free electrons (spin degeneracy included). For a general quadratic dispersion, we can choose our Cartesian axes to align with the principal axes of the effective mass tensor $m^*_{\alpha\beta}$. These axes are necessarily orthogonal because the effective mass tensor is symmetric. Along principal axes, the conduction band dispersion is given by $E(k) = \frac{\hbar^2 k_x^2}{2m^*_x} + \frac{\hbar^2 k_y^2}{2m^*_y} + \frac{\hbar^2 k_z^2}{2m^*_z}$, and we may rescale each $k_\alpha$ in the DOS integral by $\sqrt{m^*_\alpha}$, yielding $g(\varepsilon_F) = \frac{\text{det}^{1/3}(m^*) (3\pi^2 n)^{1/3}}{\pi^2 \hbar^2}$. 

[4] Cyclotron resonance in Si and Ge – Both Si and Ge are indirect gap semiconductors with anisotropic conduction band minima and doubly degenerate valence band maxima. In Si, the conduction band minima occur along the (100) (⟨ΓX⟩) directions, and are six-fold degenerate. The equal energy surfaces are cigar-shaped, and the effective mass along the ⟨ΓX⟩ principal axes (the ‘longitudinal’ effective mass) is $m^*_l \approx 1.0 m_e$, while the effective mass in the plane perpendicular to this axis (the ‘transverse’ effective mass) is $m^*_t \approx 0.20 m_e$. The valence band maximum occurs at the unique Γ point, and there are two isotropic hole branches: a ‘heavy’ hole with $m^*_{lh} \approx 0.49 m_e$, and a ‘light’ hole with $m^*_{lh} \approx 0.16 m_e$.

In Ge, the conduction band minima occur at the fourfold degenerate L point (along the eight ⟨111⟩ directions) with effective masses $m^*_l \approx 1.6 m_e$ and $m^*_t \approx 0.08 m_e$. The valence band maximum again occurs at the Γ point, where the hole masses are $m^*_{lh} \approx 0.34 m_e$ and $m^*_{lh} \approx 0.044 m_e$. Use the following figures to interpret the cyclotron resonance data shown below. Verify whether the data corroborate the quoted values of the effective masses in Si and Ge.

(See the following four figures.)

Solution:
We found that $\sigma_{\alpha\beta} = n e^2 \Gamma^{-1}_{\alpha\beta}$, with

$$
\Gamma_{\alpha\beta} \equiv (\tau^{-1} - i \omega) m_{\alpha\beta} \pm \frac{e}{c} \epsilon_{\alpha\beta\gamma} B^\gamma \\
= \begin{pmatrix}
(\tau^{-1} - i \omega)m^*_x & \pm e B_z/c & \mp e B_y/c \\
\mp e B_z/c & (\tau^{-1} - i \omega)m^*_y & \pm e B_x/c \\
\pm e B_y/c & \mp e B_x/c & (\tau^{-1} - i \omega)m^*_z
\end{pmatrix}.
$$

The valence band maxima are isotropic in both cases, with

$$
\begin{align*}
    m^*_{lh}(\text{Si}) &\approx 0.49 m_e & m^*_{lh}(\text{Ge}) &\approx 0.34 m_e \\
    m^*_{lh}(\text{Si}) &\approx 0.16 m_e & m^*_{lh}(\text{Ge}) &\approx 0.044 m_e.
\end{align*}
$$
Figure 2: Constant energy surfaces near the conduction band minima in silicon. There are six symmetry-related ellipsoidal pockets whose long axes run along the \( \langle 100 \rangle \) directions.

Figure 3: Cyclotron resonance data in Si (G. Dresselhaus et al., Phys, Rev, 98, 368 (1955).) The field lies in a (110) plane and makes an angle of 30° with the [001] axis.

With isotropic bands, the absorption is peaked at \( \omega = \omega_c = eB/m^*c \), assuming \( \omega_c \tau \gg 1 \).
Figure 4: Constant energy surfaces near the conduction band minima in germanium. There are eight symmetry-related half-ellipsoids whose long axes run along the \(\langle 111\rangle\) directions, and are centered on the midpoints of the hexagonal zone faces. With a suitable choice of primitive cell in \(k\)-space, these can be represented as four ellipsoids, the half-ellipsoids on opposite faces being joined together by translations through suitable reciprocal lattice vectors.

Writing \(\omega = 2\pi f\), the resonance occurs at a field

\[
B(f) = 2\pi f \cdot \frac{m^*c}{e}
\]

\[
= \frac{hc}{e} \cdot \frac{m^*}{me} \cdot \frac{1}{2\pi a_B^2} \cdot \frac{hf}{(e^2/a_B)}
\]

\[
= 3.58 \times 10^{-7} \text{ G} \cdot \frac{m^*}{me} \cdot f [\text{Hz}]
\]

\[
= 8590 \text{ G} \cdot \frac{m^*}{me}
\]

where we have used

\[
\frac{hc}{e} = 4.137 \times 10^{-7} \text{ G} \cdot \text{cm}^2
\]

\[
a_B = \frac{h^2}{me e^2} = 0.529 \text{ Å}
\]

\[
h = 4.136 \times 10^{-15} \text{ eV} \cdot \text{s}
\]

\[
\frac{e^2}{a_B} = 27.2 \text{ eV} = 2 \text{ Ry}
\]

\[
f = 2.40 \times 10^{10} \text{ Hz}
\]
Thus, we predict

\[ B_{hh}(Si) \approx 4210 \text{ G} \quad B_{hh}(Ge) \approx 2920 \text{ G} \]
\[ B_{lh}(Si) \approx 1370 \text{ G} \quad B_{lh}(Ge) \approx 378 \text{ G} \]

All of these look pretty good.

Now let us review the situation with electrons near the conduction band minima:

**Si:** 6-fold degenerate minima along \( \langle 100 \rangle \)

**Ge:** 4-fold degenerate minima along \( \langle 111 \rangle \) (at L point)

\[ m_{t}^{*}(Si) \approx 1.0 \, m_{e} \quad m_{t}^{*}(Ge) \approx 1.6 \, m_{e} \]
\[ m_{t}^{*}(Si) \approx 0.20 \, m_{e} \quad m_{t}^{*}(Ge) \approx 0.08 \, m_{e} \]

The resonance condition is that \( \sigma_{\alpha \beta} = \infty \), which for \( \tau > 0 \) occurs only at complex frequencies, i.e. for real frequencies there are no true divergences, only resonances. The location of the resonance is determined by \( \det \Gamma = 0 \). Taking the determinant, one finds

\[
\det \Gamma = (\tau^{-1} - i\omega) \, m_{t}^{*} \cdot \left\{ (\tau^{-1} - i\omega)^{2} m_{t}^{*2} + \frac{e^{2}}{c^{2}} B_{z}^{2} + \frac{m_{t}^{*2}}{m_{t}^{*} c^{2}} (B_{x}^{2} + B_{y}^{2}) \right\}.
\]

Assuming \( \omega \tau \gg 1 \), the location of the resonance is given by

\[
\omega^{2} = \left( \frac{eB_{\parallel}}{m_{t}^{*} c} \right)^{2} + \frac{m_{t}^{*2}}{m_{t}^{*} c} \left( \frac{eB_{\perp}}{m_{t}^{*} c} \right)^{2},
\]
where $B_\parallel \equiv B_x$ and $B_\perp \equiv B_x \hat{x} + B_y \hat{y}$. Let the polar angle of $B$ be $\theta$, so $B_\parallel = B \cos \theta$ and $B_\perp = B \sin \theta$. We then have
\[
\omega^2 = \left( \frac{eB}{m_1^*} \right)^2 \left\{ \cos^2 \theta + \frac{m_e^*}{m_1^*} \sin^2 \theta \right\}
\]
\[
B(f) = 8600 \text{ G} \cdot \frac{\left( \frac{m_e^*}{m_1^*} \right)}{\sqrt{\cos^2 \theta + \frac{m_e^*}{m_1^*} \sin^2 \theta}},
\]
where again we take $f = \omega/2\pi = 2.4 \times 10^{10} \text{ Hz}$.

According to the diagrams, the field lies in the (110) plane, which means we can write
\[
\hat{B} = \sqrt{\frac{3}{8}} \sin \chi \hat{e}_1 - \sqrt{\frac{3}{8}} \sin \chi \hat{e}_2 + \cos \chi \hat{e}_3,
\]
where $\chi$ is the angle $\hat{B}$ makes with $\hat{e}_3 = [001]$.

\text{under}text Ge

We have
\[
\frac{m_e^*}{m_1^*} = 0.082 \quad \frac{m_e^*}{m_1^*} = 0.051,
\]
and we are told $\chi = 60^\circ$, so
\[
\hat{B} = \sqrt{\frac{3}{8}} \hat{e}_1 - \sqrt{\frac{3}{8}} \hat{e}_2 + \frac{1}{2} \hat{e}_3.
\]

The conduction band minima lie along $\langle 111 \rangle$, which denotes a set of directions in real space:
\[
\begin{align*}
\pm[111] : \hat{n} = \pm \frac{1}{\sqrt{3}} (\hat{e}_1 + \hat{e}_2 + \hat{e}_3) & \Rightarrow \cos^2 \theta = (\hat{B} \cdot \hat{n})^2 = \frac{1}{12} \Rightarrow B = 1950 \text{ G} \\
\pm[\bar{1}11] : \hat{n} = \pm \frac{1}{\sqrt{3}} (\hat{e}_1 + \hat{e}_2 - \hat{e}_3) & \Rightarrow \cos^2 \theta = (\hat{B} \cdot \hat{n})^2 = \frac{1}{12} \Rightarrow B = 1950 \text{ G} \\
\pm[\bar{1}11] : \hat{n} = \pm \frac{1}{\sqrt{3}} (-\hat{e}_1 + \hat{e}_2 + \hat{e}_3) & \Rightarrow \cos^2 \theta = (\hat{B} \cdot \hat{n})^2 = \frac{7+2\sqrt{6}}{12} \Rightarrow B = 1510 \text{ G} \\
\pm[1\bar{1}1] : \hat{n} = \pm \frac{1}{\sqrt{3}} (\hat{e}_1 - \hat{e}_2 + \hat{e}_3) & \Rightarrow \cos^2 \theta = (\hat{B} \cdot \hat{n})^2 = \frac{7+2\sqrt{6}}{12} \Rightarrow B = 710 \text{ G}.
\end{align*}
\]
All OK!

\text{Si:}

Again, $B$ lies in the (110) plane, this time with $\chi = 30^\circ$, so
\[
\hat{B} = \sqrt{\frac{3}{8}} \hat{e}_1 - \sqrt{\frac{3}{8}} \hat{e}_2 + \sqrt{\frac{3}{4}} \hat{e}_3.
\]

The conduction band minima lie along $\langle 100 \rangle$, so
\[
\begin{align*}
\pm[001] : \hat{n} = \pm \hat{e}_3 & \Rightarrow \cos^2 \theta = (\hat{B} \cdot \hat{n})^2 = \frac{3}{4} \Rightarrow B = 1820 \text{ G} \\
\pm[010] : \hat{n} = \pm \hat{e}_2 & \Rightarrow \cos^2 \theta = (\hat{B} \cdot \hat{n})^2 = \frac{1}{8} \Rightarrow B = 2980 \text{ G} \\
\pm[100] : \hat{n} = \pm \hat{e}_1 & \Rightarrow \cos^2 \theta = (\hat{B} \cdot \hat{n})^2 = \frac{1}{8} \Rightarrow B = 2980 \text{ G}.
\end{align*}
\]
These also look pretty good.