PHYSICS 211B : CONDENSED MATTER PHYSICS
HW ASSIGNMENT #5

(1) Consider the following model of a mesoscopic Josephson junction:

$$\hat{H} = -J\cos(\phi_1 - \phi_2) + 2e^2 \sum_{i,j} C^{-1}_{ij}\, M_i\, M_j - 2\sum_i \mu_i\, M_i \quad .$$

Here $i, j \in \{1, 2\}$, μ_i is the chemical potential on grain i, and C_{ij} is the capacitance matrix, which is real and symmetric.

(a) Find the equations of motion.

(b) Show that the total number of Cooper pairs is conserved.

(c) Defining $M = \frac{1}{2}(M_1 + M_2)$, $N = \frac{1}{2}(M_1 - M_2)$, and $\chi = \phi_1 + \phi_2$, and $\varphi \equiv \phi_1 - \phi_2$, find the equations of motion for these variables. Confirm your result from (b).

(d) Treating M as a constant, show that the dynamics for N and φ form a closed system of equations. By eliminating N, show that φ obeys the equation of motion of a pendulum.

(2) Consider a Josephson junction between two conventional superconductors. The junction has a square cross section of side length a. A magnetic field $\boldsymbol{H} = H_0(\cos\alpha\,\hat{\boldsymbol{x}} + \sin\alpha\,\hat{\boldsymbol{y}})$ lies in the plane of the junction and makes an angle α with respect to one of the sides of the square.

(a) Compute the critical current $I_c(\Phi, \alpha)$ as a function of the magnetic field H_0 and the angle α. It is convenient to measure the field H_0 in units of the flux $\Phi = H_0(\lambda_1 + \lambda_2 + d)a$, where λ_1 and λ_2 are the penetration depths of the superconductors forming the junction and d is their separation. Identify all the symmetries of $I_c(\Phi, \alpha)$ with respect to the junction orientation.

(b) Your result should reduce to the familiar $I_c(\Phi) = I_0(T)\,|\sin(\pi\Phi/\phi_L)/(\pi\Phi/\phi_L)|$, with $\phi_L = hc/2e$ the London quantum, when the field lies along one of the principal axes of the square. Check that this is so. Then consider the case $\alpha = \pi/4$ where the field is oriented along the diagonal. How does the pattern change? Plot I_c/I_0 *vs.* Φ/ϕ_L for $\alpha = 0$ and $\alpha = \frac{1}{4}\pi$ for $0 \le \Phi/\phi_L \le 3$.

(c) Compute $I_c(\Phi)$ when the junction has a circular cross section of radius a.