PHYSICS 211B : CONDENSED MATTER PHYSICS
HW ASSIGNMENT #1

(1) Define the operator

$$\Pi_N = \frac{1}{N!} \int\limits_{\mathbb{R}^{dN}} d^d x_1 \cdots d^d x_N \, |\, \boldsymbol{x}_1 \cdots \boldsymbol{x}_N \,\rangle\langle\, \boldsymbol{x}_1 \cdots \boldsymbol{x}_N \,| \quad ,$$

where

$$|\, \boldsymbol{x}_1 \cdots \boldsymbol{x}_N \,\rangle = \psi^\dagger(\boldsymbol{x}_1) \cdots \psi^\dagger(\boldsymbol{x}_N) \,|\, 0 \,\rangle \quad ,$$

where $\big[\psi(\boldsymbol{x})\,,\,\psi^\dagger(\boldsymbol{x}')\big]_\mp = \delta(\boldsymbol{x}-\boldsymbol{x}')$ for bosons $(-)$ and fermions $(+)$. Here each $\boldsymbol{x}_j \in \mathbb{R}^d$.

(a) Show that Π_N is a projector onto the totally symmetric and totally antisymmetric parts of the N-body Hilbert space for bosons and fermions, respectively.

(b) Show that one can also write

$$\Pi_N \equiv \int\limits_{\Delta_N} d^d x_1 \cdots d^d x_N \, |\, \boldsymbol{x}_1 \cdots \boldsymbol{x}_N \,\rangle\langle\, \boldsymbol{x}_1 \cdots \boldsymbol{x}_N \,| \quad ,$$

where Δ_N is defined to be the subset of $\mathbb{R}^{dN}$ for which

$$\Delta_N = \Big\{ (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \,|\, x_1^{(1)} < x_2^{(1)} < \cdots < x_N^{(1)} \Big\} \quad .$$

(2) Compute the Hartree-Fock self-energy $\Sigma(\boldsymbol{k})$ for a two-dimensional electron gas with interactions $u(\boldsymbol{r}) = e^2 \ln(a/r)$ (where a is some fixed length scale), and the one-dimensional electron gas, with interactions $u(x) = -e^2\,|x|$. Take note of any divergences you encounter as a function of $\boldsymbol{k}$.

(3) Consider a polarized electron gas (three dimensions, Coulomb interactions) in which N_σ denotes the number of electrons with spin polarization σ.

(a) Find the ground state energy to first order in the interaction potential as a function of $N = N_\uparrow + N_\downarrow$ and the magnetization $M = N_\uparrow - N_\downarrow$.

(b) Prove, to this order in the interaction, that the ferromagnetic state ($M = N$) has a lower energy than the unmagnetized state ($M = 0$) provided $r_{\rm s}$ exceeds a critical value $r_{\rm s,1}$. Find that critical value $r_{\rm s,1}$.

(c) Define $\varepsilon(\zeta) = E/N$ with $\zeta = M/N$. Show that $\varepsilon''(0) < 0$ when $r_{\rm s}$ exceeds a critical value $r_{\rm s,2}$. Find $r_{\rm s,2}$. You should find $r_{\rm s,1} < r_{\rm s,2}$. What happens for $r_{\rm s} \in [r_{\rm s,1}, r_{\rm s,2}]$?