PHYSICS 239.c : CONDENSED MATTER PHYSICS
HW ASSIGNMENT #1

(1) Consider the Euclidean Lagrangian density

$$\mathcal{L}_{\mathsf{E}}(\boldsymbol{\phi},\boldsymbol{\nabla}\boldsymbol{\phi},\partial_\tau\boldsymbol{\phi}) = i\,\phi_1\,\partial_\tau\,\phi_2 - i\,\phi_2\,\partial_\tau\,\phi_1 + \tfrac{1}{2}K(\boldsymbol{\nabla}\phi_1)^2 + \tfrac{1}{2}K(\boldsymbol{\nabla}\phi_2)^2 + V\big(\phi_1^2+\phi_2^2\big) \quad .$$

Express $\mathcal{L}_{\mathsf{E}}$ in terms of the complex scalar field $\psi = \phi_1 + i\phi_2$.

(2) Consider the $\mathsf{U}(1)$ Ginsburg-Landau theory with

$$F = \int\! d^d\!r\,\Big[\tfrac{1}{2}a\,|\Psi|^2 + \tfrac{1}{4}b\,|\Psi|^4 + \tfrac{1}{2}\kappa\,|\boldsymbol{\nabla}\Psi|^2\Big]\ .$$

Here $\Psi(\boldsymbol{r})$ is a complex-valued field, and both b and κ are positive. This theory is appropriate for describing the transition to superfluidity. The order parameter is $\langle\Psi(\boldsymbol{r})\rangle$. Note that the free energy is a functional of the two independent fields $\Psi(\boldsymbol{r})$ and $\Psi^*(\boldsymbol{r})$, where Ψ^* is the complex conjugate of Ψ. Alternatively, one can consider F a functional of the real and imaginary parts of Ψ.

(a) Show that one can rescale the field Ψ and the coordinates $\boldsymbol{r}$ so that the free energy can be written in the form

$$F = \varepsilon_0\!\int\! d^d\!x\,\Big[\pm\tfrac{1}{2}\,|\psi|^2 + \tfrac{1}{4}\,|\psi|^4 + \tfrac{1}{2}\,|\boldsymbol{\nabla}\psi|^2\Big]\ ,$$

where ψ and $\boldsymbol{x}$ are dimensionless, ε_0 has dimensions of energy, and where the sign on the first term on the RHS is $\mathsf{sgn}(a)$. Find ε_0 and the relations between Ψ and ψ and between $\boldsymbol{r}$ and $\boldsymbol{x}$.

(b) By extremizing the functional $F[\psi,\psi^*]$ with respect to ψ^*, find a partial differential equation describing the behavior of the order parameter field $\psi(\boldsymbol{x})$.

(c) Consider a two-dimensional system ($d = 2$) and let $a < 0$ (*i.e.* $T < T_{\mathrm{c}}$). Consider the case where $\psi(\boldsymbol{x})$ describe a *vortex* configuration: $\psi(\boldsymbol{x}) = f(r)\,e^{i\phi}$, where (r,ϕ) are two-dimensional polar coordinates. Find the ordinary differential equation for $f(r)$ which extremizes F.

(d) Show that the free energy, up to a constant, may be written as

$$F = 2\pi\varepsilon_0\!\int\limits_0^R\! dr\,r\,\left[\tfrac{1}{2}\big(f'\big)^2 + \frac{f^2}{2r^2} + \tfrac{1}{4}\big(1-f^2\big)^2\right]\ ,$$

where R is the radius of the system, which we presume is confined to a disk. Consider a *trial solution* for $f(r)$ of the form

$$f(r) = \frac{r}{\sqrt{r^2+a^2}}\ ,$$

where a is the variational parameter. Compute $F(a,R)$ in the limit $R\to\infty$ and extremize with respect to a to find the optimum value of a within this variational class of functions.

(3) Consider the Ginzburg-Landau free energy density

$$f = \tfrac{1}{2}\alpha t m^2 + \tfrac{1}{6} d m^6 + \tfrac{1}{2}\kappa(\boldsymbol{\nabla} m)^2 \quad ,$$

where $t = (T - T_c)/T_c$ is the reduced temperature.

(a) Find the equilibrium order parameter $m_0(t)$ as a function of reduced temperature.

(b) Find expressions for the correlation length in the cases $t < 0$ and $t > 0$.

(4) Verify the inequalities proven at the end of §1.5.3 of the Lecture Notes,

$$\begin{aligned} d = 1 : &\quad m < C_1 \left(\frac{h\mathcal{J}}{(k_B T)^2} \right)^{1/3} \\ d = 2 : &\quad m < C_2 \, \frac{(\mathcal{J}/k_B T)^{1/2}}{\ln^{1/2}(\mathcal{J}/hm)} \\ d > 2 : &\quad m < C_d \, \mathcal{J}/k_B T \quad , \end{aligned}$$

and provide estimates for the constants C_d.