Chapter 4: Fields of Moving Charges

Lienard-Wiechert potential & field

To determine the field due to a point charge in arbitrary motion, given by a specified trajectory \( y^0(x) \) (we use \( y^0 = x^0 \)), set up an auxiliary point (the argument of \( \Phi \)) and use the retarded Green's function \( \Phi(x,x_0) \). Recall

\[
\Phi(x,x_0) = \frac{G(x,x_0)}{\gamma_{\perp}}
\]

with \( \frac{G(x,x_0)}{\gamma_{\perp}} = \frac{\delta(\vec{r})}{\gamma} \)

so that \( G_{\perp}(x) = \frac{1}{4\pi c_0^2} \delta(x^0 - x_0) \)

And

\[
\hat{A}_\mu(x) = \hat{A}_{\mu}^i(x) + \int_{x^0 - \infty}^{x^0} d\tau^0 \, G_{\perp}(x,x_0) \frac{\partial u^\alpha(x)}{\partial \tau^0}(x_0)
\]

(\( \alpha \)Call \( \partial^2 \hat{A}_\mu = \frac{\mu}{c^2} \) in Lorentz gauge)

Notice that the \( \delta \)-function in \( \Phi \) means

that the field at \( (x',x') \) is determined by the charge at the retarded time \( x_r = x_0 + \frac{1}{c} r \), given by \( x_0^0 - x_r^0 = \tau_0 - \frac{1}{c} r \)

where \( r \) is the distance from the charge to \( x' \) at time \( t_0 \).

\( \hat{A}_\mu(x) \) has \( \delta \hat{A}_\mu = 0 \) and has a simple interpretation: if the charge is infinitely far away as \( t - \infty \), then the only contribution to \( \hat{A}_\mu(x) \) is from \( \hat{A}_\mu \) near the "initial" time \( t = \infty \) of \( \hat{A}_\mu(x) \), specified at \( t = \infty \). We set it to zero (can add it back at no cost).

The 4-current of the point charge \( q \) is

\[
\mathcal{J}^\mu(x) = \frac{q}{c} \delta^\mu_0 \delta(x-y)
\]

Trick: multiply by \( \int dx^0 \delta(x^0 - y^0) \frac{d\mathcal{J}}{dx^0} = 1 \) to obtain a covariant expression and, more usefully, a \( \delta \)

\[
\delta(x) = \int \frac{d^4k}{(2\pi)^4} \frac{\delta^\mu_0(\vec{k})}{\omega_k} \delta^0(x - y) \] where \( \omega_k = \frac{d\mathcal{J}}{dx^0} \)

So we have

\[
\hat{A}_\mu(x) = \int \frac{d^4k}{(2\pi)^4} \frac{\delta^\mu_0(\vec{k})}{\omega_k} \frac{d\mathcal{J}}{dx^0} G_{\perp}(x,x_0) \frac{\partial u^\alpha(x)}{\partial \tau^0}(x_0)
\]

Then we have two ways to do the integral. The elegant way involves expressing \( G_{\perp}(x,x_0) \) in a Lorentz invariant form: since \( x^0 = \infty \) is on the future light cone, consider

\[
\delta(x^0 - \infty) = \delta(x^0 - \infty) \Theta(x^0) = \frac{\delta(x^0 - \infty) \Theta(x^0)}{2\tau_0}
\]

So

\[
\hat{A}_\mu(x) = \frac{q}{c} \int \frac{d^4k}{(2\pi)^4} \frac{\delta^\mu_0(\vec{k})}{\omega_k} \frac{d\mathcal{J}}{dx^0} G_{\perp}(x,x_0) \frac{\partial u^\alpha(x)}{\partial \tau^0}(x_0)
\]

\[
\Rightarrow \hat{A}_\mu(x) = \frac{q}{c} \int \frac{d^4k}{(2\pi)^4} \frac{\delta^\mu_0(\vec{k})}{\omega_k} \frac{d\mathcal{J}}{dx^0} G_{\perp}(x,x_0) \frac{\partial u^\alpha(x)}{\partial \tau^0}(x_0)
\]

Lienard-Wiechert potential
Here, evaluated at $\lambda_0$ mean, as anticipated, at retarded time: $\lambda_0$ is the solution to

$$(x - y(\lambda_0)) = 0 \quad \text{with} \quad x > y(\lambda_0)$$

which of course is just

$$y(\lambda) = x^0 - |x - y(\lambda)|$$

The alternative way is to use $G_{\mu\nu}(x) = \frac{1}{\sqrt{-g}} \delta(x - \hat{x})$ directly:

$$A_\mu(x) = \frac{q}{\sqrt{-g}} \int d^4y \frac{\epsilon_\mu}{\sqrt{-g}} \delta(x - y)$$

where

$$\frac{d}{d\lambda}(y_\mu + x_\lambda) = \frac{\partial y_\mu}{\partial \lambda} + \frac{\partial x_\lambda}{\partial \lambda}$$

This equals the previous expression, since

$$\int d^4y \frac{1}{\sqrt{-g}} \int [\frac{\partial y_\mu}{\partial \lambda} - \frac{\partial x_\mu}{\partial \lambda}] = \frac{1}{\sqrt{-g}} \frac{d}{d\lambda}(y - x) = \frac{d}{d\lambda}(x - y)$$

Let's write the potentials in terms of the velocity $\vec{\beta}$ of the charge and the distance $R$ from retarded charge to $x$. Using $x = y^0 \frac{dy^0}{dy^0} = (1, \vec{\beta})$. As above,

$$A_\mu(x) = \frac{q}{\sqrt{-g_{\mu\lambda}}} \frac{1}{\sqrt{-g}} \frac{\partial y_\mu - \partial x_\mu}{\partial \lambda} = q \frac{(1, \vec{\beta})}{R(1 - \vec{\beta}^2)}$$

where $\vec{\beta}$ is at $\lambda_0$ (retarded).

We can also compute $\vec{E}$ and $\vec{B}$. The complication is that in this $\frac{d}{d\lambda}$ we are changing not just $x$, but also $\lambda_0$ (think of $\lambda_0 = \lambda_0(x)$ determined by $\lambda(x(\lambda)) = 0$).

$$(x - y(\lambda)) = 0 \quad \text{and} \quad (x + \delta x - y(\lambda + \delta \lambda)^2 = 0 \quad \text{or} \quad (x + \delta x - y(\lambda + \delta \lambda))^2 = 0$$

For $F_{\mu\nu}$ we need $\frac{d}{d\lambda}$, and for $\mu$'s:

$$\frac{\partial y_\mu}{\partial \lambda} = \frac{\partial x_\mu}{\partial \lambda}$$

For $E_\mu$ we need:

$$\frac{d}{d\lambda}(x - y) = \frac{d}{d\lambda}(x - y) = 0$$

and

$$\frac{d}{d\lambda} y_\mu = \frac{d}{d\lambda} y_\mu = 0$$

where $a_\mu = \frac{\partial y_\mu}{\partial \lambda} = \frac{\partial y_\mu}{\partial \lambda} = 0$. So finally $y_\mu = (x - y) \frac{dy_\mu}{dy_\mu}$
Then \[ F_{\mu \nu} = \partial_{\mu} \left( \frac{e_q \phi(x)}{\sqrt{1 - v^2 \cdot v}} \right) - \mu_{\mu}^{} \nu \]

\[ = \frac{q}{\sqrt{(x-y) \cdot v}} \left\{ (x-y)_\mu \nabla_{\nu} \left[ \partial_{\mu} \phi(x) - \frac{e_q \phi(x) \cdot v}{\sqrt{(x-y) \cdot v}} \right] \right\} - \mu_{\mu}^{} \nu \]

This gives \( \vec{E} \) and \( B \) in terms of retarded \( \phi(x) \), \( v(x) \), and \( \rho(x) \). Note that \( \vec{E} \) is reparameterization invariant. It is useful to write \( \phi \) more explicitly in terms of \( \vec{B}, \vec{R} \) and \( t \). So the \( \gamma = x^0. \)

Let us separate this into the \( x \)-dependent part \( F_{\mu \nu}^{\text{acc}} \) and \( x \)-invariant \( F_{\mu \nu}^{\text{rel}} \),

\[ F_{\mu \nu}^{\text{acc}} = \frac{q}{\sqrt{(x-y) \cdot v}} \left( \alpha_{\mu} - \nu_{\nu} \frac{(x-y)_\mu}{\sqrt{(x-y) \cdot v}} \right) - \mu_{\mu}^{} \nu \]

\[ F_{\mu \nu}^{\text{rel}} = \frac{q}{\sqrt{(x-y) \cdot v}} \left( \frac{(x-y)_\mu \nu_{\nu} - (x-y)_\nu \nu_{\mu}}{\sqrt{(x-y) \cdot v}} \right) \]

Then \( \vec{u}^1 = \vec{R} = (1, \vec{0}) \), \( (x-y) \cdot v = 15 \cdot \sqrt{v^2} \) \( = (2-v \cdot \vec{0}) \) \( \vec{p} = \vec{R} \) \( = (1, \vec{R}) \)

\[ F_{\mu \nu}^{\text{rel}} = \frac{q}{\delta^2} \frac{v^2 (x+y) \cdot v - v^2}{(1-v^2 \vec{R})^2} \Rightarrow \vec{E} \cdot \vec{B} = -\frac{q}{\delta^2} \frac{1}{(1-v^2 \vec{R})^2} \vec{R} \cdot \vec{B} = \frac{q}{\delta^2} \frac{\vec{R} \cdot \vec{B}}{(1-v^2 \vec{R})^2} \]

and \( \vec{B} \cdot \vec{B} = \vec{E} \cdot \vec{E} = \frac{q}{\delta^2} \frac{\vec{R} \cdot \vec{B}}{(1-v^2 \vec{R})^2} \frac{\vec{R} \cdot \vec{E}}{(1-v^2 \vec{R})^2} = \frac{q}{\delta^2} \frac{\vec{R} \cdot \vec{B} \cdot \vec{E}}{(1-v^2 \vec{R})^2} \]

which we recognize as the \( \vec{E} \times \vec{B} \) fields of a moving charge with \( \vec{p} \) constant.

But for \( \vec{B} \) we have additional terms. We need \( \vec{A}^1 = \vec{B} \) to be a solution of the equations of motion, so we need \( \vec{A}^1 = \frac{\partial}{\partial y} \phi(x) = \frac{\partial}{\partial y} \phi(0, 0) = 0, \vec{A} \)

\[ \vec{E}^{\text{acc}} = -\frac{q}{\delta^2} \frac{1}{(1-v^2 \vec{R})^2} \left[ \vec{R} \left( \vec{a} + \frac{\vec{B} \cdot \vec{a}}{1-v^2 \vec{R}} \right) - \vec{R} \left( \vec{a} + \frac{\vec{E} \cdot \vec{a}}{1-v^2 \vec{R}} \right) \right] \]

\[ = -\frac{q}{\delta^2} \frac{1}{(1-v^2 \vec{R})^2} \left[ \vec{R} \left( \vec{a} + \frac{\vec{B} \cdot \vec{a}}{1-v^2 \vec{R}} \right) - \vec{R} \left( \vec{a} + \frac{\vec{E} \cdot \vec{a}}{1-v^2 \vec{R}} \right) \right] \]

\[ = -\frac{q}{\delta^2} \frac{1}{(1-v^2 \vec{R})^2} \left[ \vec{R} \left( \vec{a} + \frac{\vec{B} \cdot \vec{a}}{1-v^2 \vec{R}} \right) - \vec{R} \left( \vec{a} + \frac{\vec{E} \cdot \vec{a}}{1-v^2 \vec{R}} \right) \right] \]

Note that \( \vec{B}^{\text{acc}} = \vec{R} \times \vec{E}^{\text{acc}} \) and \( \left| \vec{E}^{\text{acc}} \right| \approx \frac{1}{R} \). So \( \vec{E} \approx \frac{1}{R} \frac{e}{R} \) as a "radiation field"

\[ \vec{B}^{\text{acc}} = \frac{\vec{R}}{R} \times \frac{e}{(1-v^2 \vec{R})^2} \left[ \vec{R} \left( \vec{a} + \frac{\vec{B} \cdot \vec{a}}{1-v^2 \vec{R}} \right) - \vec{R} \left( \vec{a} + \frac{\vec{E} \cdot \vec{a}}{1-v^2 \vec{R}} \right) \right] \]

\[ = \frac{\vec{R}}{R} \left( \vec{a} + \frac{\vec{B} \cdot \vec{a}}{1-v^2 \vec{R}} \right) \]
At long distances only the radiation field is significant. It has \( \vec{E} \perp \vec{B} \) perpendicular to \( \vec{R} \), the retarded position vector, and are perpendicular to each other.

**Non-relativistic limit:** \( E_{\text{acc}} = \frac{\mu}{R} \frac{\vec{R} \times (\vec{R} \times \vec{a})}{c^2 R} = \frac{\mu}{c^2 R} \vec{a} \times (\vec{R} \times \vec{a}) \)

For linear motion, \( \vec{a} = \beta \dot{\vec{a}} = \dot{E}_{\text{acc}} = \frac{\mu}{c^2 R} \frac{\vec{a} \times \beta}{(1 - \beta^2)^{3/2}} \left[ \vec{R} \times (\vec{R} \times \vec{a}) \right] \)

**Power Radiated**

Power flux: energy flux

\[
\vec{S} = \frac{c}{u} \vec{E} \times \vec{B} = \frac{c}{u} |\vec{E}|^2 \hat{R} \quad (u = \frac{c}{\sqrt{1 + \beta^2}}, \, \text{with} \, \beta = \frac{u}{c})
\]

Consider a sphere centered at \( R, \) the retarded charge position \( \vec{x}(R) \).

The energy

\[
\text{d}^3 \rho = \text{power} \quad \text{per unit time} \quad \text{through area} = \left( \vec{S} \cdot \hat{R} \right) \left( \rho \text{d}^2 \Omega \right)
\]

\[
= \frac{c}{u} \frac{\text{d}^3 \rho}{\text{d} S^2} = \frac{c}{u} |\vec{E}|^2 \left( \frac{1}{(1 - \beta^2)^{3/2}} \left| \vec{R} \times (\vec{R} \times \vec{a}) \right| \right)^2
\]

Noting that \( R \to \infty \) the contribution of \( E_{\text{rad}} \) vanishes, here we have neglected it here.

This expression gives us energy per unit time in the inertial frame. We would like to lower radiated as seen in a frame simultaneously at rest with the moving particle, \( \text{d} \rho = \text{d} \rho \times \vec{x} \times \vec{v} \). Now, recall

\[
\frac{\partial x^o}{\partial x^i} = \frac{(x - x^o)}{(x - x^o)_0} \quad \text{so that} \quad \frac{\partial x^o}{\partial x^i} = \frac{1}{1 - \beta}
\]

and we have then

\[
\frac{\text{d} \rho}{\text{d} \Omega} = \frac{c}{u} \frac{1}{(1 - \beta^2)^{3/2}} \left| \vec{R} \times (\vec{R} \times \vec{a}) \right|^2
\]

(The reason for using time as seen by particle \( x^o \) is (i) unit inertial power radiated between particles at \( x^o \) and \( x^o \), and (ii) \( \text{d} \rho \) is then a Lorentz invariant: see below).
The relativistic expression presents some challenges with 
by integration:

\[
\frac{d\mathbf{p}}{d\tau} = \frac{e q^2}{4\pi} \frac{1}{(1-\beta^2)^{3/2}} \left[ \mathbf{a} \cdot (1-\hat{\mathbf{a}} \cdot \hat{\mathbf{p}}) + \frac{1}{2} \mathbf{a} \cdot (\hat{\mathbf{a}} \cdot (1-\hat{\mathbf{a}} \cdot \hat{\mathbf{p}})) \right]^{2}
\]

To write this in terms of angles we pick a frame (\(\hat{\mathbf{e}} = \hat{x} \hat{z}\) in xz plane)

\[
\mathbf{a} = \alpha (\sin \theta, 0, \cos \theta)
\]

\[
\mathbf{p} = \rho (0, 0, 1)
\]

\[
\hat{\mathbf{a}} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)
\]

\[
\mathbf{p} \cdot \hat{\mathbf{a}} = \rho \alpha \sin \theta \cos \phi \sin \theta \sin \phi \cos \theta
\]

\[(\text{and } \mathbf{a} \cdot \mathbf{p} \text{ is independent of } d\Omega)\]

Now \(\int d\Omega\) is trivial, with \(\chi = \alpha \theta\)

\[
P = \frac{e q^2}{2} \int d\lambda \frac{1}{(1-\beta^2)^{3/2}} \left[ \mathbf{a} \cdot (1-\hat{\mathbf{a}} \cdot \hat{\mathbf{p}}) + \frac{1}{2} \mathbf{a} \cdot (\hat{\mathbf{a}} \cdot (1-\hat{\mathbf{a}} \cdot \hat{\mathbf{p}})) \right]^{2}
\]

(We used \(\int d\theta \int d\phi (0, \sin \theta, \cos \theta) = 2 \pi (0, 0, 1).\) The remaining integral is straightforward but tedious (and we have Mathematica):

\[
P = \frac{e q^2}{2} \left[ \frac{1}{2} \mathbf{a} \cdot \left(1 - \frac{s^2}{(1-\beta^2)^2}\right) \right]
\]

where \(s^2 = \sin^2 \beta = 1 \Gamma \hat{\mathbf{p}}^2\)

\[
\text{or } \quad P = \frac{2}{3} \frac{e q^2}{2} \left[ \mathbf{a}^2 - (\mathbf{p} \cdot \mathbf{a})^2 \right]
\]

Lienard (1933)

In the non-relativistic limit, \(\beta \ll 1\), we obtain "Larmor's formula":

\[
P = \frac{2}{3} \frac{e q^2}{2} \mathbf{a}^2
\]

(NE limit, "Larmor's formula").
Comment: It is easy to obtain Larmor's formula from the non-relativistic limit of

\[ \frac{d\mathbf{p}'}{dt} = \frac{e}{m} \left[ \mathbf{\Omega} \times (\mathbf{R} - \mathbf{p}) \times \mathbf{\dot{a}} \right] \]

Then \( \mathbf{p}' = \mathbf{p} \) and integrating over \( d\Omega \) (recall \( \mathbf{a} = \frac{\mathbf{v}}{c} \))

\[ \mathbf{p} = \frac{2}{3} \frac{q^{2}a^{2}}{c^{2}} \]

Many textbooks use the Lighthill's formula as follows: argue that \( \mathbf{p} \) is frame invariant, then find a Lorentz scalar limit reduces to Larmor's formula in the NR limit.

The 1st part of the argument is Huygens' law \( \frac{d\mathbf{p}}{dt} = \frac{\mathbf{F}}{m} \) as measured by comoving oberrer.

2nd part: substitute \( \dot{\mathbf{a}} = \frac{1}{m} \frac{d\mathbf{F}}{dt} \) in Larmor's:

\[ \mathbf{a} \times = \frac{1}{m} \frac{d\mathbf{F}}{dt} \times \mathbf{a} = \frac{1}{m} \frac{d}{dt} \left[ \mathbf{a} \frac{d\mathbf{p}}{ds} \frac{d\mathbf{F}}{ds} \right] \]

where the last step is valid in the NR limit (we'll verify below).

\[ \frac{d^{2}\mathbf{F}}{ds^{2}} \frac{d\mathbf{p}}{ds} = -\frac{c^{2}}{m^{2}} \frac{d}{dt} \left[ \left( \frac{d\mathbf{p}}{dt} \right)^{2} - \left( \frac{d\mathbf{m}}{dt} \mathbf{\dot{\mathbf{p}}} \right)^{2} \right] \]

\[ = -c^{2}\gamma^{2} \left[ \left( \gamma^{2} \mathbf{\dot{p}} \right) \left( \gamma^{2} \mathbf{\dot{p}} \right) - \left( \gamma^{2} \mathbf{\dot{p}} \right) \left( \gamma^{2} \mathbf{\dot{p}} \right) \right] \]

\[ = -c^{2}\gamma^{2} \left[ \gamma^{4} \left( \mathbf{\dot{p}} \right)^{2} - 2 \gamma^{4} \mathbf{\dot{p}} \mathbf{\dot{p}} \right] \]

\[ = c^{2}\gamma^{2} \left[ \mathbf{\dot{p}}^{2} \left( \mathbf{\dot{p}} \right) + \left( \mathbf{\dot{p}} \right)^{2} \right] \]

\[ = c^{2}\gamma^{2} \left[ \mathbf{\dot{p}}^{2} - \left( \mathbf{\dot{p}} \mathbf{\dot{p}} \right) \right] = c^{2}\gamma^{2} \left[ \mathbf{\dot{p}}^{2} - \left( \mathbf{\dot{p}} \mathbf{\dot{p}} \right) \right] \]

\[ \mathbf{p} = \frac{2}{3} \frac{q^{2}a^{2}}{c^{2}} \] as before.
Angular Distribution

We already have the basic equation for this in
\[
\frac{d\Phi}{d\Omega} \quad \text{given in terms of } \theta \times \phi \text{ angles in } p \leq q \text{ above.}
\]

We look at special cases:

1. Linear motion: \( \vec{p} \parallel \vec{\alpha} \).

\( \frac{d\Phi}{d\Omega} \) is independent of \( \alpha \) (symmetric under rotations about an axis defined by \( \vec{p} \)).

Explicitly
\[
\frac{d\Phi}{d\Omega} = \frac{c q^2}{4 \pi} \left( \frac{1}{1-\beta^2} \right)^5 \left| \frac{\hat{\alpha} \times (\hat{\alpha} \times \vec{p})}{\vec{p}} \right|^2 \\
= \frac{c q^2}{4 \pi} \frac{\alpha^2 \sin^2 \theta}{(1-\beta \cos \theta)^5}
\]

The direction of maximum radiation can be found analytically, but can also be quickly approximated by noting that for \( \beta \gg 1 \), it is at small \( \theta \) and the denominator is small for
\[
1-\beta \cos \theta \approx 1-\beta + \frac{1}{2} \cos^2 \theta = \text{small}, \quad \text{for } \frac{1}{\beta} = \frac{1-\beta}{1+\beta} \approx \frac{2}{1+\beta}
\]

\( \Rightarrow \theta^2 \approx \frac{1}{\beta^2} \Rightarrow \frac{d\Phi}{d\Omega} \) is peaked at \( \theta = \frac{1}{\beta} < 1 \), but \( \Phi \) vanishes at \( \theta = 0 \)

\[
\frac{d\Phi}{d\Omega} \approx \frac{c q^2}{4 \pi} \left( \frac{1}{1-\beta} \right)^5 \frac{\beta}{(1+\beta)^5}
\]

or
\[
\frac{d\Phi}{d\Omega} \approx \frac{c q^2}{4 \pi} \frac{\beta^2}{(1+\beta)^5}
\]

In the opposite limit, \( \beta < 1 \)
\[
\frac{d\Phi}{d\Omega} \approx \frac{c q^2}{4 \pi} \sin^2 \theta \quad \text{(Larmor), so } \phi_{\text{max}} = \frac{\pi}{2}
\]

Radiation pattern

(\( \text{The distance from origin represents } \frac{d\Phi}{d\Omega} \))
Another case of interest is when \( \vec{\beta} \perp \vec{\alpha} \), as in circular motion. Then, from previous lecture

\[
\frac{d\mathcal{P}}{d\Omega} = \frac{ca^2}{\mathcal{A}} \frac{\alpha^4}{(1 - \vec{\alpha} \cdot \vec{\beta})^5} \left[ \left( -\vec{\alpha} \cdot \vec{\beta} \right)^2 - (\vec{\alpha} \cdot \vec{\beta}) (1 - \vec{\alpha} \cdot \vec{\beta}) + 2 \vec{\alpha} \cdot \vec{\beta} \vec{\hat{\alpha}} \cdot \vec{\hat{\beta}} (1 - \vec{\alpha} \cdot \vec{\beta}) \right]
\]

\[
= \frac{ca^2}{\mathcal{A}} \frac{\alpha^4}{(1 - \vec{\alpha} \cdot \vec{\beta})^5} \left[ (1 - \vec{\alpha} \cdot \vec{\beta})^3 - (\vec{\alpha} \cdot \vec{\beta})^3 (1 - \vec{\alpha} \cdot \vec{\beta}) \right]
\]

In the coordinate system we used earlier (\( \vec{\beta} = \hat{z}, \vec{\alpha} = x \) plane), we have now \( \hat{\alpha} = \hat{x} \), and \( \hat{\beta} = \cos \theta \hat{x} \). \( \hat{\alpha} = \sin \theta \cos \phi \)

\( \hat{\alpha} \), \( \hat{\beta} \), and \( \hat{\alpha} \cdot \hat{\beta} \) are given in the figure.

\[
\frac{d\mathcal{P}}{d\Omega} = \frac{ca^2}{\mathcal{A}} \frac{\alpha^4}{(1 - \vec{\alpha} \cdot \vec{\beta})^5} \left[ \frac{1}{(1 - \rho \cos \theta)^3} - \frac{(1 - \rho \cos \theta)^3}{(1 - \rho \cos \theta)^5} \right]
\]

Using \( 1 - \rho \cos \theta \approx \frac{1}{2} (1 - \rho \cos \theta)^2 \) for \( \rho \approx 1 \), we have

\[
\frac{d\mathcal{P}}{d\Omega} \approx \frac{2ca^2}{\mathcal{A}} \frac{\alpha^4}{\rho} \left[ 1 - \frac{4 \rho \cos \theta \rho \sin \phi}{\rho^2 + (\rho \cos \theta)^2} \right] \frac{\rho}{\rho^2 + (\rho \cos \theta)^2} \frac{d\Omega}{d\Omega}
\]

The radiation is emitted preferentially in the direction of \( \vec{\beta} \), to within a cone of angular size \( \theta \approx \frac{\alpha}{\rho} \). There is slightly more power radiated off the \( \vec{z} - \vec{\beta} \) plane (i.e. \( \phi = \frac{\pi}{2} \)) than on plane (\( \rho = 0 \)), but the difference is order \( \frac{1}{\rho^4} \).
At small $\beta$, retaining lowest order in $\beta$:

\[
\frac{d\mathbf{p}}{d\Omega} = \frac{e^2}{4\pi} \left[ 1 - 5\sin^2\theta \cos^2 \phi + 5\cos \theta \left( 3 - 5 \sin^2 \theta \cos^2 \phi \right) \right]
\]

\[
\frac{e^2}{4\pi} \begin{cases} 
0, & \phi = 0 \\
\frac{\rho}{3\rho \cos \phi}, & \phi = \frac{\pi}{2}
\end{cases}
\]

Patterns:

\[
\begin{array}{c}
\text{\textbf{v}} \\
\text{\textbf{\rho}}
\end{array}
\]

\[
\begin{array}{c}
\text{\textbf{\rho}} \\
\text{\textbf{\rho}}
\end{array}
\]

How about polarization? Recall we first derived (Lienard-Weichert):

\[
\mathbf{E}_{rad} = -\frac{q}{R} \left( \frac{1}{1-\beta^2} \right) \left[ \mathbf{k} - \frac{\mathbf{k} \cdot \mathbf{\hat{r}}}{R} \mathbf{\hat{r}} + \frac{\mathbf{\hat{r}}}{R} \right]
\]

For $\phi = 0$, radiation is mostly along $\mathbf{\hat{r}} = \mathbf{\hat{r}}$, so focusing on $\mathbf{\hat{r}}$ direction and noting $\mathbf{\hat{r}} \cdot \mathbf{\hat{r}} = 0$ for circular motion, $\mathbf{E}_{rad} \approx -\frac{q}{R} \frac{1}{1-\beta^2} \mathbf{\hat{r}}$

That is, fully in the plane of the circular motion to good approximation.
Arbitrary motion with \( r \gg 1 \) and into a spectral decomposition

\[
\text{Heuristic discussion of radiation from } \quad \text{Ref: Jackson 14.11}
\]

\[
\text{Arbitrary relativistic motion } (r \gg 1): \quad \text{Radiation in forward cone } \omega \sim \frac{1}{f}
\]

For a fixed observer

as the particle transits a small section of the curved path, there is radiation within the cone \( \omega \sim 1/f \) at observer.

\[
\text{so burst of emission over time } \Delta t = \frac{d}{v} = \frac{r}{c}\beta
\]

Front edge of radiation moves a distance \( l = c\Delta t = \frac{r}{\beta} \) by time the back edge of radiation "plane" is emitted from particle. Note

mored distance \( d \), lost emission

\[
\text{of plane } = l - d = \frac{r}{\beta} - \frac{r}{f} = \frac{r}{f}\left(1 - \beta^2\right) = \frac{r}{f^2}
\]

"Width"

\[
\text{Observed intensity } f^2 \sim \frac{1}{c^2 f^2}
\]

\[
\text{Fourier } \quad \Delta \omega = \frac{1}{\Delta t} = \frac{\Delta f}{c} \frac{f^3}{c^2}
\]

For circular motion \( \frac{\xi}{r} = \omega_f \) angular frequency we observe frequencies

\[
\omega \sim \Delta \omega \sim \frac{\Delta f}{c}\omega_f
\]
The amplification factor \( Y^3 \) is important.

For example, for \( \omega = \text{MHz} \), one can produce 10 keV X-rays \( (\omega \sim 10^{19} \text{s}^{-1}) \) with \( \gamma \sim (10^9/10^{11})^2 \sim 10^4 \). For electrons with \( m_c^2 = 1 \text{MeV} \), this requires \( E \sim 10 \text{GeV} \) — see energies of synchrotrons used for X-ray sources!

---

Spectral Analysis

Clearly of interest to have a more quantitative analysis.

We would like to have an expression for

\[
\frac{dI}{d\Omega dw} = \text{observed intensity of radiation} / \text{solid angle} \cdot \text{frequency}
\]

Now \( \frac{dI}{d\Omega} \) is just \( \frac{d\rho}{dt} \) integrated over time. We use \( \rho' \) which refers to per time of lab frame, i.e., observer's time as is appropriate for this question. The expression is in notes (p.4).

\[
\frac{dI}{d\Omega} = \int_0^\infty dt \frac{d\rho}{dt}
\]
Parserval's theorem
\[ f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{f}(k) \quad \Rightarrow \quad \int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} \frac{dk}{2\pi} |\tilde{f}(k)|^2 \]

Physical proof:
\[
\int_{-\infty}^{\infty} |f(x)|^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{f}(k) e^{-ikx} \tilde{x}(k) \]
\[
= \int \frac{dk}{2\pi} \int \frac{dk^\prime}{2\pi} \tilde{f}(k) \tilde{x}(k) \tilde{x}(k^\prime) \int_{-\infty}^{\infty} e^{i(k-k^\prime)x} \]
\[
= \frac{2\pi}{2\pi} \delta(k-k^\prime) \]
\[
= \int \frac{dk}{2\pi} |\tilde{f}(k)|^2 \]

Since we had
\[
\frac{dP'}{ds^2} = \frac{e^2}{4\pi} \left( \frac{r|E|^2}{v} \right) = \frac{e^2}{4\pi} \left[ \frac{q}{v} \left( \hat{r} \cdot \hat{p} \right) \times \hat{z} \right] \]

we write this as
\[
\frac{dP'}{dt} = \left| \vec{P}(t) \right|^2 \quad \vec{P}(t) = \frac{e^2}{4\pi} \left[ \frac{q}{v} \left( \hat{r} \cdot \hat{p} \right) \times \hat{z} \right] \]

Then
\[
\frac{d\vec{P}}{d\Omega} = \int \frac{d\omega}{2\pi} |\vec{P}(\omega)|^2 \quad \text{and} \quad \frac{d\vec{P}}{d\omega} = \frac{1}{2\pi} \left[ |\vec{P}(\omega)|^2 + |\vec{P}(-\omega)|^2 \right] \]

where the step 1 is valid, and \( \frac{d\vec{P}}{d\omega} = |\vec{P}(\omega)|^2 \).
Useful approximations:

\[ \int_{-\infty}^{\infty} dt e^{iwt} \left( \text{func} \right) \approx e^{i\omega t} \int_{-\infty}^{\infty} dt \left( \text{func} \right) \]

Consider a time \( t - t' = R/c \)

Change integration variable \( dt \to dt' \). Then \( \text{func} = \text{func}(t') \)

\[ \int_{-\infty}^{\infty} dt' \left( 1 - \hat{r} \cdot \hat{p} \right) e^{i\omega (t' + R/c)} \left( \text{func}(t') \right) \]

Now, if \( \hat{r} \) changes, little so \( \hat{r} \) changes.

(1) Choose origin somewhere in blob (i.e., particle trajectory).

(2) If observer is \( a + \hat{a} \)

\[ R(t') = |\vec{r} - \vec{\gamma}(t')| = \vec{r} - \hat{r} \cdot \vec{\gamma}(t') \]

So (drops primes and \( \epsilon^{\infty} = \text{constant} \)) initial trajectory of \( q \).

\[ \vec{p}(\omega) = \left( \frac{\epsilon}{m} \right) \int_{-\infty}^{\infty} dt e^{i\omega (t - \hat{r} \cdot \vec{\gamma}(t))} \frac{\hat{p} x \left[ (\hat{r} - \hat{\gamma}) \times \hat{\omega} \right]}{(1 - \hat{r} \cdot \hat{p})^2} \]

and

\[ \frac{d^2 T}{d\Omega dw} = \frac{e^4}{4\pi^2} \left( \int_{-\infty}^{\infty} dt e^{i\omega t - \hat{r} \cdot \vec{\gamma}(t)} \frac{\hat{p} x \left[ (\hat{r} - \hat{\gamma}) \times \hat{\omega} \right]}{(1 - \hat{r} \cdot \hat{p})^2} \right)^2 \]
This integral is hard. There is a trick that simplifies significantly. Note that

\[
\frac{\hat{r} \times \left( (\hat{r} \cdot \hat{p}) \hat{r} \times \hat{p} \right)}{(1 - \hat{r} \cdot \hat{p})^2} = \frac{i}{c} \frac{d}{dt} \left[ \frac{\hat{r} \times \left( (\hat{r} \cdot \hat{p}) \hat{r} \times \hat{p} \right)}{1 - \hat{r} \cdot \hat{p}} \right]
\]

\[
\text{proof: } \frac{i}{c} \frac{d}{dt} \left[ \frac{\hat{r} \times (\hat{r} \cdot \hat{p})}{1 - \hat{r} \cdot \hat{p}} \right] = \frac{\hat{r} \cdot \hat{p}}{(1 - \hat{r} \cdot \hat{p})^2} \hat{r} \times \left( \hat{r} \times \hat{p} \right) + \frac{\hat{r} \times (\hat{r} \cdot \hat{p})}{1 - \hat{r} \cdot \hat{p}}
\]

\[
= \frac{1}{(1 - \hat{r} \cdot \hat{p})^2} \left[ \hat{r} \times (\hat{r} \cdot \hat{p} - \hat{p}) - \hat{r} \times (\hat{r} \cdot \hat{p}) - \hat{r} \cdot \hat{p} \left( \hat{r} \times \hat{p} \right) \right]
\]

\[
= \frac{1}{(1 - \hat{r} \cdot \hat{p})^2} \left[ (\hat{r} \cdot \hat{p}) \hat{p} - (\hat{r} \cdot \hat{p}) \hat{p} + \hat{r} \times (\hat{r} \cdot \hat{p}) \hat{p} \right]
\]

\[
= \frac{1}{(1 - \hat{r} \cdot \hat{p})^2} \left[ -\hat{r} \times (\hat{r} \cdot \hat{p}) + \hat{r} \times (\hat{r} \cdot \hat{p}) \right]
\]

Now, we willy-nilly integrate by parts

\[\tilde{\Phi}(\omega) = \frac{\sqrt{\frac{\omega}{4\pi}}}{c} \int_0^\infty dt \ e^{i\omega (t - \hat{r} \cdot \hat{p} t / c)} \frac{1}{c} \frac{d}{dt} \left[ \frac{\hat{r} \times \left( (\hat{r} \cdot \hat{p}) \hat{r} \times \hat{p} \right)}{1 - \hat{r} \cdot \hat{p}} \right] \]

\[= -i \frac{\sqrt{\frac{\omega}{4\pi}}}{c} \int_0^\infty dt \ dt \ e^{i\omega (t - \hat{r} \cdot \hat{p} t / c)} \frac{\hat{r} \times \left( (\hat{r} \cdot \hat{p}) \hat{r} \times \hat{p} \right)}{1 - \hat{r} \cdot \hat{p}} \]

and noting that \( \frac{d}{dt} \left[ t - \hat{r} \cdot \hat{p} t / c \right] = t - \hat{r} \cdot \hat{p} \), we have

\[\tilde{\Phi}(\omega) = -i \omega \sqrt{\frac{\omega}{4\pi}} \frac{\hat{r} \times \left( (\hat{r} \cdot \hat{p}) \hat{r} \times \hat{p} \right)}{c} \]

This is much simpler. Now

\[\frac{d^2}{d\sigma d\omega} = \frac{2 \omega^2}{4\pi^2c} \left| \int_0^\infty dt \ e^{i\omega (t - \hat{r} \cdot \hat{p} t / c)} \hat{r} \times \left( \hat{r} \cdot \hat{p} \right) \right|^2 \]
Spectrum of circular motion
(Synchrotron radiation)

Note: I use θ for angle with x-axis (not ϑ).

\[ \vec{y}(t) = \rho \left( \sin \left( \frac{vt}{p} \right), -\cos \left( \frac{vt}{p} \right), 0 \right) \]

(putting \( \gamma(0) = \rho \left( 0, 1, 0 \right) \))

and no argument of exp in \( \beta \) i,

\[ \omega \left( t - \frac{\vec{y}(t)}{c} \right) = \omega \left( t - \frac{\rho \sin \left( \frac{vt}{p} \right)}{c} \cos \theta \right) \]

\[ = \frac{c}{\gamma} \left( \left( \frac{\gamma - 1}{\beta^2} \right) t + \frac{c^2}{\beta^2} \gamma^2 \right) \]

where we used our previous heuristic discussion that for \( \gamma \gg 1 \)

The radiation is in a cone \( \theta = \frac{v}{c} \) around \( \beta \), and since the observer is in the xz plane, the burst of radiation that reaches him/her is from when \( \beta = \frac{v}{c} \), i.e., from a small time interval around

\[ \vec{y}(t) \approx \rho \left( \frac{vt}{p}, 0, 0 \right) \Rightarrow \text{expand about } \sin \left( \frac{vt}{p} \right) = 0; \text{ and of course, } \]

expand about \( \theta = 0 \).

The higher order terms are suppressed by either \( \theta \gg 1 \) or

by \( \left( \frac{vt}{p} \right)^2 \sim \left( \frac{vt}{p} \right)^4 \) with \( \gamma t \sim \frac{p}{\beta} \) from our previous arguments.

So \( \left( \frac{vt}{p} \right)^2 \sim \frac{1}{\beta^2} \).

Note that this makes sense if we only integrate over one cycle of the circular motion. If we really integrate over all
times, we get infinitely many equal contributions $\to \infty$.

The reason is clear: it is periodic motion, with fixed

frequency = angular frequency = $\omega_0 = \frac{\nu}{\rho}$

The proper mathematical analysis is then to do a Fourier
series rather than an integral:

$\vec{\rho}(t) = \sum_{n=-\infty}^{\infty} e^{i n \omega_0 t} \vec{\rho}_n$

with

$\vec{\rho}_n = \frac{1}{2 \pi} \int_0^{2 \pi} dt \ e^{i n \omega_0 t} \vec{\rho}(t)$

Physically we expect $n \omega \approx \omega_0$, the number of modes

that contribute is huge, the approximation above of just retaining

one pulse replaces the large (but exact) number of discrete modes by

a continuum:

and

$\frac{1}{2 \pi} \sum_n |\vec{\rho}_n|^2 \to \frac{1}{2 \pi} \int d\omega |\vec{\rho}(\omega)|^2$.
The approximation to $\vec{\Phi}'$ is $\vec{\Phi}' \propto \vec{r} \times (\vec{r} \cdot \hat{p}) \hat{x}.$

is $\perp$ to $\vec{r}$: physically the EM wave at the observer has $\vec{E}$ (and $\vec{B}$) $\perp$ to line of sight. We can decompose $\vec{E}$ into 2 polarizations in the plane $\perp$ to $\vec{r}$:

- $\vec{E}_\|$: in the $xy$ plane
- $\vec{E}_\perp$: in direction, $\vec{E}_\perp = \hat{r} \times \vec{E}_\|$.

Since radiation is from $\hat{p} = \hat{x}$ ($t \vec{x} + \hat{r} \cdot \vec{r} = 0$)

Recall

Then $\vec{E}_\perp = \hat{r} \times \vec{E}_\| = (\sin \theta, \cos \theta, 0) \times (0, 1, 0) = (\sin \theta, 0, \cos \theta)$

Decompose

Then $\hat{r} \times (\hat{r} \times \hat{p}) = \hat{p}_\perp \vec{E}_\| + \hat{p}_\| \vec{E}_\perp$

$\hat{p}_\| = \vec{E}_\| \cdot (\hat{r} \times (\hat{r} \times \hat{p})) = (\vec{E}_\| \times \hat{r}) \cdot (\hat{r} \times \hat{p}) = -\vec{E}_\| \cdot (\hat{r} \times \hat{p})$

$= -(\vec{E}_\| \times \hat{r}) \cdot \hat{p} = -\vec{E}_\| \cdot \hat{p} = -\sin (\frac{v}{c} \tau) \hat{z}$

$\hat{p}_\perp = \vec{E}_\perp \cdot (\hat{r} \times (\hat{r} \times \hat{p})) = (\vec{E}_\perp \times \hat{r}) \cdot (\hat{r} \times \hat{p}) = \vec{E}_\perp \cdot (\hat{r} \times \hat{p})$

$= (\vec{E}_\perp \times \hat{r}) \cdot \hat{p} = -\vec{E}_\perp \cdot \hat{p} = \rho \cos (\frac{v}{c} \tau) \sin \theta = \Theta$

Then

$$\int_{0}^{\infty} \frac{d^3 \Omega}{d\Omega d\omega} = \frac{q^2 \omega^2}{4 \pi^2} \int_{0}^{\infty} dt = e^{-\omega t} \left\{ \left[ \frac{1}{2} \epsilon^{*} \rho \left( \frac{v}{c} t - \frac{1}{2} \frac{v^2}{c^2} t^2 \right) \right]^2 \left[ \frac{c^2 t - \epsilon \rho \epsilon_\| + \Theta \epsilon_\|}{\epsilon_\|} \right]^2 \right\}$$
The integral can be expressed in terms of the Airy function

\[ \text{Ai}(x) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{i(xt - \frac{1}{3} t^3)} = \frac{1}{\pi} \int_{0}^{\infty} dt \cos(xt + \frac{1}{3} t^3) \]

and its derivative

\[ \text{Ai}'(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt t e^{i(xt - \frac{1}{3} t^3)} \]

Our integral is of the form \[ \int_{-\infty}^{\infty} dt e^{i(at - \frac{1}{3} b t^3)} \]

So rescale \( t = \frac{1}{b^{1/3}} T \), \( T = \frac{1}{b^{1/3}} \int_{-\infty}^{\infty} dt e^{i(aT - \frac{1}{3} b T^3)} = \frac{2\pi}{b^{1/3}} \text{Ai}(\frac{a}{b^{1/3}}) \]

And \( \int_{-\infty}^{\infty} dt t e^{i(at - \frac{1}{3} b t^3)} = \frac{2\pi}{b^{1/3}} \text{Ai}'(\frac{a}{b^{1/3}}) \)

In our case \( a = \frac{\omega^2}{\rho^2} \left( \frac{1}{d^2} + \theta^2 \right) \) and \( b = \frac{\omega^2}{\rho^2} \frac{c^2}{d^2} \). Let \( z = \frac{\omega^2}{\rho^2} \frac{c^2}{d^2} \left( \frac{1}{d^2} + \theta^2 \right) \)

or \( z = \left( \frac{\omega^2}{\rho^2} \right)^{1/2} \left[ \frac{1}{d^2} + \theta^2 \right] \)

With this

\[
\frac{d^2 I}{d\Omega d\omega} = \frac{g^2 \omega^2}{4 \rho^2 c^2} \left| -\frac{\omega^2}{\rho^2} \frac{c^2}{d^2} \frac{1}{2} \text{Ai}'(z) \overline{E}_x + \theta \frac{2\pi}{(\omega^2/c^2)^{1/3}} \text{Ai}(z) \overline{E}_y \right|^2
\]

\[
= \frac{g^2}{c^2} \left[ (\omega^2/c^2)^{7/2} [\text{Ai}'(z)]^2 + \theta^2 (\omega^2/c^2)^{4/3} [\text{Ai}(z)]^2 \right]
\]

Note that the angular frequency of the circular motion is \( \omega_0 = \frac{c}{\rho} \)

so the result is given in terms of the ratio \( \frac{\omega}{\omega_0} = \frac{\omega}{\omega_0} \).
Now

\[ A_1(t) = \frac{1}{\pi} \text{Re} \int_0^\infty e^{i\frac{1}{3} t^2} dt \quad \text{Change variable } u = \frac{1}{3} t^3 \]

\((du = \frac{1}{3} dt \Rightarrow dt = \frac{du}{(3u)^{1/3}})\)

Then

\[ \int_0^\infty e^{i\frac{1}{3} t^2} dt = \int_0^\infty e^{iuv} \frac{du}{3u^{1/3}} \]

Then change \(u = \sqrt[3]{v}\) (formally consider \(\lim\) \(dt e^{it} = 0\))

\[ \frac{1}{3} \int_0^\infty e^{-\frac{1}{3} v} \frac{i}{3v^{1/3}} dv = \frac{i}{3^{1/3}} \int_0^\infty e^{-\frac{1}{3} v^{3/2}} dv = \frac{\pi}{3^{1/3}} \Gamma \left(\frac{1}{3}\right) \]

\[ s_n A_1(t) = \frac{1}{\pi} \frac{\Gamma(n/3)}{3^{n/3}} \Gamma \left(\frac{1}{3}\right), \quad \text{or using } \Gamma\left(\frac{1}{3}\right) \Gamma\left(\frac{1}{3}\right) = \frac{n}{3^{1/3}} \Gamma(n/3) \]

\[ A_1(t) = \frac{1}{3^{1/3}} \approx 0.355 \]

Similarly we find \( A_1(t) = -\frac{1}{3^{1/3}} \Gamma(1/3) \approx -0.259 \)

The large \(x\) behavior can be obtained by stationary phase:

\[ \frac{d}{dt} (x + \frac{1}{3} t^2) = 0 \Rightarrow t^2 = -x \quad \frac{d^2}{dt^2} (x + \frac{1}{3} t^2) = 2t \]

\[ S_0 \quad x + \frac{1}{3} t^2 = x \sqrt[3]{-x} + \frac{1}{3} \left( -x \right)^{2/3} + \frac{1}{2} \sqrt[3]{x} (t - \sqrt[3]{x})^2 + \ldots \]

So we have

\[ \int_0^\infty dt e^{i(x + \frac{1}{3} t^2)} = \int_0^\infty dt e^{i \left( \frac{3}{2} x \sqrt[3]{-x} + \sqrt[3]{x} (t - \sqrt[3]{x}) \right)} \]
Use $F(x) = i x$ (the other solution blows up (steepest ascent)),

\[
S = e^{-\frac{2}{3} x^{3/2}} \int_{-\infty}^{\infty} dt \ e^{-\sqrt{x}(t-i\xi)}
\]

\[
e^{-\frac{2}{3} x^{3/2}} \int_{-\infty}^{\infty} dv \ e^{\sqrt{x} v^2} = \frac{e^{-\frac{2}{3} x^{3/2}}}{\sqrt{x^{1/4}}} 2 \int_{0}^{\infty} du \ e^{-u^2}
\]

\[
S = \frac{1}{\sqrt{x^{1/4}}} \int_{0}^{\infty} d\xi \ \frac{df}{d\xi} \ e^{\xi} = \frac{e^{-\frac{2}{3} x^{3/2}}}{\sqrt{x^{1/4}}} \Gamma(1/2) = \sqrt{\pi} \ \frac{e^{-\frac{2}{3} x^{3/2}}}{\sqrt{x^{1/4}}}
\]

And $Ai(x) \sim \frac{e^{-\frac{2}{3} x^{3/2}}}{\sqrt{2\pi x^{1/4}}} \quad \text{as} \quad x \to +\infty$

For $Ai'(x)$, $Ai'(x) \sim -\frac{x^{1/4}}{2^{1/2}} \ e^{\frac{2}{3} x^{3/2}}$.

For $x \to -\infty$ and relation to $K_v$ see Garg.
We can now analyze the behavior of \( \frac{d^2 I}{d \Omega \, d\omega} \).

Recall

\[
\frac{d^2 I}{d \Omega \, d\omega} = q \, \frac{1}{\varepsilon} \left[ \left( \frac{\omega}{\omega_0} \right)^{7/8} \left\{ A_i(\theta) \right\}^2 + \Theta^2 \left( \frac{\omega_0}{\omega} \right)^{7/3} \left\{ A_i(\theta) \right\}^2 \right]
\]

where \( \theta = \left( \frac{\omega}{\omega_0} \right)^{1/8} \left( \frac{1}{\theta'} + \theta' \right) \).

For fixed \( \theta' \), we have at small \( \omega \)

\[
\frac{d^2 I}{d \Omega \, d\omega} \approx q \, \frac{1}{\varepsilon} \left[ \left( \frac{\omega}{\omega_0} \right)^{7/8} \left\{ A_i(\theta) \right\}^2 + \Theta^2 \left( \frac{\omega_0}{\omega} \right)^{7/3} \left\{ A_i(\theta) \right\}^2 \right] = q \, \frac{1}{\varepsilon} \left[ A_i(\theta)^2 \left( \frac{\omega}{\omega_0} \right)^{7/3} \right]
\]

and for large \( \omega \)

\[
\frac{d^2 I}{d \Omega \, d\omega} \approx q \, \frac{1}{\varepsilon} \left[ \left( \frac{\omega}{\omega_0} \right)^{7/8} \left\{ A_i(\theta) \right\}^2 + \Theta^2 \left( \frac{\omega_0}{\omega} \right)^{7/3} \left\{ A_i(\theta) \right\}^2 \right]
\]

\[
\approx q \, \frac{1}{\varepsilon} \left[ \left( \frac{\omega}{\omega_0} \right)^{7/8} \left\{ A_i(\theta) \right\}^2 + \Theta^2 \left( \frac{\omega_0}{\omega} \right)^{7/3} \left\{ A_i(\theta) \right\}^2 \right]
\]

\[
= q \, \frac{1}{\varepsilon} \left[ \left( \frac{\omega}{\omega_0} \right)^{7/8} \left\{ A_i(\theta) \right\}^2 + \Theta^2 \left( \frac{\omega_0}{\omega} \right)^{7/3} \left\{ A_i(\theta) \right\}^2 \right]
\]

The exponential gives a rapid roll-off \( e^{-2(1+\theta')^{3/4}/\sqrt{\omega}} \).

where the critical frequency \( \omega_c = 3 \beta^2 \omega_0 \) characterizes the frequency beyond which \( (\omega > \omega_c) \) radiation is negligible even for \( \theta = 0 \).

Note also that for small \( \omega \) the polarization is largely \( \parallel \). Likewise, at \( \theta = 0 \) only \( \parallel \) contributes.
Jackson has a nice plot:

\[ \frac{dI}{d\omega} \]

where \( \omega_c = 3 \gamma^3 \omega_0 \) is defined by \( \gamma = 1 \) at \( \theta = 0 \), the critical frequency beyond which there is negligible radiation for any angle.