Chapter 4: Fields of Moving Charges


c\text{Lienard-Wiechert potential and field}

to determine the field due to a point charge in arbitrary motion given by a specified trajectory $\gamma(t)$ (we use $x'=\gamma(t)$ so as to get coupled with the argument of $A_\mu(x)$), we use the retarded Green's function $G_{\text{ret}}(x,x')$. Recall

$$G_{\text{ret}}(x,x') = G(x,x')$$

with $\delta_+ G_{\text{ret}}(x) = \delta(x)$

so that $G_{\text{ret}}(x) = \frac{1}{4\pi\epsilon_0} \delta(x-x')$.

And

$$A_\mu^R(x) = \int G_{\text{ret}}(x,x') \frac{\partial}{\partial x^\nu} \epsilon_{\mu\nu} \, dx'$$

(we call $\partial^2 A_\mu = \nabla^2 A_\mu$ in Lorentz gauge)

Notice that the $\delta$-function in $G_{\text{ret}}$ means that the field at $(x,x')$ is determined by the charge at the retarded time $x^\mu = x^\mu - x'^\mu$, given by

$$x^0 = x'^0 - \frac{1}{c} |x-x'|$$

where $|x|$ is the distance from the charge at $x$ at time $t^\nu$.

$$A^R_\mu(x)$$

no $\delta A_\mu = 0$ and has a simple interpretation: if the charge is infinitely far away as $t^\nu \to -\infty$, then the only contribution to $A^R_\mu(x)$ is from $\delta^4(x-y)_0$, the "initial" value of $A^R_\mu(x)$, specified at $t^\nu = -\infty$. We set it to zero (can add it back at no cost).

The $\delta$-moment of the point charge is $\delta^4(x-y)_0 = \epsilon_0 \frac{\partial}{\partial x^\nu} \delta^4(x-y)_0$

Trick: multiply by $\int \delta(x^0 - y^0) \frac{\partial}{\partial x^\nu} = 1$ to obtain a covariant expression and, more usefully, a $\delta^4(x-y)_0 = \int \delta^4(x-y)_0 \frac{\partial}{\partial x^\nu}$ where $\nu = \frac{\partial}{\partial x^\nu}$

So we have

$$A^R_\mu(x) = \int G_{\text{ret}}(x,x') \frac{\partial}{\partial x^\nu} \epsilon_{\mu\nu} \, dx'$$

Then one has ways to do this integral. The elegant way involves expressing $G_{\text{ret}}(x)$ in a Lorentz invariant way since $x^0 = |x|$ is on the proper light cone, consider

$$\delta^4(x-y)_0 = \delta(x^0 - y^0) \Theta(x^0) = \frac{\delta(x^0 - 0)}{x^0} \Theta(x^0)$$

So $G_{\text{ret}} = \frac{1}{x^0} \delta(x^0 - x^0) \delta(x-y)$

$$= \frac{1}{x^0} \delta^4(x-y) \delta(x-y)^0 \theta(x-y)^0$$

$$= \frac{1}{x^0} \int \frac{1}{x^0} \delta(x-y) \theta(x-y)^0 \delta(x-y)^0$$

$$= \frac{1}{x^0} \int \frac{1}{x^0} \delta(x-y) \theta(x-y)^0 \delta(x-y)^0$$

$$= \frac{1}{x^0} \int \frac{1}{x^0} \delta(x-y) \theta(x-y)^0 \delta(x-y)^0$$
Here, evaluated at $\lambda_0$ means, as anticipated, for retarded time: $\lambda_0$ is the solution to

$$(x - y, \lambda_0)^2 = 0 \quad \text{with} \quad x^+ > y^+(\lambda_0)$$

which, of course, is just $y^+(\lambda_0) = x^+ - |x - y(\lambda_0)|$

The alternative way is to use $G_{\mu\nu}(x) = \frac{1}{20} \delta'(x-R)$ directly:

$$A_\mu(x) = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \int d^4y' \delta'(x-y')$$

where

$$\frac{d}{d\lambda} (y^+ + \beta x) = \frac{dy^+}{d\lambda} + \frac{(x-y) \cdot \beta}{\sqrt{-g}}$$

This equals the previous expression, since

$$\int d^4y' \frac{1}{\sqrt{-g}} \frac{\partial}{\partial \lambda} \left[ (x-y) \cdot \beta d\lambda - (x-y) \cdot \beta d\lambda \right] = \frac{1}{1-\beta^2} (x-y) \nu$$

Let's write the potentials in terms of the velocity $\beta$ of the charge and the distance $R$

how retarded charge to $x$. Using $x - y^0 \frac{dy^0}{d\lambda} = (1, \beta)$. As above,

$$A_\mu(x) = \frac{q}{\sqrt{-g(\lambda_0)}} \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left[ \frac{1}{\sqrt{-g}} \frac{1}{\sqrt{-g}} \frac{1}{\sqrt{-g}} \right] = \frac{q}{R} \frac{(1, \beta)}{1-\beta^2}$$

where $\beta$ is at $x$ (retarded).

We can also compute $E_\mu$ and $B_\mu$. The approximation is that in taking $\frac{\partial}{\partial x^\mu}$ we are changing not just $x$ but also $\lambda$ (think of $\lambda_0 = \lambda_0(x)$ determined by $(x-y(\lambda))^2 = 0$).

$$(x - y(\lambda))^2 = 0 \quad \text{and} \quad (x + \delta x - y(\lambda + \delta \lambda))^2 = 0 \Rightarrow (x + \delta x - y(\lambda + \delta \lambda))^2 = 0$$

For $E_{\mu
u}$ we need $\frac{\partial}{\partial x^\mu} \gamma_{\mu
u}$ and for $B_{\mu
u}$

$$\frac{\partial}{\partial x^\mu} \gamma_{\mu
u} = \frac{\partial}{\partial x^\mu} \frac{1}{\sqrt{-g}} \frac{1}{\sqrt{-g}} \frac{1}{\sqrt{-g}} = \frac{(x-y)^\mu}{(x-y) \nu} \frac{(x-y)^\nu}{(x-y) \nu}$$

where $\gamma_{\mu
u} = \frac{\partial}{\partial x^\mu} \frac{1}{\sqrt{-g}} = \frac{\delta_{\mu\nu}}{(x-y) \nu}$ since $\frac{\partial}{\partial x^\mu} \frac{1}{\sqrt{-g}} = \frac{(x-y)^\mu}{(x-y) \nu}$
Then \( \mathbf{F}_{\nu} = \partial_\mu \left( \frac{q \mathbf{u}_\nu}{(u \cdot x - u \cdot x)} \right) - \mu \mathbf{u}_\nu \)

\[
- \frac{q}{[u \cdot x - u \cdot x]^2} \left\{ u \cdot x \mathbf{u}_\nu \left[ \left( \mathbf{u} \cdot \mathbf{u} - (u \cdot x)^2 \right) \mathbf{u}_\mu + \left( u \cdot x \mathbf{u}_\mu \right) u \cdot x \right] \right\} - \mu \mathbf{u}_\nu
\]

This gives \( \mathbf{E} \) and \( \mathbf{B} \) in terms of retarded \( \mathbf{e}(\mathbf{x}, t) \), \( \mathbf{u}(\mathbf{x}, t) \), and \( \mathbf{a}(\mathbf{x}, t) \). Note that it is reparameterization invariant. It is useful to write this more explicitly in terms of \( \mathbf{\beta}, \mathbf{R} \) and \( \mathbf{e}_\nu \). So the \( \gamma = \gamma^0 \).

Let's separate this into the \( x \)-dependent part of \( \mathbf{F}_{\nu} \), \( \mathbf{F}_{\nu}^{\text{acc}} \), and \( \mathbf{F}_{\nu} \), \( \mathbf{F}_{\nu}^{\text{rel}} \),

\[
\mathbf{F}_{\nu}^{\text{acc}} = \frac{q}{[u \cdot x - u \cdot x]^2} \left( \mathbf{u} \cdot \mathbf{u} - (u \cdot x)^2 \right) \mathbf{u}_\nu
\]

\[
\mathbf{F}_{\nu}^{\text{rel}} = \frac{q \mathbf{u}_\nu}{[u \cdot x - u \cdot x]^3} \left[ (u \cdot x)^2 \mathbf{u}_\nu - (u \cdot x)^3 \mathbf{u}_\nu \right]
\]

Then, \( (x, y) \cdot u = (1, \mathbf{\beta}) \), \( (x, y) \cdot \mathbf{u} = (1 - \mathbf{\beta}) \), \( \mathbf{R} = \mathbf{R}(1 - \mathbf{\beta}) \)

\[
\mathbf{F}_{\nu}^{\text{rel}} = \frac{q}{\delta^2 \mathbf{R}^2 (1 - \mathbf{\beta})} \Rightarrow \mathbf{E}_{\nu}^{\text{rel}} = -\mathbf{F}_{\nu}^{\text{rel}} = -\frac{q}{\delta^2 (1 - \mathbf{\beta})^2} \mathbf{R} \mathbf{R}^\dagger - \mathbf{R} = \frac{q}{\delta^2 (1 - \mathbf{\beta})^2} \mathbf{R} \mathbf{R}^\dagger
\]

and \( \mathbf{B}_{\nu}^{\text{rel}} = \mathbf{e}^1 (1, \mathbf{\beta}) \mathbf{e}^2 (1, \mathbf{\beta}) = \frac{q}{\delta^2 (1 - \mathbf{\beta})^2} \mathbf{R} \mathbf{R}^\dagger = \frac{q}{\delta^2 (1 - \mathbf{\beta})^2} \mathbf{R} \mathbf{R}^\dagger \)

which we recognize as the \( \mathbf{E} \) and \( \mathbf{B} \) fields of a moving charge with \( \beta = \) constant.

But for \( \beta \neq 0 \) we have additional terms. We need

\[
\mathbf{F}_{\nu}^{\text{acc}} = -\frac{q}{\mathbf{R} (1 - \mathbf{\beta})} \left[ \mathbf{R} \left( \mathbf{a}_\nu + \mathbf{\beta} \hat{\mathbf{a}}_{\nu} - \mathbf{\beta} \mathbf{R} \hat{\mathbf{a}}_{\nu} \right) \right] = -\frac{q}{\mathbf{R} (1 - \mathbf{\beta})^2} \left[ \mathbf{R} \mathbf{R}^\dagger \mathbf{R} \mathbf{R}^\dagger \right]
\]

\[
\mathbf{F}_{\nu}^{\text{acc}} = \frac{q}{\mathbf{R} (1 - \mathbf{\beta})^2} \mathbf{R} \left[ \mathbf{R} \mathbf{R}^\dagger \mathbf{R} \mathbf{R}^\dagger \right]
\]

Note that \( \mathbf{B}_{\nu}^{\text{acc}} = \mathbf{R} \mathbf{E}_{\nu}^{\text{acc}} \) and \( \mathbf{E}_{\nu}^{\text{acc}} \approx \mathbf{E}_{\nu} \). So \( \mathbf{E} \approx \frac{1}{\mathbf{R} (1 - \mathbf{\beta})^2} \mathbf{R} \mathbf{E} \mathbf{R} \mathbf{R} \mathbf{R}^\dagger \) "radiation field"

\[
\mathbf{E}_{\nu} \approx \mathbf{R} \left[ \left( \mathbf{R} \mathbf{R}^\dagger \right) \mathbf{E}_{\nu} \right] = \left( \mathbf{R} \mathbf{R}^\dagger \right) \mathbf{R} \mathbf{R}^\dagger \mathbf{R} \mathbf{E}_{\nu} = \left( \mathbf{R} - \mathbf{\beta} \mathbf{R} \mathbf{R}^\dagger \mathbf{R} \mathbf{E}_{\nu} \right) = \left( \mathbf{R} - \mathbf{\beta} \mathbf{R} \mathbf{R}^\dagger \mathbf{R} \mathbf{E}_{\nu} \right) = \left( \mathbf{R} - \mathbf{\beta} \mathbf{R} \mathbf{R}^\dagger \mathbf{R} \mathbf{E}_{\nu} \right)
\]

One may write

\[
\mathbf{E}_{\nu}^{\text{acc}} = \frac{q}{\mathbf{R} (1 - \mathbf{\beta})^2} \mathbf{R} \left[ \left( \mathbf{R} \mathbf{R}^\dagger \right) \mathbf{E}_{\nu} \right]
\]
At long distances only the radiation field is significant. It has $\vec{E}$ and $\vec{B}$ perpendicular to $\vec{r}$ — the retarded position vector — and are perpendicular to each other.

**Non-relativistic limit:**

$$E^{\text{nr}} = \frac{q}{\epsilon_0} \frac{\vec{r} \times (\vec{r} \times \vec{a})}{r^3} = \frac{q}{\epsilon_0} \frac{\vec{r} x (\vec{r} \times \vec{a})}{r^3}$$

For linear motion, $\vec{\beta} = \beta \hat{\alpha} = \frac{\vec{E}^{\text{nr}}}{c^2 \sqrt{1 - \beta^2}} \frac{1}{(1 - \beta^2)^3} [\vec{r} x (\vec{r} \times \vec{a})]$

**Power Radiated**

Power per unit area:

$$P = \frac{\vec{E} \cdot \vec{B}}{c^2} dS = \frac{\vec{E} \cdot \vec{B}}{c^2}$$

Consider a sphere centered at the retarded position $\vec{x}(R)$.

The energy

$$d\mathcal{P} = \text{power} = \frac{\text{energy through area}}{\text{time}} = \frac{(\vec{E} \cdot \vec{B})}{c^2} d^2 R$$

$$d\mathcal{P}' = \frac{d\mathcal{P}}{d\Omega} = \frac{\vec{E} \cdot \vec{B}}{c^2} R [R \cdot \vec{E}]^2 - \frac{\vec{E} \cdot \vec{B}}{c^2} \frac{R \times (\vec{r} - \vec{p}) \times \vec{a}}{(1 - \vec{r} \cdot \vec{p})^3}$$

Noting as $R \to \infty$, the contribution of $E^{\text{nr}}$ vanishes, how we have neglected it here.

This expression is for energy per unit time in the inertial frame. We would like to lower radiated as seen in a frame instantaneously at rest with the moving particle, $d\mathcal{P} = d\mathcal{P}' \frac{\vec{a}}{\gamma^2}$. Now, recall

$$\frac{\partial \gamma}{\partial x^i} = \frac{(x-x)^i}{x^2} \frac{\gamma}{(x-x)^2}$$

So that $\frac{\partial \gamma}{\partial x^i} = \frac{1}{1 - \vec{r} \cdot \vec{p}}$

and we have then

$$d\mathcal{P} = \frac{\vec{E} \cdot \vec{B}}{c^2} d\Omega = \frac{\vec{E} \cdot \vec{B}}{c^2} \left[ \frac{1}{1 - \vec{r} \cdot \vec{p}} \frac{R \times (\vec{r} - \vec{p}) \times \vec{a}}{(1 - \vec{r} \cdot \vec{p})^3} \right]^2$$

(The reason for using time as seen by particle is (1) unit inertial power radiated between particles at $\vec{y}_0$ and $\vec{y}_i$, and (2) $d\mathcal{P}$ is then a Lorentz invariant; see below).
The relativistic expression present some challenges with its integration:

\[
\frac{d\mathbf{P}}{dt} = \frac{e\gamma^2}{4\pi} \left( \frac{1}{(1 - \beta^2)^{3/2}} \right) \left[ \mathbf{a} \cdot (\mathbf{\hat{r}} - \mathbf{\hat{r}} \cdot \mathbf{\hat{b}}) + (\mathbf{\hat{r}} - \mathbf{\hat{r}} \cdot \mathbf{\hat{b}})^2 \right]
\]

To write this in terms of angles, we pick a frame (\( \mathbf{\hat{r}} \) in xt plane)

\[
\mathbf{\hat{a}} = \alpha (\sin \theta, \cos \phi)
\]

\[
\mathbf{\hat{b}} = \beta (0, 0, 1)
\]

\[
\mathbf{\hat{r}} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)
\]

\[
\Rightarrow \mathbf{\hat{a}} \cdot \mathbf{\hat{b}} = \beta \cos \theta \sin \phi \]

\[
(\text{and } \mathbf{\hat{a}} \cdot \mathbf{\hat{b}} \text{ is independent of } \Delta \Omega).
\]

Now \( \int \mathbf{\hat{a}} \cdot \mathbf{\hat{b}} \) is trivial, with \( \Delta \Omega = \alpha \theta \)

\[
P = \frac{e\gamma^2}{2} \int_0^1 \sin \alpha d\alpha \int_1^{\pi/2} d\theta \sin \theta \left[ \alpha^2 (1 - \beta^2)^{-3/2} \left( (1 - \beta^2) \alpha^2 \cos^2 \beta - \frac{1}{2} \sin^2 \beta (1 - \beta^2) \right) \right]
\]

\[
\text{(we used } \int_0^{\pi/2} \sin \alpha d\alpha = 2 \sin (1, 0, 1/2), \text{ the remaining integral is straightforward but tedious and we have Mathematica)}.
\]

\[
P = \frac{e\gamma^2}{2} \left[ \frac{1}{3} \alpha^2 \frac{1 - \beta^2}{(1 - \beta^2)^{3/2}} \right]
\]

\[
\text{where } s^2 = 5\alpha^2 = |\mathbf{a}|^2
\]

\[
\Box \quad \boxed{\mathbf{P} = \frac{e\gamma^2}{3} \left[ \frac{1}{3} \alpha^2 \frac{1 - \beta^2}{(1 - \beta^2)^{3/2}} \right]} \quad \text{Lienard (1983)}
\]

In the non-relativistic limit, \( \beta < 1 \), we obtain "Larmor's formula":

\[
\mathbf{P} = \frac{\alpha^2}{3} \frac{e\gamma^2}{c^2}
\]

(Non-limit, "Larmor's formula").
Comment: It is easy to obtain Larmor’s formula from the non-relativistic limit of
\[
\frac{\alpha^{\pm}}{\alpha} = \frac{e^{\pm}}{m} \left( \frac{1}{1 - \beta^2} \right)^{\frac{1}{2}} \left[ R \times (\vec{p} - \vec{\beta} \times \vec{a}) \right] \left( \frac{1}{1 - \beta^2} \right)^{\frac{1}{2}} = \frac{e^{\pm}}{m} \left( \frac{1}{1 - \beta^2} \right)^{\frac{1}{2}} \left[ R \times \vec{a} \right] = \frac{e^{\pm}}{m} \sin \vartheta
\]

Then \( \mathbf{p}' = \mathbf{p} \) and integrating over \( d\Omega \) (recall \( \vec{a} = \frac{e}{c^2} \overrightarrow{\mathbf{E}} \))
\[
\mathbf{p} = \frac{2}{3} \frac{g^2 q^2}{c^2}
\]

Many textbooks use Larmor’s formula as follows: Argue that \( \theta \) is some invariant, then find a Lorentz scalar limit that reduces to Larmor’s formula in the NR limit.

The 1st part of the argument is this: when \( dp = \frac{\mathbf{d} E}{dy} \), this is everywhere as measured by comoving observer.

2nd part: Substitute \( \vec{a} = \frac{1}{m} \frac{\mathbf{d} \vec{a}}{dt} \) in Larmor's:
\[
\vec{a}^2 = \frac{1}{m} \frac{\mathbf{d} \vec{a}}{dt} \cdot \frac{\mathbf{d} \vec{a}}{dt} = \frac{1}{m^2} \left[ \mathbf{C} \frac{\mathbf{d} \vec{a}}{ds} \right]
\]

where the last step is valid in the NR limit (we'll verify below).

Now
\[
\frac{\mathbf{d} \vec{a}}{ds} = \frac{\mathbf{d} \vec{a}}{dt} \left[ \left( \frac{dt}{ds} \right)^2 - \left( \frac{dA}{dt} \right)^2 - \left( \frac{dC}{dt} \right)^2 \right]
\]
\[
= -c^2 \frac{d^2 \vec{a}}{ds^2} - c^2 \left( \frac{d (m \vec{C})}{dt} \right)^2 - \left( \frac{dA}{dt} \right)^2
\]
\[
= -c^2 \left( \frac{d (m \vec{C})}{dt} \right)^2 - c^2 \left( \frac{dA}{dt} \right)^2
\]
\[
= -c^2 \left( \frac{d (m \vec{C})}{dt} \right)^2 - c^2 \left( \frac{dA}{dt} \right)^2
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= c^2 \left( \frac{d (m \vec{C})}{dt} \right)^2 - c^2 \left( \frac{dA}{dt} \right)^2
\]
\[
= c^2 \left( \frac{d (m \vec{C})}{dt} \right)^2 - c^2 \left( \frac{dA}{dt} \right)^2
\]
\[
\mathbf{p} = \frac{2}{3} \frac{g^2 q^2}{c^2} \left[ \vec{a}^2 - \left( \vec{p} \times \vec{a} \right) \right] \text{ as before.}
Angular Distribution

We already have the basic equation for this in

\[ \frac{d\Phi}{d \Omega} \]

given in terms of \( \theta \times \phi \) angles in \( p \leq q \) above.

We look at special cases:

1. Linear motion: \( \vec{p} \parallel \vec{\omega} \).

\[ \frac{d\Phi}{d \Omega} \text{ is independent of } \theta, \text{ (symmetric under rotations about any axis defined by } \vec{p}). \]

Explicitly

\[ \frac{d\Phi}{d \Omega} = \frac{c^4}{4 \pi} \frac{1}{(1-\beta \sin \zeta)^5} \left| \hat{\mathbf{p}} \times (\hat{\mathbf{p}} \times \hat{\mathbf{\omega}}) \right|^2 \]

\[ = \frac{c^4}{4 \pi} \frac{\zeta \sin \zeta}{(1-\beta \sin \zeta)^5} \]

The direction of maximum radiation can be found analytically, but can also be quickly approximated by noting that for \( \rho \approx 1 \) it is at small \( \theta \) and the denominator is small for

\[ 1-\beta \sin \zeta \approx 1-\beta(1+\beta^2) \approx 1-\beta + \frac{1}{2} \beta^2 = \text{small}, \text{ so put } \frac{1}{2} \beta = (1-\beta)(1+\beta) = 2(1-\rho) \]

\[ \Rightarrow \theta^2 \approx \frac{1}{\beta}, \Rightarrow \frac{d\Phi}{d \Omega} \text{ is peaked at } \theta_{\text{max}} \approx \frac{1}{\beta}, \text{ but decreases at } \theta = 0 \]

\[ \frac{d\Phi}{d \Omega} \approx \frac{c^4}{4 \pi} \frac{1}{(1+\beta)^5} \]}

\[ \text{or } \frac{d\Phi}{d \Omega} \approx \frac{c^4}{4 \pi} \frac{\beta^4}{(1+\beta)^5} \]

In the opposite limit, \( \beta \ll 1 \)

\[ \frac{d\Phi}{d \Omega} \approx \frac{c^4}{4 \pi} \sin^3 \theta \text{ (Larmor)}, \text{ so } \theta_{\text{max}} = \frac{\pi}{2} \]

Radiation pattern

\[ \text{Note: Figures of revolution (i.e., keplerian) is a forward cone).} \]