\[ E \] waves

We consider time dependent (propagating) solutions of Maxwell's equations in free space (no sources).

Consider 1st the 4-vector potential. We have established that in free space

\[ \partial^2 A_\mu + \partial_\nu (\partial^{\nu} A_\mu) = 0 \]

We choose to impose the covariant gauge condition ("Lorentz gauge") \( \partial^\mu A_\mu = 0 \)

so that

\[ \nabla A_\mu = 0 \quad \text{(wave equation)} \]

\underline{Plane-wave solution}

A simple solution is

\[ A_\mu(x) = a_\mu e^{-ik^2 x} = e^{-ik^2 x} + e^{+ik^2 x}, \quad k^2 = (k_x, k_y, k_z, k_0) = (\omega, p) \]

where \( a_\mu \) and \( k_\mu \) are constants. Note that \( A^\mu_\mu = 0 \) requires \( \omega = 0 \)

or \( \omega = \pm k_0 \) or \( \omega = \pm k_1 \pm \sqrt{k_0^2 - \vec{k}^2} \)

A as written is complex. We always implicitly assume we take the real part:

\[ \bar{A}_\mu = \text{Re}(a_\mu e^{+ik^2 x}) = a_\mu^* e^{-ik^2 x} + a_\mu e^{+ik^2 x} \]

Now, we insist that \( \partial^\mu a_\mu = 0 \Rightarrow \partial^\mu (a_\mu e^{+ik^2 x}) = -ik^2 a_\mu e^{+ik^2 x} = 0 \]

\[ \Rightarrow k^2 a_\mu = 0 \]

For example, for wave propagation in \( x \)-direction \( k^2 = (\omega, 0, 0, k) = k(x) \)

and \( k \cdot a_\mu = 0 \Rightarrow a_\mu = a^2 \)

The \( E \) field is \( E^{\mu_0} \). Compute \( \vec{E}_0 = \partial_\nu A_\nu - \partial_\nu A_\nu = -ik_0 (a_0 - k_0 a_0) e^{-ik^2 x} \)

or \( E^2 = p^{\mu \nu} = -i (k^0 k^0 - k^2) e^{-ik^2 x} \)

For this example \( k^2 = (\omega, 0, 0) \) and \( a^2 = a^2 \) so \( E^2 = -i (k^0 a^0 - k^0 a^0) \)

and \( E^{\mu_1} = (k^1 a^1 - k^0 a^1) e^{-ik^2 x} = i k_1^0 e^{ik^2 x} \)

Also \( \vec{B}^2 = -E^2 = -i (k^0 a^0 - k^0 a^0) e^{ik^2 x} = -i k_0^0 e^{ik^2 x} \)

and \( B^2 = -E^2 = i (k^1 a^1 - k^0 a^1) e^{ik^2 x} = i k_1^0 e^{ik^2 x} \)

Note that \( \vec{B}^2 = \vec{E}^2 \), \( \vec{B} \cdot \vec{E} = 0 \). So \( \vec{E} = \pm \vec{B} \), \( |\vec{E}| = |\vec{B}| \) and hence \( \vec{E} \times \vec{B} = k \times \vec{E} \)

with \( k = \frac{e}{c} \)
Clearly only $a^a$ have physical meaning. Why do we still have $a^0 = a^0$ in $A^\mu$?

Recall if $A^\mu$ satisfies the covariant gauge condition $\partialA = 0$
we can still make a gauge transformation

$$A^\mu \rightarrow A^\mu' = A^\mu + \partial^\mu \lambda$$

with $\partial A' = 0$ provided $\partial^\mu \lambda = 0$. So we can add to our solution

$$\lambda(x) = \lambda e^{ikx} \Rightarrow \partial^\mu \lambda = k^\mu e^{ikx}$$

so that

$$a^\mu' = a^\mu + k^\mu$$

or

$$A^\mu = (a^0, e^\mu, a^i, a^k)$$

and choosing $\lambda = a^0 k$ we obtain $a^0' = (0, a^0, a^i, 0)$

So we may as well drop $a^0 = a^0$, and drop the prime.

It is convenient to define a unit vector $\vec{e} = \frac{\vec{e}}{|\vec{e}|}$. It satisfies $k \cdot \vec{e} = 0$. Also $e^0 = (0, \vec{e})$ satisfies $k \cdot \vec{e} = 0$, $e^2 = -1$

$\vec{e}$ is a "polarization" vector. We can write generally the solutions found for the propagation example:

$$\vec{E} = E^0 \hat{e} e^{-ikx} \Rightarrow \vec{B} = \hat{k} \times \vec{E} \Rightarrow \vec{k} \cdot \vec{E} = 0, \quad E^0 = 1$$

but now $\vec{e}$ is arbitrary.

Plane polarized waves: there are two independent solutions $\vec{e}_1, \vec{e}_2$ to $k \cdot \vec{e} = 0, \vec{e}^2 = 1$

that is two unit vectors in the plane $\perp \hat{k}$.

E.g. $\hat{e} = (0, 0, 1) \Rightarrow \vec{e}_1 = (1, 0, 0), \vec{e}_2 = (0, 0, 1)$

These, and any linear combination with relatively real coefficients are plane polarized waves

At a fixed point in space, $\vec{E} = (E^0, \vec{E}) e^{j\omega t}$ oscillates along $\vec{e}$.

For general $\hat{k}$ direction, choose a vector $\vec{e}_i$ i.e. $\vec{e}_i \cdot \hat{k} = 0$. Then take $\vec{E}_i = \hat{k} \times \vec{E}$.
Circular Polarization: In terms of $\vec{E}_z$ above, let

$$\vec{E}_x = \frac{\vec{E}_z + j \vec{E}_y}{\sqrt{2}}$$

Then $\vec{E}_x \cdot \vec{E}_x = \vec{E}_z \cdot \vec{E}_z$ and $\vec{E}_x \cdot \vec{E}_y = 0$, and $\vec{E}_y = \vec{E}_x$.

Now $\vec{E}(t) = \vec{E}_x e^{j \omega t}$

as seen at fixed point is a vector $\vec{E}$ of fixed magnitude rotating with

angular speed $\omega$ in the $\vec{E}_x$-$\vec{E}_y$ plane

with $E = |E| e^{j \phi}$  $Re \vec{E} = \frac{1}{\sqrt{2}} [E_x \cos(\omega t - \phi) - i E_y \sin(\omega t - \phi)]$

Elliptical polarization is a simple extension: $\vec{E} = (E_x, E_y, i E_z) e^{j \omega t}$

with $E_x/E_y = \tan \alpha$. The $E_y = 0$ case is circular.

Exercise: for $\hat{z} = \hat{x}$, $\vec{E}_z = \vec{E}_x$ verify that $\frac{\langle \vec{E}(t) \cdot \vec{E}^*(t) \rangle}{\langle |\vec{E}(t)|^2 \rangle} = 1$ which describes an ellipse.
Energy
For any $\mathbf{r}$ these plane waves

$$ U = \frac{1}{8\pi} \left( \mathbf{E}^2 + \mathbf{B}^2 \right) = \frac{1}{4\pi} \mathbf{E}^2 \quad \text{energy density} $$

$$ \mathbf{E} = \frac{\mathbf{E} \cdot \mathbf{B}}{\left( \mathbf{E} \cdot \mathbf{B} \right)} = \frac{\mathbf{E}^2}{4\pi} \mathbf{B} = \mathbf{c} \mathbf{u} \mathbf{k} \quad \text{Poynting vector (energy flux)} $$

The momentum density $\mathbf{\mathbf{S}}$ is related to the energy density ($\mathbf{\mathbf{u}}$ plane wave):

$$ \mathbf{\mathbf{S}} = \frac{1}{3} \mathbf{\mathbf{E}} = \frac{1}{3} \mathbf{\mathbf{E}} $$

The relation $U = gc$ parallels $E_{cm} = \sqrt{mc^2 + (mc^2)^2} \rightarrow (g/c)$ for the massless particle case. Such a particle can move only at the speed of light. Since

$$ E_{cm} = \frac{mc^2}{\sqrt{1-v^2/c^2}}, \quad \mathbf{p} = \frac{mc}{\sqrt{1-v^2/c^2}} $$

can stay non-zero as $m \to 0$ if we take $g/c$ simultaneously.

So $\mathbf{\mathbf{S}} = \frac{1}{3} \mathbf{\mathbf{E}}$ can be interpreted as the momentum and energy made up of many non-interacting massless particles, all propagating at the constant $c$. This interpretation is fortified in its QM-version, where particles = photons.

Finally, $\mathbf{\mathbf{S}}_j = -\frac{1}{4\pi} \left( \mathbf{E} \cdot \mathbf{\mathbf{E}} + \mathbf{B} \cdot \mathbf{\mathbf{B}} - \frac{1}{2} \mathbf{\mathbf{B}}^2 / (\mathbf{\mathbf{E}} \cdot \mathbf{\mathbf{B}}) \right)$

For $\mathbf{\mathbf{E}} = \mathbf{E}$, clearly vanishes unless $\mathbf{\mathbf{E}} = \mathbf{E}$ for which $\mathbf{\mathbf{S}}_j = \frac{1}{8\pi} \left( \mathbf{\mathbf{E}}^2 \right) = U$

Recall this is force on surface $\mathbf{\mathbf{E}} \cdot (n \times \mathbf{B})$ along $\mathbf{n}$: if we put a perfectly absorbing plane parallel to $\mathbf{\mathbf{E}}$, $\mathbf{\mathbf{E}}$, plane, it will experience force/area = $U = \underline{\text{radiation pressure}}$. 
Notes:

1. The waves are \( \omega \equiv \omega_0 \) and \( k \cdot a = 0 \). We may shift \( q \to q + 2\pi \) freely, in particular to choose to make \( q = 0 \), just as above. It is customary to expand in the two independent polarization vectors \( \epsilon^{(\nu)} \), thus:

\[
\mathbf{A}_\nu (x) = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left( \epsilon^{(\nu)}(k) \mathbf{e}_\nu (k) \, e^{ik \cdot x} + \text{c.c.} \right)
\]

(In fact, for second quantized this is the starting point, except that there usually

\[
\mathbf{A}_\nu (x) = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left( \gamma^{(\nu)}(k) \mathbf{e}_\nu (k) \, e^{ik \cdot x} + \text{c.c.} \right)
\]

The factor of \( \gamma \) is for Lorentz invariance \( \frac{\gamma^2}{\gamma} \). To see this, recall \( k = \frac{\omega}{v} \)

\[
\int d^3k \, \delta (k^0) \theta (k^1) = \int d^3k \frac{d^3k}{\omega}
\]

2. One can work directly from Maxwell's equations and get \( E \times B \) waves: probably due in your U3 course, but see next page.

3. For complex vectors, the magnitude is \( \bar{\mathbf{A}} \mathbf{A} \) or \( q q^* \). Beware of metric:

\[
\varepsilon_\mu \varepsilon^\nu = -1
\]
Waves in media; reflection and refraction at interfaces; polarization of interfaces.

We will study electromagnetic media next quarter. For you we draw from previous knowledge. While we could still use a vector potential — since the homogeneous Maxwell equations are unchanged — the presence of a medium in which $E$ and $B$ propagate breaks Lorentz invariance. It defines a preferred frame.

We introduce dimensions for permittivity $\epsilon = \epsilon 0 \hat{E}$ and permeability $\mu 0 \hat{B}$ so that the source free Maxwell equations read

\[
\begin{align*}
\nabla \cdot \vec{E} &= 0 \quad \text{(M1)} \\
\nabla \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} &= 0 \quad \text{(M2)} \quad \text{and} \\
\n\nabla \times \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} &= 0 \quad \text{(M3)} \\
\mu \nabla \cdot \vec{B} &= 0 \quad \text{etc.}
\end{align*}
\]

Taking $\nabla \times$ of (3) and using $\left( \nabla \times (\nabla \times \vec{A}) \right) = \epsilon 0 \nabla \left( \delta \nabla \cdot \vec{A} \right) - \nabla \left( \nabla \cdot \vec{A} \right)$, we get

\[
\nabla \times (\nabla \times \vec{E}) - \nabla \left( \frac{1}{c} \frac{\partial \vec{B}}{\partial t} \right) = 0
\]

and using (2) in this, $\nabla \times \vec{E} - \nabla \left( \frac{\mu 0}{\epsilon 0} \frac{\partial \vec{B}}{\partial t} \right)$, and (1) ($\nabla \vec{E} = 0$), obtain

\[
\frac{\mu 0}{\epsilon 0} \frac{\partial \vec{B}}{\partial t} - \nabla \left( \nabla \cdot \vec{B} \right) = 0
\]

This is the wave equation with velocity $v = \frac{c}{\sqrt{n}}$. With $n = \mu 0 / \epsilon 0$, this is $v = c / n$.

$n$, we will call, is the index of refraction of the medium.

Similarly, taking $\nabla \times$ of (2)

\[
\nabla \times (\nabla \times \vec{B}) - \nabla \left( \frac{\epsilon 0}{\mu 0} \frac{\partial \vec{E}}{\partial t} \right) = 0
\]

and using (4) and (5)

\[
\nabla \cdot \vec{E} - \nabla \left( \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \right) = 0
\]

again the wave equation with velocity $v = \frac{c}{\sqrt{n}}$.

Plane wave solutions are as before,

\[
\begin{align*}
E(x, t) &= E_0 e^{i(k x - \omega t)} + \text{c.c.} \\
B(x, t) &= B_0 e^{i(k x - \omega t)} + \text{c.c.}
\end{align*}
\]

where $\vec{k}^2 - \frac{\omega^2}{c^2} = 0$ and $\vec{k}^2 - \frac{\omega^2}{c^2} = 0$

These most so satisfy (1) - (4):

\[
\begin{align*}
(1) & \quad \nabla \cdot \vec{E} = 0 \Rightarrow \vec{k} \cdot \vec{E} = 0 \\
(4) & \quad \nabla \cdot \vec{B} = 0 \Rightarrow \vec{k} \times \vec{B} = 0
\end{align*}
\]
which requires \( E' = \vec{E}, \ \omega' = \omega \) and
\[ \vec{E} \times \vec{B}_o = \frac{\omega}{c} \vec{B}_o = \frac{\omega}{c} \vec{B}_o = \frac{1}{c} \vec{B}_o, \] or
\[ \vec{k} \times \vec{E}_o = \frac{1}{c} \vec{B}_o. \]

(4) \[ \vec{E} \times \vec{B} - \frac{\omega^2}{c^2} \vec{E} = 0 \Rightarrow \vec{k} \times \vec{B}_o + \omega \vec{E}_o = 0 \Rightarrow \vec{k} \times \vec{B}_o = -\frac{1}{\omega} \vec{E}_o. \]

Summarizing, \( \vec{E}_o, \vec{B}_o \) and \( \vec{k} \) are mutually perpendicular, with \( \vec{k}_o = n \vec{E}_o \) and \( \vec{k} = \vec{B}_o \).

In terms of polarization vectors
\[ \vec{E} = E_0 e^{i(\vec{k} \cdot \vec{x} - \omega t)} \]
\[ \vec{B} = n E_0 (\vec{k} \times \vec{E}) e^{i(\vec{k} \cdot \vec{x} - \omega t)}. \]

with \( \vec{E} \cdot \vec{B} = 0 \) and \( \vec{k} \cdot \vec{E} = 0. \)

Two media and interference

\[ \begin{align*}
\text{Incident wave:} & \quad \vec{E}_1 = E_{01} e^{i(k_{11} x - \omega t)} \quad \vec{B}_1 = n_{1} (\vec{k} \times \vec{E}_1) e^{i(k_{11} x - \omega t)} \\
\text{Reflected wave:} & \quad \vec{E}_r' = E_{0r} e^{i(k_{r1} x - \omega t)} \quad \vec{B}_r' = n_{1} (\vec{k} \times \vec{E}_r) e^{i(k_{r1} x - \omega t)} \\
\text{Reflected wave:} & \quad \vec{E}'' = E_{0r} e^{i(k_{r1} x - \omega t)} \quad \vec{B}'' = n_{2} (\vec{k} \times \vec{E}''') e^{i(k_{r1} x - \omega t)}
\end{align*} \]

Let the boundary be the plane \( \vec{x} = 0 \). The fields \( \vec{E}, \vec{B} \) are given by any of these at the boundary so we must have \( \omega = \omega = \omega \) for common time dependence. Note that this means
\[ |E_1'| = |E_1| = \frac{n_1}{c} \quad \text{and} \quad |E_1'| = |E_1| = \frac{n_1}{c} \quad |E_1'| = |E_1| = \frac{n_1}{c}. \]

Moreover, at \( z = 0 \)
\[ \vec{E}_1 \cdot \hat{x} = E_{11} = E_{11} \]

Then is, the projection on the \( xy \) plane is equal:
\[ |E| \sin \theta = |E| \sin \theta = |E| \sin \theta. \]

It follows that \( \theta = \theta \) and \( \sin \theta = \frac{|E|}{|E|} = 1 \) or \( \sin \theta = n \sin \theta \) or \( \sin \theta = n \sin \theta \),

Snell's Law of Refraction.
Polarization: There is additional information coded in the boundary conditions. For this we need:

1. Normal component of \( \vec{E} = e \vec{E}_0 \) and \( \vec{B} \) are continuous.
2. Tangential components of \( \vec{E} \) and \( \vec{H} = \mu \vec{B} \) are continuous.

These will be derived when we study continuow media. For now, we have: let \( \vec{n} \) be normal to the boundary. Then:

\[
\vec{n} \cdot \left( \vec{E}_0 - \vec{E}_0' \right) = 0
\]

\[
\vec{n} \cdot \left[ \mu \left( \vec{E}_0 - \vec{E}_0' \right) - \nu \left( \vec{E}_0 x - \vec{E}_0' x \right) \right] = 0
\]

\[
\vec{n} \times \left( \vec{E}_0 + \vec{E}_0' \right) = 0
\]

\[
\vec{n} \times \left[ \frac{\mu}{\nu} \vec{E}_0 + \frac{\nu}{\mu} \vec{E}_0' - \frac{\mu \nu}{\mu + \nu} \vec{n} \times \vec{E}_0 \right] = 0
\]

Two cases (the general one is a linear combination of these):

(i) Polarization vector \( \vec{E}_0 \) of incident wave (linearly polarized) perpendicular to plane of incidence. Unit vector of \( \vec{n} \) and \( \vec{E} \): Let \( \vec{E} \) into the plane.

We have 1 + 2 + 2 equations for 3 + 3 unknowns \( (\vec{E}_0, \vec{E}_0') \): The system is overdetermined. We look for solutions with the polarizations all into the plane (we could drive this). Then:

\[
\vec{n} \cdot \left( \vec{E}_0 + \vec{E}_0' \right) = 0
\]

\[
\vec{n} \times \left( \vec{E}_0 - \vec{E}_0' \right) = 0
\]

Solving for \( \vec{E}_0', \vec{E}_0'' \):

\[
\text{(ii) } \vec{E}_0' = \vec{E}_0 + \vec{E}_0''
\]

From (ii) \( \vec{n} \times \vec{E}_0 + \vec{n} \times \vec{E}_0'' = 0 \)

\[
\Rightarrow \vec{E}_0'' = \frac{\nu}{\mu} \left( \frac{\nu}{\mu} \vec{E}_0 - \frac{\mu}{\nu} \vec{E}_0' \right)
\]

\[
\Rightarrow \vec{E}_0'' = \frac{\nu}{\mu} \vec{E}_0 - \frac{\mu}{\nu} \vec{E}_0'
\]

\[
\text{Now } \frac{\vec{E}_0'}{\vec{E}_0} = 1 + \frac{\vec{E}_0''}{\vec{E}_0} = \frac{\mu}{\nu} \cos \theta + \frac{\nu}{\mu} \cos \theta' \Rightarrow \frac{\vec{E}_0'}{\vec{E}_0} = \frac{2 \cos \theta}{\mu \cos \theta + \nu \cos \theta'}
\]

\[
\vec{E}'' \perp \vec{n}, \vec{k} \text{ plane}
\]

One often writes \( \cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - \mu^2 \sin^2 \theta} = \frac{1}{\mu} \sqrt{1 - \mu^2 \sin^2 \theta} \) so that it is all expressed in terms of incident data.
What about (3)?

\[ \mathbf{E}_0 \cdot \mathbf{n} = n \sin \theta \Rightarrow n(\mathbf{E}_0 + \mathbf{E}_0') \cdot \mathbf{n} \sin \theta - n' \mathbf{E}_0' \cdot \mathbf{n} \sin \theta = 0 \]

$$ \Rightarrow \frac{n}{n'} = \frac{n \sin \theta}{n' \sin \theta} \quad \text{Snell's Law again} $$

(\nu) \mathbf{E}_0 \parallel \text{plane of incidence}

$$ \mathbf{E}_0' \perp \text{plane of incidence} $$

Now (3) and (\nu) give

\[ (\mathbf{E}_0 - \mathbf{E}_0') \cos \theta - \mathbf{E}_0' \cos \theta' = 0 \]

\[ \frac{n}{n'} (\mathbf{E}_0 + \mathbf{E}_0') - \mathbf{E}_0' = 0 \]

Algebra: (\mu): \[ \mathbf{E}_0' = \frac{\mathbf{n}}{n'} (\mathbf{E}_0 + \mathbf{E}_0') \] in (3) gives

\[ (\mathbf{E}_0 - \mathbf{E}_0') \cos \theta \cos \theta' = 0 \quad \Rightarrow \quad \mathbf{E}_0' = \frac{\mathbf{E}_0 \cos \theta - \frac{n}{n'} \mathbf{n} \cos \theta'}{\cos \theta - \frac{n}{n'} \mathbf{n} \cos \theta'} \]

Then \[ \frac{E_0'}{E_0} = \frac{n}{n'} \left( 1 - \frac{E_0'}{E_0} \right) \]

Brewster angle: in thin case the reflected wave vanishes, for \( \cos \theta_b = \frac{\mathbf{n}}{n'} \cos \theta = \frac{n}{n'} \sqrt{1 - n' \mathbf{n} \mathbf{n} \mathbf{n} \mathbf{n}} \)

Consider simpler (but often found) case \( n'^{-1} > 1 \). Then

\[ \left[ \frac{n}{n'} \cos \theta_b \right] = \frac{1}{1 - n'^{-1}} \sin \theta_b = \frac{1}{n'} \sin \theta_b - 1 - \frac{n}{n'} \cos \theta_b \]

\[ \cos \theta_b = \frac{1 - \frac{n}{n'} \mathbf{n} \mathbf{n} \mathbf{n} \mathbf{n}}{1 - \frac{n}{n'} \mathbf{n} \mathbf{n} \mathbf{n} \mathbf{n}} = 1 - \frac{n'}{n} \frac{\mathbf{n} \mathbf{n} \mathbf{n} \mathbf{n} \mathbf{n}}{n} = 1 - \frac{n'}{n} \frac{\mathbf{n} \mathbf{n} \mathbf{n} \mathbf{n} \mathbf{n}}{n} = \frac{1}{1 - n'^{-1}} = 1 - \sin \theta_b \]

\[ \tan \theta_b = \left( \frac{n}{n'} \right)^{-1} \quad \Rightarrow \quad \theta_b = \tan^{-1} \left( \frac{n}{n'} \right) \]

For water \( n=1.33 \) and air \( n=1.00 \), \( \theta_b = 0.93 \) or 53°.
Total internal reflection.

You know from elementary courses that for (using Snell's Law)

\[
\sin \Theta' = \frac{n \sin \Theta}{n'} > 1 \quad \text{critical} \quad \sin \Theta_c = \frac{n}{n'}
\]

There is no refracted wave. This is the phenomenon of "total internal reflection." We can understand some aspects of this in more detail with the tools developed.

The refracted wave is \( \mathbf{E}' = \mathbf{E}_0 \ e^{i(k' \mathbf{r} - \omega t)} \)

\[
\text{Now} \quad \mathbf{r} = \mathbf{r}_0 + \mathbf{v} t = \mathbf{r}_0 + \mathbf{v} \sin \Theta \quad \Rightarrow \quad \mathbf{r}' = \mathbf{r}_0 + \mathbf{v} \sin \Theta \quad \Rightarrow \quad \mathbf{v}' = \mathbf{v} \quad \Rightarrow \quad \omega' = \omega \quad \Rightarrow \quad \mathbf{E}' = \mathbf{E}_0 \ e^{i(k' \mathbf{r} - \omega t)}
\]

with \( \sin \Theta' = \sqrt{1 - \sin^2 \Theta} = \sqrt{\frac{\sin^2 \Theta}{\sin^2 \Theta} - 1} \)

so \( e^{i k' \mathbf{r}} = e^{-i k' \left( \frac{\sin \Theta}{\sin \Theta'} \right) - \frac{k' \mathbf{r}}{2} - \frac{i k' \mathbf{r}}{2} \left( \frac{\sin \Theta}{\sin \Theta'} \right) - \frac{1}{2} \}
\]

The wave is exponentially damped into the interior of the medium \( n' \).

There is no energy flux into \( n' \): \( \mathbf{S} \propto \mathbf{E} \times \mathbf{B} \) (actually \( \mathbf{E} \times \mathbf{B} = \mathbf{E} \times \mathbf{B}^* \))

so \( \hat{n} \cdot (\mathbf{E} \times \mathbf{B}) = \text{c.c.} \alpha 2 Re [\hat{n} \cdot (\mathbf{E}^* \times \mathbf{B})] = 2 Re [\hat{n} \cdot (\mathbf{E}^* \times \mathbf{B})] = 2 Re [\hat{n} \cdot (\mathbf{E}^* \times \mathbf{B})] = 2 \Re (\hat{n} \cdot \mathbf{E}^* \times \mathbf{B})]
\]

But \( \hat{n} \cdot \mathbf{B} = \cos \Theta' \) is purely imaginary for \( \Theta > \Theta_c \Rightarrow \hat{n} \cdot \mathbf{S} = 0 \)

Note: \( \Rightarrow |\mathbf{E}'| = 1 \) for each polarization, e.g., for \( \mathbf{E}_0 r \) to plane of incidence

\[
\frac{E_{0}'}{E_0} = \frac{\beta_{0} \sin \Theta - \beta_{1} \cos \Theta}{\beta_{0} \sin \Theta + \beta_{1} \cos \Theta}
\]

The reflected wave has same intensity, but experiences a phase shift. The phase shift is different for each of the two plane polarizations, so this can be used to polarize waves.