Electrodynamics:
\[ F^{\mu \nu} = -F^{\nu \mu} \]

\[ F_{\mu \nu} = -F^{\mu \nu} = \varepsilon^{\mu \nu \rho \sigma} F_{\rho \sigma} \]

I do not climb much is gained by writing this as a sum as above but most textbooks do:

\[ F_{\mu \nu} = \begin{pmatrix} 0 & E^t & E^z & B^z \\ E^t & 0 & -B^z & -E^z \\ E^z & B^z & 0 & -E^t \\ B^z & -E^z & E^t & 0 \end{pmatrix} \]

Charge-current \( \mathbf{J} \) - much like 4-momentum \( \mathbf{p} = (\varepsilon, \mathbf{p}) \).

\[ J^\mu = (\varepsilon \mathbf{p}, \mathbf{J}) \]

Ex: Check this with (ie, engineering dimensions) \([\varepsilon \mathbf{p}] = \frac{\mathbf{J}}{\varepsilon} \).

Charge conservation
\[ \frac{\partial \varepsilon}{\partial t} + \nabla \cdot \varepsilon \mathbf{J} = 0 \]

Reminder:

\[ \int_{s_1} \mathbf{J} \cdot d\mathbf{s} = \text{gauge choice} \]
\[ \int_{s_2} \mathbf{J} \cdot d\mathbf{s} = \text{gauge choice} \]

\[ \text{invariant volume} \]
\[ \text{constant charge density} \]
\[ = \varepsilon_0 \frac{\partial \varepsilon}{\partial t} + \varepsilon_0 \nabla \cdot \varepsilon \mathbf{J} \]
\[ = \varepsilon_0 \frac{\partial}{\partial t} (\varepsilon \mathbf{p}) + \varepsilon_0 \nabla \cdot \varepsilon \mathbf{J} \]
\[ = \varepsilon_0 \left( \varepsilon_0 \nabla \cdot \mathbf{J} + 2 \varepsilon \frac{\partial \varepsilon}{\partial t} \right) = \varepsilon_0 \nabla \cdot \mathbf{J} \]

Maxwell's Equations
\[ \frac{\partial}{\partial t} F_{\mu \nu} = \frac{4\pi}{c} J^\nu \quad \text{non-homogeneous} \]
\[ \text{Gauss} \]
\[ \varepsilon_0 \partial_\mu F_{\mu \nu} = 0 \quad \text{homogeneous} \]
\[ \text{Faraday} \]
\[ \text{Gauss} \]
\[ \text{Abraham} \]
\[ \text{Magnetic} \]

Exercise: Verify this by writing the 4 equations corresponding to the 2 cases \( \mu = 0 \) and \( \nu = 0 \) (x 2 equations).

Current conservation is required for consistency of Maxwell's equations:

\[ \partial_\mu J^\mu = \partial_\mu \left( \partial_\nu F^{\mu \nu} \right) = \partial_\mu \partial_\nu F^{\mu \nu} \]

In terms of symmetries:

\[ \text{symmetric} \quad \text{antisymmetric} \]

Notes: This equation is, equivalently, \( \partial_\nu F_{\mu \nu} + \partial_\mu F_{\nu \mu} - \partial_\nu F_{\nu \mu} = 0 \).
We are finally in a position to say something interesting in a trivial sort of way:

Maxwell equations are form-invariant Lorentz transformation:

\[ F_{\mu\nu}'(x') = \Lambda_{\rho}^{\mu} \Lambda_{\lambda}^{\nu} F_{\rho\lambda}(\Lambda^{-1}x) \]

\[ j_{\mu}'(x') = \Lambda_{\rho}^{\mu} j_{\rho}(\Lambda^{-1}x) \]

Exercise: although fairly explicit, you should know how to verify this statement!

We can explore some consequences by looking at components for specific transformation.

We know everything (?) about alphas, so concentrate on boosts. Take boosts in \( x = x \)-dir

\[-E^{1}_1 = F^{01} = \Lambda^{0}_{\rho} \Lambda^{1}_{\nu} F^{\rho\nu} \quad \text{with} \quad \Lambda = \begin{pmatrix} c^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

\[-E^1 = \sqrt{c^2 - s^2} = \frac{c}{\sqrt{c^2 - s^2}} s \]

Skip the following tedious calculations in class:

\[-E^1 = \Lambda^{0}_{\rho} \Lambda^{1}_{\nu} F^{\rho\nu} = \Lambda^{0}_{0} \Lambda^{1}_{0} F^{00} + \Lambda^{1}_{0} \Lambda^{0}_{0} F^{00} = c^2 (E^1) + s^2 (E^0) = -E^0 \]

\[-E^2 = \Lambda^{0}_{\rho} \Lambda^{1}_{\nu} F^{\rho\nu} = \Lambda^{0}_{0} \Lambda^{1}_{0} F^{00} + \Lambda^{1}_{0} \Lambda^{0}_{0} F^{00} = c^2 (E^2) + s^2 (cE^1 + E^0) \]

\[-E^3 = - \text{term, note: } E^3 = B^3 \]

\[-E^4 = - \text{term, note: } E^4 = B^0 \]

\[-B^0 = F^{01} = \Lambda^{0}_{\rho} \Lambda^{1}_{\nu} F^{\rho\nu} = cE^1 + E^2 \]

\[-B^1 = F^{12} = \Lambda^{1}_{\rho} \Lambda^{2}_{\nu} F^{\rho\nu} = cE^2 + E^3 \]

\[-B^2 = F^{23} = \Lambda^{2}_{\rho} \Lambda^{3}_{\nu} F^{\rho\nu} = cE^3 + E^4 \]

\[-B^3 = F^{31} = \Lambda^{3}_{\rho} \Lambda^{1}_{\nu} F^{\rho\nu} = -B^1 \]

\[-B^4 = F^{42} = \Lambda^{4}_{\rho} \Lambda^{2}_{\nu} F^{\rho\nu} = -B^2 \]

\[ \Phi = \frac{1}{\gamma} (\text{velocity of unpinned system in pinned one}) = \frac{1}{\gamma} \text{v}_1 \text{ of pinned system } a \text{ seen from unpinned} \]

\[ \textit{Note: } \vec{E} \perp \vec{B} \text{ clearly NOT components of same } \vec{E} \text{ and } \vec{B} \text{ 4-vectors.} \]
Properties under discrete transformations $P \times T$.

To determine transformation properties of $\mathbf{E} \times \mathbf{B}$, start from $p = j^0$. We know $p' = p$ under $T$ but not under $P$.

So $\mathbf{E}' = (j^0, \mathbf{p})$ is a pseudo-tensor under $T$ but not under $P$.

So we should have $\mathbf{E}' = -\mathbf{E}$, and this is physically sensible; $P$ reverses current direction, as does $T$ (running movie backward).

Now $\mathbf{B}$ is a vector (not pseudo). So in

\[ \beta^i = \frac{\beta^i}{c} \]

we'll better have that $\mathbf{B}' = (\epsilon^{ij}, \beta^i)$ for any $LT$. $\mathbf{B}$' is also not pseudo.

So what does this mean for $\mathbf{E} \times \mathbf{B}$?

\[ E^i = F^i = -F^{i0} = -E^j \text{ under } P, \quad E^i \text{ under } T \]

\[ B^j = F^j = F^{jk} = B^k \text{ under } P, \quad B^j = -B^j \text{ under } T \]

In accord with intuition.

Exercise: verify that the Lorentz force equation is covariant under $P \times T$.

That laws of nature are invariant under any symmetries is an empirically determined fact. And it be that they are only invariant under proper LT’s but not under $\!P \times \!T$? Yes, weak interactions are do not respect either.

But electromagnetic, strong, and gravitational interactions do respect $P \times T$. This will come handy in solving problems, modeling systems, etc.
Kinematics of relativistic particle

World-line: or particle trajectory

\[ x^0 \rightarrow \quad x^\nu = x^\nu(\lambda) \]

Not every collection of 4 functions \((x^\nu(\lambda))\) is acceptable as describing the motion of a point particle:

(i) Move forward in time, or \(\frac{dx^0}{d\lambda} > 0\)

(ii) Move no faster than speed of light: \(|v| \leq c \quad \text{or} \quad \frac{dx^\mu}{d\lambda} \leq c \quad \text{or} \quad \frac{dx^\mu}{d\lambda} dx^\mu \leq 0\)

\((i) + (ii) \Rightarrow \text{it lies in future light cone at each point of world line}\)

Above we used condition \((i)\) to implicitly invert \(x^0 = x^0(\lambda)\) to give \(\lambda = \lambda(x^0)\)

(or \(\lambda = \lambda(x^0)\)) to write \(\frac{dx^\mu(\lambda(x^0))}{dx^0} = \frac{dx^\nu}{dx^0} \cdot \frac{dx^\mu}{dx^\nu}\).

We could have equally chosen a different parameterization of the same physical trajectory in the same coordinate frame:

Say the functions \(x_1^\nu(\lambda_1)\) and \(x_2^\nu(\lambda_2)\) are physically the same.

It must be that if we write \(\lambda_2\) in terms of \(\lambda_1\), then \(x_2(\lambda_2)\) agains gives \(x_1(\lambda_1)\):

\[ x_2^\nu(\lambda_2(\lambda_1)) = x_1^\nu(\lambda_1) \]

That is, they both give the same trajectory \(x(t)\) as function of time \(t = x^0/\gamma\).

Two common preferred parameters:

(i) \(\lambda = x^0\), so \(x^\nu(x) = (x^0, x^\nu(x^0))\)

(ii) \(\lambda = t\) (proper time) \(x^\nu(x)\), so \(x^0 = x^0(x)\)

\(dx^\mu = dx^\mu dt\)
To be clear, \( ds^2 = g_{\mu\nu} dx^\mu dx^\nu \). We often use the symbol \( ds^2 = g_{\mu\nu} dx^\mu dx^\nu \) for general intervals, and reserve \( ds \) for intervals that satisfy \( ds^2 > 0 \).

Given a parameterization \( x^\mu = x^\mu(\lambda) \), we can give the proper distance between two events \( x^\mu(\lambda_1) \) and \( x^\mu(\lambda_2) \) along \( x^\mu(\lambda) \) by

\[
S(\lambda_1, \lambda_2) = \int_{\lambda_1}^{\lambda_2} ds = \int_{\lambda_1}^{\lambda_2} \sqrt{\frac{dx^\mu}{d\lambda} \frac{dx^\mu}{d\lambda}} d\lambda
\]

\[
d\lambda \quad i, \text{ a tangent vector to } x^\mu(\lambda)
\]

When \( x^\mu \) is parameterized by \( \tau \) we call this a 4-velocity

\[
u^\mu = \frac{dx^\mu}{d\tau}
\]

It satisfies \( \nu^2 = 1 \)

Its 2 derivative is the 4-acceleration \( a^\mu = \frac{d\nu^\mu}{d\tau} \)

Since

\[
\frac{d(\nu^2)}{d\tau} = \frac{d}{d\tau} \nu \cdot \nu = 0 \quad \text{we have } a^\mu \nu_\mu = 0
\]

A particle of mass \( m \) has 4-momentum \( p^\mu = mc\nu^\mu \)

It satisfies \( p^2 = m^2c^2 \). \( m \) is a Lorentz invariant

Since \( \nu^\mu \) is a 4-vector and we want our definition of \( p^\mu \) to also give a 4-vector, the zeroth component is called energy

\[
p^0 = (p^0, p^i) = (\frac{E}{c}, p^i)
\]
We often reserve the symbol $\mathbf{v}$ for a 3-vector that corresponds to the velocity of the particle: $\mathbf{v} = \frac{d\mathbf{x}}{dt}$

\[
\mathbf{v}^2 = \frac{dx^1}{dt} \frac{dx^1}{dt} = \frac{dx^2}{dt} \frac{dx^2}{dt} = \frac{dx^3}{dt} \frac{dx^3}{dt} \Rightarrow \mathbf{v} \cdot \mathbf{v} = v^2
\]

Since $u^2 = 1$, $u^0 - u^i = u^0 \left( \frac{x^i}{u^0} \right) = 1 \Rightarrow u^0 = \frac{1}{\sqrt{1 - \beta^2}}$

It is customary (and sometimes confusing, beware!) to use the same symbols here as for Lorentz transformations: $\beta = \frac{v}{c}$, $\delta = \frac{1}{\sqrt{1 - \beta^2}}$

so that

\[
\mathbf{u}^0 = (\gamma, \gamma \beta^i) \quad \text{and} \quad p^0 = (\gamma mc, \gamma \beta^i mc)
\]

or $E = \gamma mc^2$ and $p = \frac{mc^2}{\sqrt{1 - \beta^2}}$

This Lorentz-boost notation is natural in that one can obtain these results from boosting to an arbitrary frame with velocity $-\mathbf{v}$ from the (possibly instantaneous) rest frame

\[
\mathbf{u}^0 = (1, 0)
\]

In the instantaneous rest frame, $\beta = 0$, $E = mc^2$ and $p^0 = (0, 0)$

\[\textbf{Lorentz Force}\]

We are ready to write $\mathbf{F} = q \left( \mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \right)$ in covariant form. It is easy to “derive” its covariant form. Since $\frac{\partial F^i}{\partial x^0} = F^i$, we look for an equation with

\[
\frac{\partial F^0}{\partial \mathbf{u}} = \mathbf{F} \quad \text{vector mode of} \quad \mathbf{E}, \mathbf{B} \quad \text{and} \quad \mathbf{v}
\]

Now $\mathbf{E}, \mathbf{B}$ are encoded in the 2-index tensor $F^{ij}$, and $\mathbf{v}$ in the 4-vector $\mathbf{u}^i$. The 4-vector we search for must be linear in $F^{ij}$ (since $E^i$ is linear in $E^j$, $B^j$); it is not necessarily linear in $\mathbf{u}^i$ since $\mathbf{u}^2 = 1$. Possibilities

\[
F^{\alpha\beta} \mathbf{u}_\alpha \mathbf{u}_\beta = 0, \quad F^{\alpha\beta} \mathbf{u}_\alpha \mathbf{u}_\beta \mathbf{u}^\gamma = 0
\]

Thus the last two vanish by $F^{\alpha\beta} = -F^{\beta\alpha}$, $\mathbf{u}^\gamma \mathbf{u}^\gamma = \mathbf{u}^0 \mathbf{u}^0$, $\mathbf{u}^\gamma = \mathbf{u}^\gamma$. So

\[
\frac{\partial F^\gamma}{\partial \mathbf{u}} \propto F^\gamma \mathbf{u}_\alpha \mathbf{u}_\beta \mathbf{u}^\gamma
\]

The proportionality constant is $q$:

\[
\frac{\partial F^\gamma}{\partial \mathbf{u}} = q \frac{F^\gamma}{\mathbf{u}^0} \mathbf{u}_\alpha \mathbf{u}_\beta \mathbf{u}^\gamma
\]

It is understood that the field $F^\gamma(x)$ is $\mathbf{u}^\gamma(x)$.
\[ \frac{d\mathbf{p}^i}{dt} = q \left( \mathbf{F}^i \cdot \mathbf{U}^0 - \mathbf{F}^i \cdot \mathbf{U} \right) \\
= q \left( \mathbf{E}^i \cdot \mathbf{U}^0 + \epsilon_{ijk} \mathbf{B}^k \cdot \mathbf{U} \right) \\
= q \mathbf{U}^0 \left( \mathbf{E} + \frac{\mathbf{V}}{c} \times \mathbf{B} \right) \]

and since \( \frac{d\mathbf{p}^i}{dt} = \mathbf{F}^i \cdot \mathbf{U}^0 \Rightarrow \frac{d\mathbf{F}^i}{dt} = q \left( \mathbf{E} + \frac{\mathbf{V}}{c} \times \mathbf{B} \right) \)

What about \( \frac{d\mathbf{U}^0}{dt} = \mathbf{E} \cdot \mathbf{U}^0 \)? It is not really an independent equation.

Since \( \mathbf{U}^0 \cdot \mathbf{U}^0 = 0 \Rightarrow \mathbf{U}^0 \cdot \frac{d\mathbf{U}^0}{dt} = \mathbf{U}^0 \cdot \frac{d\mathbf{U}^0}{dt} \Rightarrow \frac{d\mathbf{E}}{dt} = \frac{c \mathbf{U}^0 \cdot d\mathbf{U}^0}{dt} = \mathbf{V} \cdot \frac{d\mathbf{U}^0}{dt} = \frac{d}{dt} \left( \mathbf{E} \cdot \mathbf{U}^0 \right) \)

The right hand side is \( q \left( \mathbf{E}^i \cdot \mathbf{U}^0_i = q / \mathbf{E} \right) = q \mathbf{E} \times \mathbf{U}^0 \)

So \( \frac{d\mathbf{E}}{dt} = q \frac{\mathbf{V}}{c} \mathbf{E} \)

(E on LHS is energy, \( \mathbf{E} \) on RHS is Electric field).

This follows trivially from \( \frac{d}{dt} \left( \frac{1}{2} \mathbf{U}^0 \mathbf{U}^0 \right) = \mathbf{V} \cdot \mathbf{F} \) when \( \mathbf{F} = \frac{d}{dt} (\mathbf{U}^0 \mathbf{F}) \).

Note that \( \frac{d\mathbf{E}}{dt} = q \left( \mathbf{E} + \frac{\mathbf{V}}{c} \times \mathbf{B} \right) \) is NOT \( \frac{m d^2 x}{dt^2} = q \left( \mathbf{E} + \frac{\mathbf{V}}{c} \times \mathbf{B} \right) \)

because \( \frac{d\mathbf{E}}{dt} = m \frac{d}{dt} \left( \frac{\mathbf{V}}{c} \right) \)

Exercise: Show

\[ m \frac{d^2 x}{dt^2} = q \sqrt{1 - \frac{\mathbf{V}^2}{c^2}} \left[ \mathbf{E} - \left( \frac{\mathbf{V}}{c} \mathbf{E} \right) \frac{\mathbf{V}}{c} + \frac{\mathbf{V}}{c} \times \mathbf{B} \right] \]
**Vector Potential**

Recall from electrostatics, \( \mathbf{\nabla} \times \mathbf{E} = 0 \Rightarrow \mathbf{E} = -\mathbf{\nabla} \phi \) for some electrostatic potential \( \phi = \phi(x) \).

We'd like to generalize this to include \( \mathbf{B} \) field and a time dependence.

\[
\mathbf{\nabla} \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0 \quad (3)
\]

We can also use the other homogeneous equation

\[
\mathbf{\nabla} \cdot \mathbf{B} = 0 \quad (4)
\]

Start from (4): since \( \mathbf{\nabla} \cdot (\mathbf{\nabla} \times \mathbf{A}) = 0 \) for any \( \mathbf{A} \), we propose \( \mathbf{B} = \mathbf{\nabla} \times \mathbf{A} \).

Can this hold for any \( \mathbf{B} \)? Let \( \mathbf{A} = \mathbf{A}' + \mathbf{B}/\nabla \cdot \mathbf{B} \).

So, provided \( \nabla \cdot \mathbf{B} = 0 \) and \( \nabla \cdot \mathbf{A} \) is invertible (i.e. \( \mathbf{B} \) is well behaved) then \( \mathbf{B} = \mathbf{\nabla} \times \mathbf{A} \) is always possible. It is not unique:

\[
\mathbf{B} = \mathbf{\nabla} \times \mathbf{A}' = \mathbf{\nabla} \times \mathbf{A} \Leftrightarrow \nabla \cdot (\mathbf{A}' - \mathbf{A}) = 0 \quad \text{we know the solution to \( \nabla \cdot \mathbf{A} \).}
\]

It is just like electrodynamics: \( \mathbf{A}' - \mathbf{A} = \mathbf{\nabla} \omega \) for some scalar \( \omega \).

Now for (3): using \( \mathbf{B} = \mathbf{\nabla} \times \mathbf{A} \)

\[
\mathbf{\nabla} \times \mathbf{E} + \frac{1}{c^2} \frac{\partial \mathbf{B}}{\partial t} = 0 \quad \text{or} \quad \nabla \times \left( \mathbf{E} + \frac{1}{c^2} \frac{\partial \mathbf{A}}{\partial t} \right) = 0 \Rightarrow \mathbf{\nabla} \cdot \mathbf{E} + \frac{1}{c^2} \frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial \phi}{\partial x} \quad \text{a before.}
\]

**Summary:** (3) & (4) insomuch we can write

\[
\mathbf{E} = -\mathbf{\nabla} \phi - \frac{1}{c^2} \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \mathbf{\nabla} \times \mathbf{A}
\]

**Names:**

\( \mathbf{A} \) “vector potential”

**Note:** if \( \mathbf{A}' = \mathbf{A} - \mathbf{\nabla} \phi \) and \( \phi \rightarrow \phi = \phi + \frac{\partial \phi}{\partial t} \), i.e., \( \mathbf{E} \rightarrow \mathbf{E} + \frac{1}{c^2} \frac{\partial \mathbf{A}}{\partial t} = \mathbf{B} \rightarrow \mathbf{B} \).

This is called “gauge invariance” of \( \mathbf{E} \& \mathbf{B} \).
Relativistic extension

We can combine the above expressions as follows:

\[ F_{\mu \nu} = 2 A_\mu - \partial_\mu A_\nu \]

Check:
\[ F_{\mu \nu} = E^\nu, \quad F_{\nu i} = \partial A_i - \partial_i A_\nu = -\partial_i A^\nu - \partial_\nu A^i = -\frac{1}{c^2} \partial^\nu - \partial_i \phi \]

provided \( \phi = A^0 = A_0 \); \( \partial_i \mathbf{A} = \mathbf{A}^i \)

\[ F_{\mu j} = -\delta_{ij} B^k = \partial A_j - \partial_j A_i = -\left( \partial_i A^j - \partial_j A^i \right) \]

or \( B^i = \partial_i A_j - \partial_j A_i \) etc.

Alternatively, \( \Box \Box \phi = 0 \implies F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \)

(This is the 4D generalization of \( \nabla \mathbf{B} = 0 \implies \mathbf{B} = \nabla \mathbf{A} \)).

\( A_\mu \) is also called "vector potential", often also called a "gauge field".

Gauge invariance: \( A_\mu \rightarrow A_\mu + \partial_\mu \lambda \)

Trivially seen directly:
\[ F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \]

\begin{align*}
\text{since} \quad \partial^\mu \partial_\mu \chi - \partial_\mu \partial^\mu \chi &= 0.
\end{align*}

Non-homogeneous equations in terms of \( A_\mu \):

\[ \Box A_\mu = \partial_\mu J^\mu \]

Here \( \Box = \partial_\mu \partial^\mu = \frac{1}{c^2} \partial_\mu \partial^\mu \) is also "\( \Box \)" or "\( \Box \)" in textbooks

is the "d'Alembert operator", or "d'Alembertian" which appears prominently in the wave equation.

(Also \( \Box = \partial_\mu \partial^\mu \)). So Maxwell-like (4) are

\[ \Box A_\mu - \Box_\nu (\partial_\nu A) = \frac{4 \pi}{c} \frac{J_\mu}{\partial_\mu} \quad (X) \]
An application of gauge invariance: since $F_{\mu\nu}$ is invariant under $A\to A'^\mu = A_\mu + \omega_{\mu}$, given a solution $A_\mu(x)$ above we can find another one, $A'^\mu$, such that $\partial_i A'^\mu = 0$, since

$$\partial_\mu A'^\mu = \partial_\mu A^\mu + \partial_\mu \omega = 0$$

provided the solution to

$$\omega = -\frac{1}{\partial_\mu} \partial_\mu A$$

exists.

This is generally the case. So we can look for solutions to (X) that satisfy both (X) and $\partial_i A = 0$. But the latter means the former is

$$\partial^\nu A_\mu = \frac{4\pi}{\sqrt{-g}} \to A_\mu = \frac{4\pi}{\sqrt{-g}} \frac{1}{\partial_\mu} \partial^\nu \phi$$

The procedure above is called "gauge fixing", and the field $A_\mu$ that satisfies $\partial_i A = 0$ is said to be in "covariant" or "Lorentz" gauge.
Investing differential operators, Green functions, Distributions.

We have seen a need for involving $\nabla^2 \phi$. Let's look at $\nabla^2$. For $\nabla^2$ we may as well put it in the important context of solving the Poisson Equation. This arises in electrodynamics (and in Newtonian gravitation).

Since $\vec{E} = -\nabla \phi$ for static fields, we have then:

Gauss's law $\nabla \cdot \vec{E} = 4\pi \rho \Rightarrow -\nabla^2 \phi = 4\pi \rho \Rightarrow \phi = -\int \frac{4\pi \rho}{\nabla^2} \, d\phi$.

Solve $\nabla^2 \phi = -4\pi \rho$.

"Solve" means given $\rho = \rho(x)$, determine $\phi(x)$; there are other related problems, in which we specify $\rho = \rho(x)$ and a boundary to the value under consideration, and specify a boundary condition. In our case, we want $\rho(x)$ with compact support and $\phi = \phi(x)$ everywhere in space, up to solutions to $\nabla^2 \phi = 0$.

$\rho$ has compact support, and is sufficiently well behaved, so $\int dx \rho^2 < \infty$, we can find its Fourier transform

$\hat{\rho}(k) = \int \frac{dx}{(2\pi)^3} e^{ik \cdot x} \rho(x) = \rho(x) = \frac{1}{(2\pi)^3} \int d^3k e^{-ik \cdot x} \hat{\rho}(k)$

Do this for $\phi$ as well. Then:

$\nabla^2 \phi = \frac{1}{(2\pi)^3} \int d^3k e^{ik \cdot x} \hat{\rho}(k)$

$\Rightarrow -\nabla^2 \phi = -4\pi \int d^3k \hat{\rho}(k)$

$\Rightarrow \phi(x) = -\frac{1}{4\pi} \int \frac{\hat{\rho}(k)}{k^2} = \frac{\hat{\rho}(k)}{k^2}$

and

$\phi(x) = \frac{\hat{\rho}(k)}{k^2} \Rightarrow \int d^3k e^{ik \cdot x} \frac{\hat{\rho}(k)}{k^2}$

We can express this back in terms of $\rho(x)$:

$\phi(x) = \frac{\hat{\rho}(k)}{k^2} \Rightarrow \int d^3k e^{ik \cdot x} \frac{\hat{\rho}(k)}{k^2} = \int dy G(x,y) \rho(y)$

Here $G(x,y)$ is a Green function, satisfying $\nabla^2 G(x,y) = \delta(y) - \delta(x)$

Where $\nabla^2$ means it is the Laplacian with respect to $x$ (to distinguish from $y$), and $\delta^{(3)}(x)$ is the 3D Dirac-delta function.
\[ \delta^{(3)}(x) = \delta(x) \delta(y) \delta(z), \quad \text{with} \quad \delta(x) = 0 \text{ for } x \neq 0 \text{ and } \int_{-\infty}^{\infty} \delta(x) \, dx = 1 \]

so that \[ \int_{-\infty}^{\infty} \delta^{(3)}(x) \, dx = 1 \]

Just like \[ \int_{-\infty}^{\infty} e^{iat} \, dt = 2\pi \delta(t) \], we have \[ \int d^3k \, e^{ik \cdot x} = (2\pi)^3 \delta^{(3)}(x) \]

Now, we need to take this up. From above

\[ G(r,\mathbf{\Sigma}) = \frac{4\pi}{(2\pi)^3} \int d^3k \, e^{ik \cdot (\mathbf{\Sigma} - \mathbf{x})} \frac{1}{k^2} \]

from which we learn that \( G(\mathbf{r},\mathbf{\Sigma}) = G(\mathbf{r}-\mathbf{\Sigma}) \) reflecting invariance

\[ \mathbf{r} \rightarrow \mathbf{r} + \mathbf{\Sigma} \quad \text{is} \quad \text{covariant vector of Minkowski equation} \]

Exercise: verify that \( G \) satisfies \( \nabla^2 G = -4\pi \delta^{(3)} \) by direct computation

Note that we can add to \( G \) any solution to the associated homogeneous equation.

If \( \nabla^2 F = 0 \) then \( \nabla^2 (G + F) = -4\pi \delta^{(3)} \) (provided \( \nabla \cdot F = 0 \)).

We'll take a brief look at \( \nabla^2 \phi = 0 \) and boundary value problems in electrostatics later.

**Physical interpretation:** compare \( \nabla^2 G(r,\Sigma) = -4\pi \delta^{(3)}(x-\Sigma) \) with \( \nabla^2 \phi = -4\pi \rho \)

\( \Rightarrow G \) is the electric potential due to a charge distribution \( \rho(x) = \delta^{(3)}(x-\Sigma) \)

That is \( \rho = 0 \) everywhere except at \( x = \Sigma \), and \( \int \rho \, dV = 1 \)

\( \Rightarrow \) a unit charge at \( \Sigma \).

But we know \( \phi = \frac{\rho}{4\pi} \) for charge at origin.

The constant of proportionality comes from Coulomb's law

the constant of proportionality comes from \( \int \nabla \cdot \mathbf{E} \, dV = \int \mathbf{E} \cdot d\mathbf{a} \) using \( \nabla \cdot \mathbf{E} = 4\pi \rho \)

so that \( \text{LHS} = 4\pi q \) and \( \mathbf{E} = -\nabla \phi \) with \( \mathbf{V} = \text{a sphere so RHS} = \int \hat{r} \, d\Omega \left( \frac{q}{4\pi r^2} \right) \left|_{r=R} \right. = 4\pi q \)

\( \Rightarrow \phi = \frac{q}{4\pi r} \) for charge \( q \) at \( \Sigma \). Setting \( q = 1 \), \( G(x,\Sigma) = \frac{1}{|x-\Sigma|} \)

**Exercise:**

(i) Check \( \frac{1}{|x|} = 0 \) for \( x \neq 0 \), and \( \int \frac{1}{|x|} \, dx = -4\pi \)

(ii) Verify by direct integration of \( G \). We get \( G(x) = \frac{1}{|x|} \).
Now, let's try to do for $\partial^2$ what we did for $\nabla^2$:

Let $-\partial^2 \phi = \mu \rho$ (which is Gauss's law in Lorentz gauge).

We try the same technique

$$\tilde{\rho}(k) = \int dx \ e^{ik \cdot x} \ \rho(x) \quad \text{when } k^\mu \text{ is a 4-vector, } k \cdot x = k^\mu x_\mu.$$  

Incidentally, $k \cdot x = k^0 x^0 - k^1 x^1 = \omega t - k \cdot x$ so $\omega = k^0 c$ has the interpretation of frequency (while $\mathbf{k}$ is still a wave-vector).

and

$$\rho(x) = \frac{1}{(2\pi)^4} \int d^4 k \ e^{ik \cdot x} \ \tilde{\rho}(k).$$

As before,

$$\partial^2 e^{ik \cdot x} = -k^2 e^{ik \cdot x}$$

and

$$\tilde{\phi}(k) = \int d^4 x \ G(x, y) \rho(y)$$

with $G(x, y) = G(x - y)$ and $\delta^4 G(x) = -4\pi \delta^4(y)(x).$

But there is one interesting twist: while formally

$$G(x) = \frac{\mu}{(2\pi)^4} \int d^4 k \ e^{ik \cdot x} \ \frac{1}{k^1} \Rightarrow \delta^4 G = \frac{\mu}{(2\pi)^4} \int d^4 k \ e^{ik \cdot x} \ (-k^1) \ \frac{1}{k^1} = -4\pi \delta^4(x),$$

the actual integral is ill-defined.

$$\frac{1}{k^1} = \frac{1}{(k^1)^2 - (k^0)^2}$$

Consider the $k^0$ integration:

$$\int \frac{d^4 k}{(k^1)^2 - (k^0)^2} e^{ik \cdot x}$$

Consider the complex variable integration $\int \frac{d z}{z^2 - R^2} e^{i z x}$

where $C$ is for $x > 0$ and $C'$ is for $x < 0$.

These choices guarantee $e^{i 2R x} = e^{-R |x|}$ on the circle and vanish exponentially fast as $R \to \infty.$
This allows us to analyze our integral using Cauchy’s theorem. The only problem is that there are poles at
\[ z = +|\beta| \quad \text{and} \quad z = -|\beta| \]
and the contour of integration goes through them.

So let’s propose the following: G is defined by a choice of deformation of the contour to go just above or just below the poles. There are 4 possibilities:

1. \( G_{\beta_+} \)
2. \( G_{\beta_-} \)
3. \( G_{\beta_0} \)
4. \( G_{\beta_0} \)

By Cauchy’s theorem, the size of the deformation does not matter. So it can be infinitesimal, and the resulting integrals should all give Green functions. The difference between any two should be a solution to the homogeneous equation. Easy to see. Take (1) - (2). Same as (1) followed by reversing direction if (2): cancel everywhere except at \( z = \pm |\beta| \)

But this is an easy integral:
\[
\int_{-\infty}^{\infty} \frac{e^{i\gamma z} e^{i\beta|x|}}{(z-\beta)(z+\beta)} \bigg|_{z=\pm |\beta|} = \frac{2\pi i e^{i|\beta|b}}{2|\beta|}
\]

and
\[
\Delta G = \frac{4\pi}{(1p)^2} \int d^3k \frac{e^{i\beta|x-k|}}{2|k|}
\]

This is a well defined integral and has \( \partial^2 (\Delta G) = 0 \). In fact

Exercise: if \( F(x) = \int d^3k \int (\mathcal{B}) e^{i\beta x - i\beta k} \) is well defined, show that

\( \partial^2 F = 0 \) provided \( \omega = \pm |\beta| \)
So all 4 Green functions are good candidates. But they have interesting physically distinguishing properties.

1. $G_{+}$. Since there are no poles inside $\gamma$ for $x^0$ corresponding to $x^0 > 0$, we have $G_{+} = 0$ for $x^0 > 0$. This is called the "Advanced" Green function.

2. $G_{-}$. Similarly $G_{-} = 0$ for $x^0 < 0$. "Retarded" Green functions appear naturally in Quantum field theory, "Feynman" Green functions and has the peculiar property that it can be deformed into an integral along imaginary axes. More of interest to us for now.

Before we talk about $G_{\text{Adv}}$ and $G_{\text{Ret}}$, explicitly, note physical interpretation:

Since $\varphi(x) = \int d^4 y \, G(x-y) \, \rho(y)$

we have $\varphi(x, t) = \int d^4 y \, G(x, y) \, \rho(y) = \int d^4 y \int d^4 \omega \, \rho(y, \omega) G_{\text{Ret}}(x-y, \omega) \rho(y, \omega)$

$\Rightarrow \varphi(x, t)$ depends on $\rho(x, \omega)$ only for $\omega$ before $t$, i.e., $\omega < t$. This makes sense for future evolution: one must know the history of the charge distribution in the past to predict the future.

Similarly $G_{\text{Adv}}$ gives $\varphi(x, t)$ how knowledge of the future charge distribution which can be of interest in, say, scattering if we know the future outcome.

Explicit computation.

We have $G_{\text{Ret}} = \frac{4\pi}{(2\pi)^4} \int \frac{d^3 k}{(2\pi)^3} e^{ikx} \int dz \, e^{iza} \frac{1}{z^2 - \omega^2}$

$$= \frac{4\pi}{(2\pi)^3} \int \frac{d^3 k}{|k|} e^{ikx} \left( \frac{e^{ikx} - e^{-ikx}}{2|k|} \right)$$

$$= \frac{i}{(2\pi)^3} \int \frac{d^3 k}{|k|} e^{-ikx} \left( e^{ikx} - e^{-ikx} \right)$$
\[ \text{Now } \int \frac{dk}{\sqrt{k}} e^{-i k^2} e^{i k \cdot x} = 2 \pi \left[ \int_0^\infty \frac{dk}{\sqrt{k}} e^{i k x} e^{i k x \cdot x} \right] \]

\[ = 2 \pi \int_0^\infty \frac{1}{i x} \left( e^{i k x} - e^{-i k x} \right) \]

Combining

\[ G_{\text{ret}} = \frac{i}{(2\pi)^2} \int \frac{dk}{i x} \left( e^{i k x} - e^{-i k x} \right) \left( e^{i k x} - e^{-i k x} \right) \]

\[ = \frac{1}{2 \pi} \int \frac{1}{i x} \left( e^{i k x} - e^{-i k x} \right) \left( e^{i k x} - e^{-i k x} \right) \]

\[ = \frac{1}{2 \pi} \int \frac{1}{i x} \left( 2 \delta \left( |x| - x^0 \right) + 2 \delta \left( i |x| - x^0 \right) \right) \]

and since this is only for \( x^0 > 0 \), \( \delta \left( |x| - x^0 \right) = 0 \) so

\[ G_{\text{ret}} (x, x^0) = \frac{\delta \left( x^0 - i |x| \right)}{i |x|} \]

**Ex:** Show that

\[ G_{\text{adv}} (x, x^0) = -\frac{\delta \left( x^0 + i |x| \right)}{i |x|} \]

A charge \( \delta \) at time \( t=0 \) at the origin, produces a potential \( \phi \) that propagates at the speed of light, spherically outward from the origin, with amplitude \( \frac{1}{i |x|} \)

Centered on that shell at \( |x| = ct \):

![Diagram of a charge distribution](image)

Cool! 😎 Moreover, easy to see real physical applications: start from charge distribution at rest and then have it start moving.
Action Integral, Lagrangian, Generation Laws

Obtaining the equation of motion (EOM) as extrema of an action integral $S$, is useful in various ways: exploring symmetries, quantization, extension to other models of reality, and so on.

One of the nice things about it (not emphasized in kindergarten) is that covariance of EOM under some transformation corresponds to invariance of $S$. This is powerful stuff: if you have a good reason to suspect nature is symmetric under a group $G$ of transformations, then make sure $S$ is invariant under $G$.

(Digression: for those of you who know group theory, the set of physical transformations on a system forms a group $G$, with multiplication given as one transformation followed by the other:

1. There is an identity element $e \in G$, the "do nothing" transformation.
2. For every transformation $g \in G$, there is an inverse $g^{-1} \in G$ such that $g \cdot g^{-1} = e$, the "undo what you did".
3. If $g \cdot e = g$, $g \cdot g = g \cdot g$).

An example is the EOM for a relativistic free particle (we'll add EM interactions later). If we write the world-line $x(s)$, we want a functional $S[x(s)]$ that is invariant under Lorentz transformations. It should also be independent of the way in which we parameterize world-line, i.e., it should be parametrization-invariant. We know the interval $ds$ is invariant, $ds^2 = \gamma ds^2$ and written this way it is independent of $\lambda$.

We therefore have our simplest invariant action integral

$$\int S[x(s)] = \int ds \ \text{for some constant}.$$

If we want to be more careful we can specify that this is for paths between $x_i$ and $x_f$ so that $x(0) = x_i$ and $x(1) = x_f$ are fixed end-points and write

$$\int S[x(0), x_f] = \int_{x_i}^{x_f} ds \sqrt{g}.$$

It often we leave $x_i$, $x_f$ implicit as understood. Note that $\sqrt{g} \ ds^2$ makes no sense (the square of an individual integral is zero).
For a path through \( x \), we have \( ds^2 = \mu^{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} = 0 \)

\[
S[x(t)] = \int_0^T dt \sqrt{\mu^{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt}}
\]

\[
M = S[x + \delta x] - S[x] = \int_0^T dt \left( \sqrt{\mu^{\alpha\beta} \left( \frac{dx^\alpha}{dt} + \frac{d\delta x^\alpha}{dt} \right) \frac{dx^\beta}{dt} \frac{d\delta x^\beta}{dt}} - \sqrt{\mu^{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt}} \right)
\]

where \( \delta x = \frac{d\delta x}{dt} \)

\[
= \int_0^T dt \, \mu^{\alpha\beta} \frac{dx^\alpha}{dt} \frac{d\delta x^\beta}{dt} \sqrt{\mu^{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt}}
\]

\[
= -M \int_0^T dt \, \delta x^\alpha \frac{d}{dt} \left[ \frac{\mu^{\alpha\beta} \frac{dx^\beta}{dt}}{\sqrt{\mu^{\alpha\beta} \frac{dx^\beta}{dt} \frac{dx^\beta}{dt}}} \right]
\]

So that the EOM is \( \frac{d}{dt} \left[ \frac{\mu^{\alpha\beta} \frac{dx^\beta}{dt}}{\sqrt{\mu^{\alpha\beta} \frac{dx^\beta}{dt} \frac{dx^\beta}{dt}}} \right] = 0 \) \( (1) \)

If we choose \( x = x^0 \), then this is \( \frac{d}{dt} \left( \frac{\mu^{00} \frac{dx^0}{dt}}{\sqrt{\mu^{00} \frac{dx^0}{dt} \frac{dx^0}{dt}}} \right) = 0 \) \( (2) \)

but no \( x^i \) equation because there is no variation \( dx^i \) of the trajectory (since \( x^0 = x^0(t) \) is fixed). Using \( x = x^0 \) we can connect to Lagrangian mechanics; generally \( S[\gamma(t_1, t_2)] = \int_{t_1}^{t_2} dt \, L(q^i, \dot{q}^i) \), Lagrangian, a function of two variables.

and EOM’s are \( \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} = 0 \) and momentum \( \frac{\partial L}{\partial q^i} = p^i \)

The relativistic particle has, from above

\( S = \mu \int_{t_1}^{t_2} \sqrt{1 - \beta^2} \, dt \), so \( L(q^i, \dot{q}^i) = \mu \sqrt{1 - \beta^2} \), \( p^i = \frac{\partial L}{\partial \dot{q}^i} = \frac{\mu}{\sqrt{1 - \beta^2}} \cdot \dot{q}^i \)

We see that with \( \nu = \frac{\mu}{\sqrt{1 - \beta^2}} \), this corresponds to momentum as found earlier (plus \( S \) has the proper dimensions). For EOM we have

\( \frac{d}{dt} \left( \frac{\mu}{\sqrt{1 - \beta^2}} \cdot \dot{q}^i \right) = 0 \) as above.

Note that this is just \( \frac{d}{dt} \frac{\mu}{\sqrt{1 - \beta^2}} = 0 \)
The non-relativistic limit \( E = mc^2 \sqrt{1-v^2/c^2} = -mc^2 \left( 1 - \frac{1}{2} \frac{v^2}{c^2} + \ldots \right) = mc^2 + \frac{1}{2} mv^2 + \ldots \)

\( mc^2 \) is an inessential, BUT never escape \( mc^2 \) constant; if we write \( E = T + V \)

where \( T = \frac{1}{2} mv^2 \) is kinetic and \( V \) is potential energies, then \( V = mc^2 \), which the student should recognize as the rest energy of a particle in special relativity.

The Hamiltonian (or energy, since there is no explicit time dependence) is

\[
H(q, p) = \frac{p^2}{2m} - L
\]

where \( q = q(q, p) \) is obtained from \( p = \frac{\partial L}{\partial \dot{q}} \).

In the present case \( H = H(q, p) \), so rewrite by solving \( \dot{q} = \dot{q} \) in terms of \( p \):

\[
H = \frac{p^2}{2m} \frac{1}{\left( \frac{1}{\sqrt{1-v^2/c^2}} \right)}
\]

\[
= \frac{mc^2}{\left( \frac{1}{\sqrt{1-v^2/c^2}} \right)} \left( \frac{1}{\sqrt{1-v^2/c^2}} \right) = \frac{mc^2}{\sqrt{1-v^2/c^2}} = E
\]

The Hamiltonian should be written in terms of \( p \),

\[
\frac{p^2}{2m} = \frac{mc^2}{\sqrt{1-v^2/c^2}} \left( \frac{1}{\sqrt{1-v^2/c^2}} \right) \Rightarrow \frac{p^2}{2m} = \frac{mc^2}{c^2} \Rightarrow \frac{p^2}{2m} = \frac{m^2 c^4}{c^2 + p^2} \Rightarrow \frac{1}{\sqrt{1-v^2/c^2}} = \frac{mc^2}{\sqrt{m^2 c^4 + p^2}}
\]

So \( E = \sqrt{(mc)^2 + p^2} \). No surprise!

Note that (Ω) above is \( \frac{dt}{dt} = c \).

The variation (functional derivative) defines the 4-momentum:

\[
\frac{\delta S}{\delta x^\mu} = -p^\mu \frac{\partial}{\partial x^\mu}
\]

This makes it clear that \( p^\mu \) transforms as a lower index 4-vector. Using \( \lambda = c \) the components of \( p^\mu \) are what we saw above:

\[
p^\mu = \left( \frac{E}{c}, \frac{p}{c} \right)
\]

so \( p^\mu \) form a 4-vector. Using the expression for \( E \) above, \( p^2 = E^2/c^2 - p^2 = (mc)^2 \).
Conservation Laws & Continuous Symmetries

Let us write generically

$$S[\chi] = \int \mathcal{L}(x^\mu, dx^\mu)$$

where $\mathcal{L}(x^\mu, u^\nu)$ is a function of these two variables. If we put matter where time $(\lambda = t)$ then $\mathcal{L}$ would correspond with the Lagrangean.

We can recover equations of motion by extremizing $S$:

$$\delta S = 0 \Rightarrow \int_0^1 \left[ \frac{\partial \mathcal{L}}{\partial x^\mu} \delta x^\mu + \frac{\partial \mathcal{L}}{\partial u^\nu} \delta u^\nu \right] d\lambda = 0$$

Here $\frac{\partial \mathcal{L}}{\partial x^\mu}$ and $\frac{\partial \mathcal{L}}{\partial u^\nu}$ are understood to be evaluated on $u^\nu = \frac{dx^\mu}{d\lambda}$. Integrating by parts:

$$\frac{\partial \mathcal{L}}{\partial x^\mu} \frac{dx^\mu}{d\lambda} \bigg|_0^1 - \int_0^1 \left[ \frac{\partial^2 \mathcal{L}}{\partial x^\mu \partial x^\nu} - \frac{\partial \mathcal{L}}{\partial u^\nu} \frac{d}{d\lambda} \frac{dx^\mu}{d\lambda} \right] dx^\nu = 0$$

The first term vanishes because $\delta x^\mu = 0$ at $\lambda = 0$,1. The second term must then vanish for any $\delta x^\nu = 0$ at $\lambda = 1$. This is a version of the Euler-Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial x^\mu} - \frac{d}{d\lambda} \left( \frac{\partial \mathcal{L}}{\partial u^\mu} \right) = 0$$

(EOM)

We define 4-momentum by

$$P^\mu = \frac{\partial \mathcal{L}}{\partial u^\mu}$$

Symmetries: suppose that $x^\alpha \to x^\alpha + e^{-\lambda} \delta x^\alpha$ is a symmetry of $S$, let $\delta S[\chi] = S[\chi'] - S[\chi]$ be infinitesimal.

Then

$$0 = \delta S[\chi] = \int d\lambda \left[ \frac{\partial \mathcal{L}}{\partial x^\mu} \delta x^\mu + \frac{\partial \mathcal{L}}{\partial u^\nu} \delta u^\nu \right]$$

Now, evaluating this for solutions of the EOM, $\frac{\partial \mathcal{L}}{\partial x^\mu} = \frac{d}{d\lambda} \left( \frac{\partial \mathcal{L}}{\partial u^\mu} \right)$ we have

$$0 = \int d\lambda \left[ \frac{d}{d\lambda} \left( \frac{\partial \mathcal{L}}{\partial u^\mu} \right) e^\lambda + \frac{\partial \mathcal{L}}{\partial u^\nu} \delta u^\nu \right] = \int d\lambda \left[ \frac{d}{d\lambda} \left( \frac{\partial \mathcal{L}}{\partial u^\mu} e^\lambda \right) + \frac{\partial \mathcal{L}}{\partial u^\nu} \delta u^\nu \right] = \frac{d}{d\lambda} \left[ \frac{\partial \mathcal{L}}{\partial u^\mu} e^\lambda \right]_0^1 = P^\mu e^\lambda \bigg|_0^1$$

or

$$P^\mu e^{\lambda}(\text{final}) = P^\mu e^0(\text{initial})$$

$$P^\mu e^{\lambda}(\text{final})$$ is arbitrary along world line $x^\alpha(\lambda)$, this just means $\boxed{P^\mu e^{\lambda}(\text{final}) = \text{const} \times e^{\lambda} \text{ along } x^\alpha(\lambda)}$
Translations: suppose \( x^\mu \rightarrow x^\mu + \xi^\mu \) with \( \xi^\mu = \text{constant} \) is a symmetry.

Thus \( p_\mu = \text{constant} \). Since \( \xi^\mu \) is an arbitrary fixed vector, this implies:

\[
p_\mu = \text{constant}.
\]

Conservation of momentum (as a result of translational invariance in space-time)

Angular-Momentum

Assume invariance under Lorentz transformations \( x'^\mu = \Lambda^\mu_\nu x^\nu \). The infinitesimal version is \( x'^\mu = x^\mu + \epsilon^\mu x^\nu \) with \( \epsilon_{\mu
u} = \epsilon_{\nu\mu} \quad (\epsilon_{\mu
u} = \eta_{\mu
u} \epsilon^\nu) \)

\[
\Rightarrow \quad \mu \epsilon^\nu x^\nu = \text{constant}.
\]

\[
= \quad \epsilon_{\nu} x^\nu p^\nu = \text{constant}.
\]

Since \( \epsilon_{\nu} \) is an arbitrary anti-symmetric tensor, it has 6 independent components that allow us to impose 6 independent conditions on \( x^\nu p^\nu \). Since

\[
x^\mu p^\nu = \frac{1}{2} \left( x^\mu p^\nu + x^\nu p^\mu \right) + \frac{1}{2} \left( x^\mu p^\nu - x^\nu p^\mu \right)
\]

and \( \epsilon_{\nu} x^\nu = 0 \) for any \( \nu \), it follows that the components of

\[
T^\mu_\nu = x^\mu p^\nu - x^\nu p^\mu
\]

are constrained by the above, \( T^\mu_\nu = \text{constant} \). This 2-index anti-symmetric tensor contains angular momentum: \( T^\mu_\nu = x^\mu \epsilon^\nu \quad \text{for} \quad \epsilon^\nu \quad \text{and} \quad L_\mu = \epsilon_{\mu\nu} \epsilon^\nu \)

or \( L = \epsilon_{\mu
u} p^\mu \), as usual. But what is \( T^\mu_\nu \)?

\[
T^\mu_\nu = x^\mu p^\nu - x^\nu p^\mu = \text{constant} \quad \Rightarrow \quad x^\mu = \left( \frac{p^\mu}{\gamma} \right) x^\mu + \text{constant}
\]

In the non-relativistic limit: \( p = \gamma m \) and \( x^\mu = \frac{p^\mu}{\gamma} \), an elementary operation.

For many particles this is interesting:

\[
T^\mu_\nu = \epsilon^\nu \Sigma^\mu = x^\nu \Sigma^\mu = \Sigma^\nu x^\nu = \Sigma^\nu \Sigma^\mu - \Sigma^\nu \Sigma^\nu = \Sigma^\nu \Sigma^\mu - \Sigma^\nu \Sigma^\nu
\]

Define the center of energy (cm) as:

\[
x_{cm}^\mu = \frac{\Sigma x^\nu x^\nu}{\Sigma x^\nu}
\]

and total energy \( p^0 = \Sigma \gamma \gamma \)

Thus \( x^\mu p^\nu = x^\mu p^\nu \) or

\[
x_{cm}^\mu = \left( \frac{p^\mu}{\gamma} \right) x^\mu = \frac{p^\mu}{\gamma} t
\]

This is the relativistic generalization of center of mass.
Interactions with $E,M$ fields.

The non-relativistic limit of a point charge interacting with (i.e., moving under the effect of a) electric field described by a potential $\phi = \phi(x)$ is described by a Lagrangian

$$L = \frac{1}{2} m \dot{\mathbf{v}}^2 - q \phi(x) \quad \text{(recall $L = T - V$)}$$

Relativistic generalization: $\frac{1}{2} m \dot{\mathbf{v}}^2$, we have seen, is how $-mc \sqrt{1 - \frac{v^2}{c^2}}$.

Now, we want $S$ to be invariant under Lorentz transformations. The kinetic term is explicitly invariant if we write again

$$S = -mc \int \sqrt{\eta_{\mu\nu} \eta^{\nu\mu}}$$

with $\mathbf{v}^\mu = \frac{dx^\mu}{dt}$.

But $\phi = \phi^\mu A^\mu$ is the 0-th component of a 4-vector. Now

$$v_0 = \frac{dx^0}{ds} \Rightarrow U^0 = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad v^i = \frac{v^i}{\sqrt{1 - \frac{v^2}{c^2}}}$$

In the NR limit $v^0 = 1$ and $v^i = v^i$, so we can write

$$-q U^\mu A_\mu = -q \phi + \phi(x)$$

The relativistic generalization is then

$$S = \int dt \left[ -mc \left( \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \right) \right]$$

or simply

$$S = \int mc ds - \frac{q}{e} \int dx^\mu A^\mu$$

$$= \int dx^\mu \left[ mc \sqrt{\eta_{\mu\nu} \eta^{\nu\mu}} - \frac{q}{e} U^\mu A^\mu \right]$$

Momentum conjugate, equation of motion (COM)

$$S = \int dt \left[ mc \sqrt{1 - \frac{v^2}{c^2}} - \frac{q}{e} \frac{dx^\mu A^\mu}{dt} \right] = L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} + \frac{q}{e} \mathbf{v} \cdot \mathbf{A} - q \phi$$

we find the canonical momentum $\mathbf{p}$:

$$p^\mu = \frac{\partial S}{\partial \dot{v}^\mu} = \dot{v}^\mu + \frac{q}{e} A^\mu$$

or

$$\mathbf{p} = \dot{\mathbf{v}} + \frac{q}{e} \mathbf{A}$$

at rest

$$N_{\mu\nu} \mathbf{H} = \mathbf{p} \cdot \mathbf{E} - \mathbf{E} \cdot \mathbf{p} = -\mathbf{p} \cdot \mathbf{E} + \frac{q}{e} \mathbf{v} \cdot \mathbf{A} - \frac{q}{e} \mathbf{v} \cdot \mathbf{A} + q \phi$$
Writing \( \mathbf{p} = \mathbf{p} - \frac{\mathbf{q}}{c} \), this is:

\[
H = \mathbf{p} \cdot \mathbf{v} + mc^2 \sqrt{1 - \frac{v^2}{c^2}} - qA^0(\mathbf{x}, t)
\]

The first two terms are as in the free case, so we can write immediately:

\[
H = \sqrt{\mathbf{p}^2 + (mc)^2} + qA^0
\]

At low velocity,

\[
L = \frac{1}{c} \mathbf{mv} + q \mathbf{v} \cdot \mathbf{A} - qA^0,
\]

and

\[
H = \frac{1}{c} \mathbf{mv}^2 - \frac{q}{c} \mathbf{v} \cdot \mathbf{A} + qA^0
\]

For the COM, we already have \( \frac{\partial L}{\partial v} = \mathbf{F} = \mathbf{F} + \frac{q}{c} \mathbf{A} \).

But now we also need \( \frac{\partial L}{\partial x} \). Recall that \( A^0 = A^0(x^0) \) and \( A^1 = A^1(x^1) \), so

we have:

\[
\frac{\partial L}{\partial x^1} = \frac{q}{c} \mathbf{v} \cdot \frac{\partial \mathbf{A}}{\partial x^1} = q \frac{\partial A^0}{\partial x^1}
\]

So we have:

\[
\frac{d}{dt} (\mathbf{p}^1 + \frac{q}{c} \mathbf{A}^0) = \frac{q}{c} \mathbf{v} \cdot \frac{\partial \mathbf{A}^0}{\partial x^1}
\]

We want an equation for \( \frac{d \mathbf{p}^0}{dt} \) (\( \mathbf{p}^0 \) is conserved in the \( \mathbf{v} \cdot \mathbf{A} = \mathbf{v} \cdot \mathbf{A}^0 \)). Now \( \mathbf{A} = \mathbf{A}(x^0, t) \) so

\[
\frac{d \mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \frac{\partial \mathbf{A}}{\partial x^0} = \partial_t \mathbf{A} - \mathbf{v} \frac{\partial A^0}{\partial x^1}
\]

implies:

\[
\frac{d \mathbf{p}^0}{dt} = \frac{q}{c} \left( \mathbf{v} \cdot \frac{\partial \mathbf{A}^0}{\partial x^1} - \frac{q}{c} \frac{\partial A^0}{\partial x^1} \right) = \frac{q}{c} \mathbf{v} \cdot \frac{\partial \mathbf{A}^0}{\partial x^1}
\]

or:

\[
\frac{d \mathbf{p}^0}{dt} = q (\mathbf{v} \cdot \mathbf{A} + \frac{\partial A^0}{\partial x^1})
\]

(Check of curl: \( \partial_{x^2} A^1 - \partial_{x^1} A^2 = \epsilon_{ijk} \epsilon_{x\nu} \partial_{x^j} A_{\nu} = \epsilon_{ijk} (g_{i\nu} A^\nu) \). Then \( \epsilon_{ijk} v^j g_{i\nu} A^\nu \neq 0 \).

One can check by direct computation:

\[
\frac{d E_{\text{kin}}}{dt} = \mathbf{v} \cdot \mathbf{E} \quad (\text{we know from } \mathbf{v} \cdot \frac{\partial \mathbf{p}^0}{\partial t} = 0 \text{ where } E_{\text{kin}} = \frac{\mathbf{v}^2}{2m})
\]

so:

\[
\mathbf{v} \cdot \frac{\partial \mathbf{A}}{\partial t} = \mathbf{v} \cdot \frac{\partial \mathbf{A}}{\partial x^1} + \mathbf{v} \cdot \frac{\partial \mathbf{A}}{\partial x^0} = \frac{d E_{\text{kin}}}{dt} = \mathbf{v} \cdot \mathbf{E} = q \mathbf{v} \cdot \mathbf{E}
\]

\( \)
Motion in constant crossed $\vec{E}, \vec{B}$ fields

Warm-up: $\vec{E} = 0$, $\vec{B}$ = constant

We know this from introductory courses: $d\vec{v}/dt = q\vec{E} + \frac{q}{c^2} \frac{d\vec{v}}{dt} \times \vec{B}$

except now $\vec{p} = \frac{mv}{\sqrt{1 - v^2/c^2}}$. However $\frac{d\vec{v}}{dt} = q\vec{E}/c - \vec{v} \times \vec{B}$, $\vec{v} = \text{constant}$

so $\frac{mv}{\sqrt{1 - v^2/c^2}} \frac{d\vec{v}}{dt} = \frac{mv}{\sqrt{1 - v^2/c^2}} \frac{q}{c} \frac{\vec{v} \times \vec{B}}{c}$

$\Rightarrow \vec{v}$ has a constant component along $\vec{B}$, plus it has a circular trajectory in the plane $\perp \vec{B}$.

Now consider $\vec{E} + \vec{B}$, with $\vec{E} \perp \vec{B}$, both uniform and constant.

It is useful to first consider this in a frame moving in the direction of $\vec{E} \times \vec{B}$. Let $\vec{E} = (0, E, 0)$ and $\vec{B} = (0, 0, B)$, and consider boost to $\vec{P} = (\beta, 0, 0)$ direction $\Lambda = (\gamma, \vec{v}/c)$ with $\gamma = \frac{1}{\sqrt{1 - \beta^2}}$ We had computed

$E'' = E_0$, $\vec{E}'' = \vec{E} - s \vec{B}$, $E''_3 = cE^3 - 5B^2 = 0$

$\vec{B}' = 0$, $\vec{E}' = c\vec{B} + s \vec{E}$, $\vec{B}'^3 = cB^3 - 5E^2$

So $\vec{E}'$, $\vec{B}'$ are still $\perp$, and along $x^1$, $x^3$ axes, respectively. Now note that we can choose a frame

$E'^2 = 0$ provided $\beta = \frac{E}{B}$ which must satisfy $\beta < 1$, if $\beta \sim 1$, provided $|E| < |B|$.

Similarly, $\beta^2 = 0$ in frame $\beta = \frac{E}{B}$, provided $|B| < |E|$. Motion

If $|E| < |B|$. In $K'$ frame, $\vec{E}' = 0$, $\vec{E}$ = constant $\Rightarrow$ circular motion in plane $\perp \vec{B}$; in $K'$ frame we have the boost of this motion

$\vec{v} = \sqrt{\vec{v}^2} \frac{\vec{P} - \vec{E} \vec{B}}{B} $ "$\vec{E} \vec{B}$ drift" 

($\vec{P}$ : "$\vec{E} \vec{B}$ drift velocity").

If $|B| < |E|$. In $K'$ frame, $\vec{B}' = 0$, $\vec{E}' = s\vec{B}$ but. Rectilinear acceleration in $K'$
Consider any one particle entering field region with velocity \( \vec{v} \). We can use our results if we first go to rest frame of particle, \( K' \). Our frame is \( K' \).

And \( K' \) is the frame \( \vec{B'} \) (or \( \vec{E'} \)) field have determined above.

Path is out of here. Jackson says only if \( \frac{\vec{v}}{c} \approx \frac{\vec{E}}{B} \) will the particle travel without deflection.