Electric and magnetic phenomena are described by Maxwell's equations:

\[
\nabla \cdot \mathbf{E} = \frac{4\pi}{c} \rho \quad \text{(1)}
\]

\[
\nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \frac{\mathbf{J}}{c} \quad \text{(2)}
\]

\[
\nabla \cdot \mathbf{E} + \frac{1}{c^2} \frac{\partial B}{\partial t} = 0 \quad \text{(3)}
\]

\[
\nabla \times \mathbf{B} = 0 \quad \text{(4)}
\]

(given in Gaussian units - more on this later) plus the Lorentz force equation

\[
\mathbf{F} = q (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad \text{(5)}
\]

In these \( \mathbf{E} = \mathbf{E}(x,t) \) and \( \mathbf{B} = \mathbf{B}(x,t) \) are fields - which I am displaying explicitly just to make sure we agree on notation,

\( \rho = \rho(x,t) = \text{electric charge density} \)

\( j = j(x,t) = \text{electric current density} \)

\( c = \text{speed of light} = 2.99 \times 10^8 \text{ m/s} \)

\( \mathbf{v} = \text{velocity of charge - q particle} \)

\( \mathbf{F} = \text{force due E, B fields on said particle} \)

And \( \mathbf{E} = (E_x, E_y, E_z) \) etc are 3-vectors (we'll distinguish 3-vectors from other 4-vectors)

\[
\nabla = (\partial_x, \partial_y, \partial_z) = (\partial_x, \partial_y, \partial_z)
\]

so that on a function \( f(x,y,z) \), \( \nabla f = \text{grad}(f) \) is a vector \( (\partial_x f, \partial_y f, \partial_z f) \) (gradient).

On a vector \( \mathbf{E} \), \( \nabla \cdot \mathbf{E} = \text{div}(\mathbf{E}) \) is a scalar (i.e., pure number) = \( \partial_x E_x + \partial_y E_y + \partial_z E_z \) (divergence).

And on a vector \( \mathbf{B} \), \( \nabla \times \mathbf{B} = \text{curl}(\mathbf{B}) \) is a vector: \( (\partial_y B_z - \partial_z B_y, \partial_z B_x - \partial_x B_z, \partial_x B_y - \partial_y B_x) \) (curl).

We start our discussion of Electrodynamics by exploring two key aspects of eqns (1)-(5):

(i) They are invariant under Lorentz transformations.

(ii) The fundamental dynamical variables are fields.

We will look at these together, moving back and forth between them. We will make contact with the more familiar aspects of special relativity (e.g., boosts on point particles) only at the end, for completeness.
Spacetime

Fields are functions of space and time. This in itself does not require us to think of space and time as part of the same continuum “space-time.” It is the invariance of Eqs. (1)-(5) under Lorentz transformations, and that these mix space and time, that lead us to consider space and time on an almost equal footing.

Warm-up: Relativity and space.

Points in space are accounted for with coordinates \( \mathbf{X} = (x, y, z) \in \mathbb{R}^3 \). There is much arbitrariness in how this is done. Given a coordinate system \( \mathbf{X} \), one can define a new one \( \mathbf{X}' = \mathbf{X}(\mathbf{R}) \) (3 function of 3 variables), with the obvious constraint that they map 1-to-1, reversible (if you know about manifolds and differential geometry, this can be done in patches). But we’d like to focus on coordinates we can assign with a meter stick; call them “Carthagian.”

Given one such coordinate system, others are obtained by:
- translations \( \mathbf{X}' = \mathbf{X} + \mathbf{a} \) (3 hard vectors)
- rotations \( \mathbf{X}' = \mathbf{R} \mathbf{X} \) \( \mathbf{R} \) = orthogonal matrix

Here is another look at this. In our “ruled” space distance is given by

\[
\begin{align*}
\Delta s^2 &= dx^2 + dy^2 + dz^2 = \sum_{j=1}^{3} (dx_j)^2 \\
\Delta s^2 &= \left( \frac{c}{\gamma} \right)^2 (\frac{c}{\gamma} \Delta t_j)^2
\end{align*}
\]

OK, real distance between 2-points, \( P_i \) and \( P_j \), is given by

\[
\sqrt{\int_{P_i}^{P_j} ds} \text{ along a straight line} \quad \text{or, equivalently, min of all paths } \int_{P_i}^{P_j} \frac{ds}{\gamma}.
\]
Question: what is the set of transformations \( \mathbf{\Sigma} \) that preserve the form of (6)?

\[
\frac{ds^2}{dx^{\mu}} = \frac{2}{\gamma} (dy^i)^2 = \frac{2}{\gamma} (dy^j)^2.
\tag{7}
\]

Since \( \mathbf{\Sigma} \rightarrow \mathbf{\Sigma}(x) \), we have

\[
\sum_{\alpha=1}^{3} \frac{2}{\gamma} \frac{d\gamma}{dx^\alpha} dx^\alpha = \sum_{\alpha=1}^{3} \frac{2}{\gamma} \frac{dy^\alpha}{dx^\alpha} dx^\alpha
\]

One can show

\[
\sum_{\alpha=1}^{3} \frac{2}{\gamma} \frac{dy^\alpha}{dx^\alpha} dx^\alpha = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \text{ (Kronecker delta)} \end{cases}.
\tag{7}
\]

Only if \( \mathbf{\Sigma} \rightarrow \mathbf{\Sigma}(x) \) is a linear transformation, \( \mathbf{\Sigma} \rightarrow \mathbf{\Sigma}(x) = a^i + \mathbf{R}^j_\alpha x^\alpha \)

where \( \mathbf{R}^j_\alpha \) a 3x3 matrix of numbers and \( a^i \) a 3-vector of numbers

(by “numbers” we mean constants, independent of \( x^\alpha \)).

Moreover it is required that

\[
\sum_{\alpha=1}^{3} R^j_\alpha R^\alpha_i = \delta_{ij}
\tag{8}
\]

which follows directly from (7).

Note: the reason for the peculiar upper/lower indices will become clear soon.

**Condition (8) defines “orthogonal” matrices.**

It is convenient to introduce a *metric tensor* \( g_{ij} \) so that

\[
\frac{ds^2}{dx^{\mu}} = \sum_{\alpha=1}^{3} \frac{2}{\gamma} g_{ij} dx^i dx^j
\tag{9}
\]

Of course, \( g_{ij} = \delta_{ij} \) in our Cartesian system. But we have already seen that if we go oblique with coordinate choices, \( y^i = y^i(x^\alpha) \) then

\[
\frac{ds^2}{dx^{\mu}} = \sum_{\alpha=1}^{3} \frac{2}{\gamma} g_{ij} dx^i dx^j \quad \text{where} \quad g_{ij} = \sum_{\alpha=1}^{3} \frac{2}{\gamma} \frac{dy^i}{dx^\alpha} \frac{dy^j}{dx^\alpha}
\tag{10}
\]

(we used \( x^i = x^i(y) \) as the inverse of \( y = y(x) \)). This can be convenient! We can relate the metric tensors in coordinate systems that are not Cartesian.

**Examples follow!**
Cylindrical coordinates:

\[ x = \rho \cos \phi \]
\[ y = \rho \sin \phi \]
\[ z = z \]

\[ ds^2 = (d\rho)^2 + \rho^2 (d\phi)^2 + (dz)^2 \]

Spherical coordinates:

\[ x = r \sin \theta \cos \phi \]
\[ y = r \sin \theta \sin \phi \]
\[ z = r \cos \theta \]

Exercise: verify \( ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \)

We will make this useful momentarily.

But let us go back to rotations: consider \( x^i \rightarrow R_{ij} x^j \) (this is often going to be shorthand, or short-speech, for let \( x^i = R_{ij} x^j \) in a function \( f(x) \)):

\[ ds^2 = \sum_{j,k} M_{jk} dx^j dx^k \rightarrow \sum_{j,k} m'_{jk} \left( \sum_{i} R^i_j dx^i \right) \left( \sum_{i} R^i_k dx^i \right) = \sum_{j,k} m'_{jk} dx^j dx^k \]

where \( m'_{jk} = \sum_{i} R^i_j R^i_k m_{ij} \)

The invariance condition \( (7) \) is now \( m'_{ij} = m_{ij} \)

Notice that since \( m_{ij} = \delta_{ij} \), condition \( (13) \) is just the same as \( (8) \). But it tells us something interesting: orthogonal transformations are those that leave the metric tensor invariant under transformations \( M \rightarrow M' \) given in \( (13) \).
Einstein convention:
Repeated indices are presumed summed over their understood range unless otherwise stated.

\[ y^i = R^i_{\ j} x^j \quad \text{stands for} \quad y^i = \sum_{j=1}^{2} R^i_{\ j} x^j \]

and
\[ ds^2 = g_{ij} dx^i dx^j \]

Some how we even simply like indices, \( y = R x \) means \( y^i = R^i_{\ j} x^j \)
And
\[ R^T g R = g \]
characterizes transformations that leave metric invariant
\[ \text{and} \quad g = 1 \quad \text{means} \quad R R = 1, \quad \text{a more familiar condition for orthogonal matrix.} \]

Physics:
let's consider a rotation on Maxwell's equations. Start from Gauss's law (4).
\[ \nabla \cdot \mathbf{E} = \mu_0 \rho \]
or rather
\[ \partial_\alpha E^\alpha = \mu_0 \rho \]
where \( \partial_\alpha = \frac{\partial}{\partial x^\alpha} \), and the lower index on \( \partial_\alpha \), upper on \( x^\alpha \), will be explained later.

Consider a coordinate change \( y^i = y^i(x^j) = R^i_{\ j} x^j \)
We want to show that there is a matrix function of \( R \), say \( O(R) \), such that \( E \) is invariant under
\[ E'^\alpha(y^\alpha, t) = O^\alpha_{\ \beta}(x) E^\beta(x, t) = O^\alpha_{\ \beta}(x) E^\beta(R x, t) \]

Since \( t \) is going along for a ride, I will omit below, "Form invariant" or plainly "invariant" means
\[ \partial_\alpha E^\alpha = \partial_\alpha (\mu_0 \rho) = 0 \]

To see \( \alpha j \) is the case, and infer \( O(R) \), compute the divergence:
\[ \partial_\alpha E^\alpha = \partial^\alpha \partial_\beta E^\beta(R^\gamma_{\ \delta}) = \partial^\alpha \partial_\beta (R^\gamma_{\ \delta} (2 E^\epsilon(x))) |_{x^\gamma = 0} = \partial^\alpha (R^\gamma_{\ \delta}) |_{x^\gamma = 0} \partial_\beta E^\epsilon \]

We want this to equal \( \mu_0 \rho \) so \( \partial^\alpha (R^\gamma_{\ \delta}) |_{x^\gamma = 0} = \partial_\beta E^\epsilon |_{x^\gamma = 0} \). Comparing we see that we need
\[ (R^\gamma_{\ \delta}) = 0 \Rightarrow \partial^\alpha = \partial^\alpha \]
\[ \text{or in matrix notation} \quad \text{O} = \mathbb{1} \]
\[ \text{(So indeed, O is a function of R, namely O(0) = R.)} \]
Recap: Eq. (1) is invariant under the change of coordinates ("rotations")
\[ \mathbf{\mathbf{\hat{y}}} = R \mathbf{\mathbf{\hat{x}}} \]
If in the new coordinate system \( \mathbf{\mathbf{E}}'(\mathbf{\mathbf{\hat{y}}}) = R \mathbf{\mathbf{E}}(\mathbf{\mathbf{\hat{x}}}) \) and \( \mathbf{\mathbf{\hat{p}}}'(\mathbf{\mathbf{\hat{y}}}) = p(\mathbf{\mathbf{\hat{x}}}) \).
Or simply, (1) is invariant under \( \mathbf{\mathbf{E}}(\mathbf{\mathbf{\hat{y}}}) = R \mathbf{\mathbf{E}}(\mathbf{\mathbf{\hat{x}}}) \) and \( \mathbf{\mathbf{p}}'(\mathbf{\mathbf{\hat{y}}}) = p(\mathbf{\mathbf{\hat{x}}}) \).

We say that \( \mathbf{\mathbf{E}} \) is a **vector** because it transforms under rotations just like \( \mathbf{\mathbf{\hat{x}}} \) does.
(namely \( \mathbf{\mathbf{x}} \rightarrow R \mathbf{\mathbf{x}}, \mathbf{\mathbf{E}} \rightarrow R \mathbf{\mathbf{E}} \)).

We say that \( p \) is a **scalar**: it transforms under rotations just as \( ds^2 \) does.
(namely \( \mathbf{\mathbf{p}} \rightarrow p \)).

**Exercise**: show that with \( c \) a scalar, and \( \mathbf{\mathbf{\hat{B}}}, \mathbf{\mathbf{\hat{p}}}, \mathbf{\mathbf{\hat{v}}} \) and \( \mathbf{\mathbf{\hat{E}}} \) vectors, eqs. (2)-(5) are also invariant.

Maxwell equations are invariant under rotations. This may seem trivial, particularly since we have written them in an explicitly covariant notation.
That is, once we know that

- (i) Dot products are **scalars** (i.e., do not transform)
- (ii) \( \mathbf{\mathbf{\hat{B}}} \) is a **vector**
- (iii) Cross products of vectors are vectors

we can "see" that each equation is invariant because both sides of the equality transform in the same way, e.g.

\[
\nabla \times \mathbf{\mathbf{\hat{B}}} - \frac{1}{c^2} \frac{\partial \mathbf{\mathbf{\hat{E}}}}{\partial t} = \frac{\mathbf{\mathbf{\hat{J}}}}{c}.
\]

Our aim is to show that Eqs. (1)-(5) are invariant under a bigger set of transformations, namely Lorentz transformations (plus translations in space-time, but these are already explicit). The problem is much simpler if we can make the symmetry explicit, i.e., we just showed for rotations in Ampère's law (Eq. (2)) we have been going through explicit computations as we did under Gauss's law (and I proposed an exercise for Eqs. (2)-(5)).
Before leaving rotations, let’s use the technique we developed to derive a couple of useful equations. 

When going to curvilinear coordinates (as, spherical) we have to be more careful in defining vectors. It is not necessarily true that new coordinates \( \tilde{y}^i = \gamma^i (\tilde{x}) \) transform as vectors. What is always true is that the infinitesimal displacement between two points is a vector, and the vector transformation is given by
\[
\delta \tilde{y}^i = \frac{\partial \tilde{y}^i}{\partial x^j} \delta x^j
\]
which corresponds to \( \delta y^i = R^i_j \delta x^j \) for transformations \( \tilde{y} = \gamma(x) \) of the form \( y = R x \).

The dot product of two vectors \( \tilde{a} \cdot \tilde{b} \) is \( \tilde{a} \cdot \tilde{b} = \tilde{a}_i \tilde{b}^i \).
Recall \( ds^2 = g_{ij} dx^i dx^j \) has
\[
g_{ij} = \frac{\partial x^k}{\partial x^i} \frac{\partial x^l}{\partial x^j} g_{kl}
\]
(14)

So if \( \tilde{a}^i = \frac{\partial x^k}{\partial x^i} a^k \) for vectors, then
\[
\tilde{a} \cdot \tilde{b} = g_{ij} a^i b^j = \left( \frac{\partial x^k}{\partial x^i} \frac{\partial x^l}{\partial x^j} g_{kl} \right) \left( \frac{\partial x^m}{\partial x^i} a^m \right) \left( \frac{\partial x^n}{\partial x^j} b^n \right) = g_{ij} a^i b^j = \tilde{a} \cdot \tilde{b}
\]

Consider the gradient \( \partial \tilde{\phi} \) of a scalar function, \( \tilde{\phi} = \phi (x) \).
\[
\partial \tilde{\phi} = \frac{\partial \phi}{\partial x^i} = \frac{\partial x^i}{\partial x^j} \frac{\partial \phi}{\partial x^j}
\]
And, This is NOT a vector! \( \partial \tilde{\phi} = \frac{\partial x^i}{\partial x^j} \frac{\partial \phi}{\partial x^j} \) instead of \( \partial \tilde{\phi} = \frac{\partial \phi}{\partial x^i} \).

In the language of differential geometry there are “1-forms” (spoken “one forms”).
In old physics language
\[
a^i = \text{contra-variant vector}
\]
\[
\partial \tilde{\phi} = \text{co-variant vector}
\]

Any \( a_i \) that transforms as \( a_i = \frac{\partial \phi}{\partial y_j} \) is a 1-form (as a covariant vector).

Note that \( a^i a^j = a_j a^i \), so the “contraction” of a 1-form and a vector is a scalar (somewhat misused to define 1-forms).

Note furthermore that \( \partial \phi = g_{ij} a^i \) transforms as (and therefore is) a 1-form.

This is why we have differentiated between upper and lower indices.
Given a 1-form $a^i$, can I make a vector $a^j$? Yes!

Let $g^{ij}$ denote the inverse matrix to $g_{ij}$, so that

$$g^{ij} g_{jk} = \delta^i_k \quad (g^{-1} g = I)$$

Then

$$a^i = g^{ij} a_j$$

is a vector.

**Exercise:** Prove the above assertion.

**Invariant integrals.**

Consider

$$\int_V \sqrt{g} \, d^3x = \int_V dx \, dy \, dz = \int_V \sqrt{J} \, dx$$

In changing variables to curvilinear coordinates $\mathcal{J} = \mathcal{J}(x)$

$$\int_V d^3x = \int_V dy \, J$$

$$J = \text{Jacobian} = \left| \det \left( \frac{\partial x^i}{\partial y^j} \right) \right|$$

Recall, if $ds^2 = g_{ij} \, dx^i \, dy^j = M_{ij} \, dx^i \, dy^j$ then (eq.(13)): $g_{ij} = \frac{\partial x^k}{\partial y^i} \frac{\partial x^l}{\partial y^j} \, M_{kl}$

That is $g = \det g_{ij} = J^2 \, det \eta \quad (\text{and } \det \eta = 1, \, h = \text{let's keep it explicit for now})$

So

$$\int_V \sqrt{g} \, d^3x = \int_V \sqrt{J} \, d^3y$$

is the invariant integration volume.

**Examples:**

(a) Cylindrical: $g = \det \left( \begin{array}{cc} 1 & \rho \sin \theta \\ 0 & \rho \cos \theta \end{array} \right) = \rho^2 \Rightarrow \text{vol} = dp \, d\theta \, dz = \rho \, dp \, d\theta \, dz$

(b) Spherical: $g = \det \left( \begin{array}{cc} 1 & r \sin \theta \\ 0 & r \cos \theta \end{array} \right) = r^2 \sin \theta \Rightarrow \text{vol} = dr \, d\theta \, d\phi = r^2 \sin \theta \, dr \, d\theta \, d\phi$
Divergence in curvilinear coordinates
looks a priori messy, but not much.
Consider \[ \int d^4 \mathbf{a} \partial_\mu \mathbf{a}_\mu \] clearly invariant under coordinate transformation.
Take \( a^0 \) to vanish at spatial \( x \) and integrate by parts (in fact, we will want \( p \) to have this support).

\[
\int d^3 \mathbf{a} \partial_\mu \mathbf{a}_\mu = - \int d^3 \mathbf{a} \partial_\mu \mathbf{a}_\mu (\mathbf{a}_0) = - \int d^4 \mathbf{a} \left[ \frac{1}{\sqrt{\mathbf{a}^2}} \partial_0 (\sqrt{\mathbf{a}^2}) \right]
\]

\[ \text{Invariant} \Rightarrow \text{must be invariant} \]

Moreover, comparing to Cartesian coordinates this invariant is what we called \( \text{div}(\mathbf{a}) \).

So

\[
\text{div}(\mathbf{a}) = \frac{1}{\sqrt{\mathbf{a}^2}} \partial_\mu (\sqrt{\mathbf{a}^2} \mathbf{a}_\mu)
\]

Examples:

1. Cylindrical: \( \text{div}(\mathbf{a}) = \frac{1}{r} \partial_r (r a_r) + \partial_\theta a_\theta + \partial_z a_z \)

2. Spherical: \( \text{div}(\mathbf{a}) = \frac{1}{r^2} \partial_r (r^2 a_r) + \frac{1}{r \sin \theta} \partial_\theta (\sin \theta a_\theta) + \partial_z a_z \)

Note: This result differs from many textbooks (e.g., Jackson or Gary).
The reason is their meaning for \( a^i \) is different—mine is better :)

Suppose you want to write \( \mathbf{a} = A \hat{a} \), where \( \hat{a} \) are orthonormal vectors in the Cartesian sense. For example, for cylindrical coordinates (not the \( Z \)-direction):

\[
\begin{align*}
\hat{e}_r \cdot \hat{e}_r &= 1, \\
\hat{e}_\theta \cdot \hat{e}_\theta &= 1, \\
\hat{e}_r \cdot \hat{e}_\theta &= 0
\end{align*}
\]

Then, for example \( \hat{a} \cdot \hat{a} = g_{ij} a^i a^j = A^i A^j \hat{a}_i \hat{a}_j = A^i A^j \delta_{ij} \). Since \( g_{ij} \) is diagonal.

(a condition for this to work) we have \( \mathbf{a}^i = \mathbf{e}^i a_i \) (no sum on \( i \)) and

\[
\text{div}(\mathbf{a}) = \frac{1}{\sqrt{\mathbf{a}^2}} \partial_\mu (\sqrt{\mathbf{a}^2} \mathbf{a}_\mu)
\]

\[
\text{cylindrical:} \quad \frac{1}{r} \partial_r (r A^r) + \frac{1}{r} \partial_\theta (A^\theta) + \partial_z A_z
\]

\[
\text{spherical:} \quad \frac{1}{r^2} \partial_r (r^2 A^r) + \frac{1}{r \sin \theta} \partial_\theta (\sin \theta A^\theta) + \frac{1}{r \sin \theta} \partial_\phi A^\phi
\]
Lesson: make sure you know what your symbols mean (especially when you use formulas from the back flap of a textbook).

**Laplacian**

A simple extension of the previous exercise: use $a^2 = g_{ij} \partial^2 f$ where $f$ is a scalar. Then

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{1}{\sqrt{g}} \partial_i (g^{ij} \partial_j f)$$

In Cartesian coordinates $\nabla^2 f = \eta^{ij} \partial_i \partial_j f = (\partial_x^2 + \partial_y^2 + \partial_z^2) f$

**Exercise:**

Cylindrical: $\nabla f = \frac{1}{r} \partial_r (r \partial_r f) + \frac{1}{r} \partial_{\theta} \partial_{\theta} f + \partial_z^2 f$

Spherical: $\nabla f = \frac{1}{r^2} \partial_r (r^2 \partial_r f) + \frac{1}{r^2 \sin \theta} \partial_{\theta} \partial_{\theta} f + \frac{1}{r^2 \sin^2 \theta} \partial^2 f$

Tensors, invariants and the peculiar cross product.

Restrict attention to rotations, $Y^i = R^i_j \ x^j$.

**Vectors** $a^i = R^i_j \ a^j$ 

$s_i b^i = R^i_j \ b^j$, $\delta^i_j = \delta^i_0 = \delta^i_2 = \delta^i_3$, $\delta^i_j = \delta^i_0 = \delta^i_1 = \delta^i_3$ 

**1-forms** $b_i^j = R_i^j b_0$

**2-index tensors:** $T^i_j = R^i_m R^j_n T^m_n$

$G_{ij} = R^m_i R^j_m \ \Omega_{mn}$

Note that $g_{im} g_{jm} T^m_n$ transforms just like $G_{ij}$

**Tensors** are defined by their transformation properties, not by having indices.

We can form a 2-index tensor from two vectors by a "tensor product" $T^i_j = a^i b_j$ 

Similarly $T^i_j = a^i b_j$ and $T^i_j = a_i b^j$ are tensors. The latter, obviously $\delta_{ai} T^i_j = R^i_m R^j_n T^m_n$.
3-index tensors: $T_{ijk} = R^l_j R^n_k R^m_i \delta_{lmn}$, etc. The generalization is

obvious: $T^{ij...}_{...} = R^k_i R^n_j R^m_k \delta_{mn}$

Definition: $S_{nm} = S_{mn}$ is a symmetric tensor.

$S_{nm} = -S_{mn}$ is an anti-symmetric tensor

with generalizations to higher index tensors, e.g. $S^{n...}_{...}$ is completely symmetric if it is invariant under permutations of the indices.

Useful: If $S^{lm} = S_{lm}$ and $A_{nm} = A_{mn} \Rightarrow S^{nm} A_{nm} = 0$.

Deriv: $S_{nm} A_{nm} = -S_{nm} A_{nm}$ (anti-symmetry of $A_{nm}$)

= $-S_{nm} A_{nm}$ (dummy variables, change labels)

= $-S_{nm} A_{nm}$ (symmetry of $S_{nm}$)

$\Rightarrow S^{nm} A_{nm} = 0 \Rightarrow S_{nm} A_{mn} = 0$

Invariant tensor: Def. $T^{\alpha_1 \alpha_2 \ldots}_{\beta_1 \beta_2 \ldots} = T^{\beta_1 \beta_2 \ldots}_{\alpha_1 \alpha_2 \ldots}$

We have already encountered one:

$\delta^{ij}_{ij} = R^i_j R^n_i \delta_{mn} = \delta_{mn}$

(where, of course, $\delta_{mn} = \delta_{mn}$).

If you like math, this generalizes to spaces in any number of dimensions: $\delta_{ij}$ is always an invariant tensor.

In 3-dimensions, there is another interesting tensor. Let

$\epsilon_{ijk} = \begin{cases} +1 & \text{(an even permutation of } (123) \text{)} \\ -1 & \text{(an odd permutation of } (123) \text{)} \\ 0 & \text{otherwise} \end{cases}$

This is the completely anti-symmetric 3-index tensor, or Levi-Civita tensor.

That is $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$ $\epsilon_{221} = \epsilon_{313} = -1$ $\epsilon_{321} = \epsilon_{133} = \cdots = \epsilon_{333} = 0$.

Consider $T_{ijk} = R^l_i R^m_j R^n_k \epsilon_{lmn}$

First note that if any two indices in $T_{ijk}$ are equal then it vanishes,

$T_{ijk} = R^l_i R^m_j R^n_k \epsilon_{lmn} \stackrel{\text{symmetric}}{=} \underbrace{R^l_i R^n_j R^m_k \epsilon_{lmn}}_{\text{antisymmetric}} \Rightarrow 0$

It is then easy to see $T_{ijk}$ is completely anti-symmetric.
Now \( T_{\mu\nu} = R_{\mu}^{\kappa} R_{\kappa}^{\alpha} R_{\alpha}^{\nu} \epsilon_{\alpha\beta\gamma\delta} = \det(R) \)

or, since \( T_{\mu\nu} \) is completely antisymmetric, like \( \epsilon_{\alpha\beta\gamma\delta} \), and \( \epsilon_{123} = +1 \),

\[
T_{\mu\nu} = \det(R) \epsilon_{\mu\nu}
\]

Furthermore \( M_{\mu\nu} = R_{\mu}^{\alpha} R_{\nu}^{\beta} M_{\alpha\beta} = R_{\mu}^{\alpha} \epsilon_{\alpha\beta\gamma\delta} (R^T)^{\gamma}_{\delta} = (R M R^T)_{\mu\nu} \)

\[
\implies \det M = \det(R M R^T) = \det(R) \det(\det M) = (\det R)^2 \det G
\]

\[
\implies \det R = \pm 1
\]

\[
\text{Under rotations } \det(R) = \pm 1 \quad \text{ \( G_{\mu\nu} \) is an invariant tensor.}
\]

Imaginably, it flips sign under rotations with \( \det(R) = -1 \).

Let \( R(r) \) be a continuous function \([0, \beta] \rightarrow \{3 \times 3\text{ orthogonal matrices}\}\)

such that \( R(0) = I \). Then \( \det(R(r)) \) is a continuous function \([0, \beta] \rightarrow \mathbb{R} \) which can only take values \(-1, -\frac{1}{2}, \text{ and } \frac{1}{2} \), and hence \( \det(R(r)) = \det(1) = +1 \) \( \implies \det(R(\beta)) = +1 \).

In words, rotations that can be reached from 1 only have \( \det R = 1 \).

If \( R_1, R_2 \) both have \( \det R = 1 \) then so does \( R_3 = R_1 R_2 \) : \( \det(R_3) = \det(R_1 R_2) = \det(R_1) \det(R_2) = 1 \times 1 = 1 \)

but if both have \( \det R = -1 \) then \( R_3 = R_1 R_2 \) has \( \det(R_3) = +1 \).

In fact, every rotation with \( \det R = -1 \) can be written as \( R^1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} R \) where \( \det(R) = 1 \).

\((-1)\), of course, is a “spatial inversion,” or “reflection,” or “parity transformation.”

Manly stuff: The set of rotations \( \{3 \times 3 \text{ matrices} | \det R = \pm 1 \text{ or } R^T R = I \} \) form a group, called \( O(3) \), for “orthogonal group in 3 dimensions.”

Obvious question: \( O(N) \) ... \( N \) dimensions.

The subset of matrices with \( \det R = 1 \) form a subgroup, \( SO(3) \), the special orthogonal.

The subset with \( \det R = -1 \) does not form a group. (Question: why?)
Cross Product:
For proper rotations (i.e., $\mathbf{R}^T\mathbf{R} = \mathbf{I}$) we have $\mathbf{E}_{ij}$ is an invariant tensor.

$\mathbf{\omega}_i = \epsilon_{ijk} a^j b^k$ transforms as a 1-form and $\mathbf{a}_i = \epsilon^{ijk} \mathbf{a}_j \mathbf{b}_k$ transforms as a vector.

Also,
1. $\mathbf{E}_{ij} = \epsilon_{ijk} b^k$ is a 2-index anti-symmetric tensor.

2. If $\mathbf{E}_{ij}$ is a 2-index anti-symmetric tensor, then $\mathbf{E}_{ik} = \epsilon_{kim} \mathbf{a}_m$ and $\mathbf{a}_{ij} = \epsilon^{ikm} \mathbf{E}_{mj}.

This is, here is a 1-to-1 invertible correspondence between vectors and anti-symmetric 2-index tensors.

Exercise: Show $\epsilon^{ijk} E_{mj} = 2 \delta^i_m$ and hence $\mathbf{a}_{ij} = \epsilon^{ikm} (\frac{1}{2} \mathbf{E}_{mj} \mathbf{a}_m)$.

In Cartesian coordinates we do not distinguish $a^i$ from $a_i$ since $a^i : \epsilon^{ijm} a_j b_m = \delta^i_j a_j.$

So $\mathbf{\omega}_i = \epsilon_{ijk} a^j b^k$ is a vector composed of $\mathbf{a} \times \mathbf{b}$ with components

$\omega_1 = a_2 b_3 - a_3 b_2, \quad \omega_2 = a_3 b_1 - a_1 b_3, \quad \omega_3 = a_1 b_2 - a_2 b_1$

denoted by $\mathbf{\omega} = \mathbf{a} \times \mathbf{b}$.

Space inversions: vectors vs pseudo-vectors (also called axial vectors).

Let $P = -1$ be a space inversion ("P" is the parity).

Vectors $\mathbf{a} \rightarrow P\mathbf{a}$ transform as $\mathbf{a} \rightarrow -\mathbf{a}$ under space inversions.

Pseudo-vectors, however, transform as $\mathbf{a} \rightarrow \mathbf{a}$ under space inversions.

This is the statement that if $\mathbf{a} \rightarrow P \mathbf{a}$ then $P\mathbf{E} = \mathbf{a} \times \mathbf{b} \rightarrow - \mathbf{a} \times (P\mathbf{b})$.

Exercise: show this.

For "improper" rotations (those with $\det \mathbf{R} = -1$) this means $\mathbf{a} \times \mathbf{b} \rightarrow (P\mathbf{a}) \times (P\mathbf{b})$.

In particular, under space inversion $\mathbf{a} \times \mathbf{b} \rightarrow (P\mathbf{a}) \times (P\mathbf{b})$.

The cross product vector $\mathbf{x}$ vector is pseudo-vector.

Vector $\times$ pseudo-vector is vector.

Pseudo-vector $\times$ pseudo-vector is pseudo-vector.
Tenors in curvilinear coordinates

The generalization is straightforward.

Recall if \( y^i = y^i(x) \) then

\[
\partial x^i = \frac{\partial x^i}{\partial y^j} \partial y^j \quad \text{(just as } \partial x^i = \frac{\partial x^i}{\partial x^j} \partial x^j)\]

and

\[
\partial y^i = \frac{\partial y^i}{\partial y^j} \partial y^j
\]

A 2-index tensor has

\[
T'_{ij} = \frac{\partial y^i}{\partial x^m} \frac{\partial y^j}{\partial x^n} T^{mn}
\]

\[
T''_{ij} = \frac{\partial x^m}{\partial y^i} \frac{\partial x^n}{\partial y^j} T_{mn}
\]

et cetera.

Exercise: check that these are consistent with \( T''_{mn} = g_{mk} T^{mk} = g^{mn} T_{mn} \).

That is, we can "raise" and "lower" indices using the metric and its inverse to consistently make other tensors.

Recall that if \( ds^2 = g_{ij} dy^i dy^j \), then \( g_{mn} = \frac{\partial x^m}{\partial y^i} \frac{\partial x^n}{\partial y^j} g_{ij} \)

⇒ the metric tensor is a tensor (not just in name).

It is not generally invariant. That \( \eta_{ij} = \delta_{ij} \) is invariant under rotations is the statement that there is a special set of coordinate transformations (rotations + translations) that leave \( \eta_{ij} \) invariant.

What about \( \varepsilon_{ijk} ? \)

By the same calculation as above

\[
\frac{\partial x^k}{\partial y^i} \frac{\partial x^m}{\partial y^j} \varepsilon_{klm} = \det \left( \frac{\partial x^k}{\partial y^i} \right) \varepsilon_{ijk} = \sqrt{g} \varepsilon_{ijk}
\]

That is, \( \varepsilon_{ijk} T^{ijk} = \varepsilon_{ijk} \frac{\partial x^m}{\partial y^i} \frac{\partial x^n}{\partial y^j} \frac{\partial x^o}{\partial y^k} \) \( T_{mno} = \det \left( \frac{\partial x^m}{\partial y^i} \right) \varepsilon_{lmn} T^{lmn} = \frac{1}{\sqrt{g}} \varepsilon_{lmn} T^{lmn}
\]

⇒ In any frame there is a 3-form, completely anti-symmetric, given by \( \omega_{ijk} = \sqrt{g} \varepsilon_{ijk} \)

This is called a (metric) volume form.
Curl and Stoke's Theorem

We first review in Cartesian coordinates

\[ \text{Curl}(A) = \nabla \times A \text{ has } (\nabla \times A)_k = \epsilon_{ijk} \frac{\partial}{\partial x_j} A_i \] (Balancing upper lower indices in an ad-hoc way). 

eq \[ \left( \nabla \times A \right)_k = \frac{\partial}{\partial x_j} A_i - \frac{\partial}{\partial x_i} A_j \]

which we prefer to write as \( (\nabla \times A)_k = \partial_j A_i - \partial_i A_j \)

If \( \mathbf{B} \) is a rotation matrix (rigid, in the sense that it's \( \mathbb{R}^3 \) independent) then

\( \text{Curl}(\mathbf{B}) \) is a vector (actually if \( \mathbf{A} \) is a vector, \( \text{Curl}(\mathbf{A}) \) is a pseudo-vector).

Exercise: Show this: (Hint: \( \mathbf{A}'(\mathbf{B}) = \mathbf{B}^{-1} \mathbf{A} \mathbf{B}(\mathbf{B}) \)).

\[ \text{Stoke's Theorem states that } \int_S (\nabla \times A) \hat{n} \, dS = \oint_{\partial S} A \cdot d\hat{r} \]

when \( \int_S \) is a surface integral over \( S \), a sum over infinitesimal \( < \circ > \)

surface normal (loosely) to \( \hat{n} \) with area \( dS \), and \( \oint_{\partial S} \) is a line integral over the boundary \( \partial S \) of the whole surface \( S \), with tangent line element \( d\hat{r} \).

\[ \text{Stoke's Theorem states that } \int_S (\nabla \times A) \hat{n} \, dS = \oint_{\partial S} A \cdot d\hat{r} \]

where but if it is due over an infinitesimal area element then it is due over the whole of \( S \) bounded by \( \partial S \) (the path is the line integral)

\[ (\nabla \times A) \hat{n} \, dS \text{ while line integral is } A \cdot d\hat{r} \]

Cancel except at the boundary \( \partial S \), so \( \int_S (\nabla \times A) \hat{n} \, dS = \oint_{\partial S} A \cdot d\hat{r} \)

So used to show infinitesimally: sum an infinitesimal surface element is flat and has normal \( \hat{n} \) everywhere. Take for simplicity \( \hat{n} = \hat{z} \) (ie, 3rd direction). Then \( dS = dydz \). If the boundaries are not aligned \( \hat{n} \), make a relation to align them: since \( \nabla \times A \) is a vector \( (\nabla \times A) \hat{n} \) won't change.

So we consider

\[ \int_{\hat{z}=1}^{\hat{z}=0} dy \int_{\hat{x}=1}^{\hat{x}=0} dx \left( \partial_x A_y - \partial_y A_x \right) = \int_{\hat{z}=1}^{\hat{z}=0} dx \left( A_x(\hat{z},0,x) - A_x(\hat{z},1,x) \right) - \int_{\hat{z}=0}^{\hat{z}=1} dy \left( A_y(\hat{z},x,0) - A_y(\hat{z},x,1) \right) \]

\[ \rightarrow \int \frac{\partial}{\partial \hat{z}} A \cdot d\hat{r} \]
While this proof may not seem quite general, the fact that we can always find a relation to put \( \mathbf{F} \) in the \( \mathbf{F} \) direction, and that this just corresponds to a change of variables makes it a truly general argument.

Note: I was a bit careless about direction of \( \mathbf{a} \) vs orientation of loop (but I did it right).

The curl in curvilinear coordinates is tricky. The reason is that when we add that it involves \( \mathbf{E}_{ij} \) we may think it involves the volume form \( \sqrt{\mathbf{g}} \mathbf{E}_{ij} \) when going curvilinear.

But since Stokes's theorem holds and should be generalized, and this involves a surface integral of the (original component) of the curl, it is actually the 'volume' form of the 2-dimensional space \( S \) that should be involved. If the metric restricted to the surface (at a point on the surface element) is \( h_{ij} \), then we want \( \frac{1}{\sqrt{h}} \mathbf{E}_{ij} \mathbf{a}_j \) for the component of \( \text{curl}(\mathbf{a}) \) along the normal.

This is simple in orthogonal coordinates (with \( g_{ij} = 0 \) if \( i \neq j \)), because the vector element \( dv = dx_i dx_j \) has a normal along \( e^{ij} \) (ie, the \( \mathbf{a} \) direction).

So the \( 3 \) components of \( \text{curl}(\mathbf{a}) \) are \( E^{ij} \mathbf{a}_j \). And as before, expressing this in terms of components of normalized metric, so \( \mathbf{h} = g_{ij} \mathbf{e}^i \otimes \mathbf{e}^j \), we have

\[
\text{curl}(\mathbf{A})^k = \mathbf{e}^{ij} \frac{1}{\sqrt{h_{ij}}} \frac{\partial}{\partial e^j} \left( g_{ij} A^i \right)
\]

Examples

Cylindrical: \( \mathbf{e}^{\rho z} \frac{1}{r \sin \theta} \left( \frac{\partial}{\partial \rho} \left( r \phi \right) - \frac{\partial}{\partial \phi} \left( r \rho \right) \right) = (r \rho \phi) \frac{1}{r \sin \theta} \left( \frac{\partial}{\partial \rho} \left( r \phi \right) - \frac{\partial}{\partial \phi} \left( r \rho \right) \right) = -\frac{\partial}{\partial \phi} \left( \rho A^\phi \right) - \frac{\partial}{\partial \rho} \left( \rho A^\phi \right)
\]

a bit distr ... \( \mathbf{e}^{\theta \phi} \frac{1}{\rho} \left( \frac{\partial}{\partial \theta} \left( \rho \phi \right) - \frac{\partial}{\partial \phi} \left( \rho \theta \right) \right) = (\rho A^\phi - \theta A^\rho) \) and \( \mathbf{e}^{\rho \phi} \frac{\partial}{\partial \rho} \left( \rho A^\phi \right) - \frac{\partial}{\partial \phi} \left( \rho A^\phi \right) \)

or \( \text{curl}(\mathbf{A}) = \frac{1}{\rho} \left( \frac{\partial}{\partial \theta} \left( \rho A^\phi \right) - \frac{\partial}{\partial \phi} \left( \rho A^\theta \right), \frac{\partial}{\partial \phi} A^\rho - \frac{\partial}{\partial \rho} A^\phi, \frac{\partial}{\partial \theta} A^\phi - \frac{\partial}{\partial \phi} A^\theta \right) \)

Spherical: \( \text{curl}(\mathbf{A})^r = \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} \left( \sin \theta \rho A^\rho \right) - \frac{\partial}{\partial \rho} \left( \rho A^\theta \right) \right] = \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} \left( \sin \theta \rho A^\rho \right) - \frac{\partial}{\partial \rho} \left( \rho A^\theta \right) \right]
\]

Exercise: Compute remaining components

\( [\text{curl}(\mathbf{A})]^\theta = \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (r A^\rho) - \frac{1}{r} \frac{\partial}{\partial \rho} A^\phi \)

\( [\text{curl}(\mathbf{A})]^\phi = \frac{1}{r} \frac{\partial}{\partial \rho} (r A^\rho) - \frac{1}{r} \frac{\partial}{\partial \rho} A^\phi \)
Gauss's Theorem:
\[ \int_{\partial V} \mathbf{A} \cdot d\mathbf{s} = \int_{V} \nabla \cdot \mathbf{A} \, dV \]

This is easy to prove for an infinitesimal cube, and then extended to finite volumes by summation, as in Stoke's case:

⇒ left as exercise (but countless textbooks have it; still you should be able to construct the proof).

Explicit sample calculations using Stoke's and Gauss's theorem are given as part of Homework #1, and in problem session.
Back to Space-Time (It's about time).

Coordinates in space-time: \((x^t, x^1, x^2, x^3)\)

A point in space-time is called "an event".

Basic property of space-time: invariance of the interval.

Interval between \((x, x)\) and \((x + dx, x + dx)\): \(ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2\)

Notation: Greek indices \(\mu, \nu, \ldots\) range 0-3 (while Latin indices \(a, b, \ldots\), range 1-3).

Einstein summation convention applies, applying to any type of index.

Metric \(g_{\mu\nu}\) is \(4 \times 4\), diagonal with \(g_{\mu\nu} = \begin{cases} +1 & \mu = 0 \\ -1 & \mu = 1, 2, 3 \end{cases}\)

So \(ds^2 = g_{\mu\nu}dx^\mu dx^\nu = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2\) is the metric in the \(x\)\(\nu\)\(\xi\) family looking:

Lorentz transformations: \(x^\mu = \Lambda^\mu_\nu x^\nu\), are defined to be those that leave \(ds^2\) form invariant. As we already did in this entry:

\(\Lambda^\mu_\nu\) is a matrix transformation \(\Rightarrow \Lambda^\mu_\nu\Lambda^\nu_\rho = \delta^\mu_\rho\) or \(\Lambda^\mu_\nu\Lambda^\nu_\rho\) for short.

It leaves the metric invariant.

Vectors \(\times\) Tensors:

A (covariant) vector transforms as \(\mathbf{a}^\mu \rightarrow \gamma^\mu_\nu \mathbf{a}^\nu\). Tensors can be expanded with \(\gamma\), e.g. \(\mathbf{a}^\mu \rightarrow \gamma^\mu_\nu \mathbf{a}^\nu\).

The inverse metric is denoted \(g^{\mu\nu}\), i.e. \(g^\mu_\nu g^\nu_\rho = \delta^\mu_\rho\).

A \(\nu\)\(\mu\)\(\xi\) vector (or covariant vector, or really a \(1\)-form) transforms:

so we get \(\mathbf{a}^\mu \rightarrow \Gamma^\mu_\nu \mathbf{a}^\nu\) (invariant)

\(\mathbf{a}^\mu' = \Gamma^\mu_\nu \mathbf{a}^\nu\)

\(\sigma^{\mu} = \mathbf{a}^\mu\)

Indices can be raised with \(\gamma\): \(a^\mu = \gamma^{\mu\nu} a_\nu\). Shorthand: \(a^\mu = \mathbf{a}^\mu = \mathbf{a}^\mu \mathbf{a}_\mu = a_\mu a^\mu\); \(a = a_\mu a^\mu\).

Tensors: \(T^\mu_{\cdot \nu \cdot \kappa} = \Lambda^\mu_\mu \cdots \Lambda^\kappa_\kappa \cdots \Lambda^\nu_\nu \cdots T^\mu_{\cdot \nu \cdot \kappa}\)

\(e.g. \ T^\mu_{\nu \cdot \rho} = \Lambda^\mu_\mu \cdots \Lambda^\rho_\rho \cdots T^\mu_{\nu \cdot \rho}\).

Again, \(\mu, \nu\) lower, raise indices, as \(T^\nu_{\cdot \mu} = \gamma^\nu_\rho T^\rho_{\cdot \mu}\).
Many generalizations from the discussion of rotations are straightforward, e.g.:

- A field is a function of spacetime. A scalar field \( \phi(x) \) (or simply \( \phi \)) satisfies
  \[
  \phi'(x') = \phi(x) \quad \text{under} \quad x' = \Lambda x
  \]
  This is often written as
  \[
  \phi'(x) = \phi(\Lambda^{-1} x)
  \]

A vector field satisfies
  \[
  \Lambda^\mu (x) = \Lambda^\mu \cdot A^\nu (\Lambda^{-1} x)
  \]
  and
  \[
  B^\mu (x) = \Lambda^\mu \cdot B^\nu (\Lambda^{-1} x)
  \]

- The gradient is a (covariant) vector: \( \nabla \phi \) transforms as \( B^\mu \) above.

Some things are a little different.
- \( E_{ijk} \) is not an invariant tensor. Neither is \( E_{\mu\nu} \).
  But \( E_{\mu\nu} \) is invariant under transformations with \( \det(\Lambda) = +1 \).

As before, \( \det(\Lambda) = +1 \).

Exercise: show \( M_1 \).

But now there are 4 connected components of the group of Lorentz transformations:
- Space inversion: \( x' = -x \) and \( t' = t \) gives \( \det(\Lambda) = -1 \)
- Time reflection: \( t' = -t \) and \( x' = x \) also gives \( \det(\Lambda) = -1 \).

Are these disconnected? To understand the meaning of the question go back to rotations for a moment. Are
  \[
  P = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad P_\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
  \]
  in disconnected components, both with \( \det(P) = -1 \)? The answer is no:
  \[
  P_\varepsilon = R_2 P = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{with} \quad R_2 \text{ a 180° rotation about the } z \text{ axis
Back to \(4\times4\) Lorentz Transformations:

Since \(M' = \rho T M \sigma_0\), \(L^0_{\nu}' = I L^0_{\nu} I\), \(L^0_{\nu}' = I L^0_{\nu} I\) \\
\Rightarrow cannot smoothly connect \(M_0 = 1\) to \(M_{\infty} = \Lambda\)

And for fixed sign \(\Lambda^0\), a space-inversion gives a flip in sign of \(\det \Lambda\)

So there are 4-connected components

\[
\begin{aligned}
T &= \text{time reflection} \\
P &= \text{space inversion} \\
\Lambda^0 < 0 &\quad \text{(future)} \\
\Lambda^0 > 0 &\quad \text{(past)} \\
\end{aligned}
\]

Since \(a^2\) is invariant, \(a^2 > 0\) means \(a^0 > |a^1|\) or \(a^0 < -|a^1|\)

Light-cone diagram

\[
\begin{aligned}
\text{future} &\quad a^0 > 0 \\
\text{light cone} &\quad a^0 = 0 \\
\text{timelike} &\quad a^0 < 0 \\
\text{spacelike} &\quad a^0 = 0 \\
\end{aligned}
\]

Remarks:
1. This diagram is for an arbitrary vector, not necessarily coordinates.
2. We will call the regions by their coordinate usage (i.e., spacelike and timelike regions). We also say \(a^0\) is timelike/lightlike/spacelike according to whether \(a^0 > 0\), \(a^0 = 0\), \(a^0 < 0\).
3. The 2-dim space above is limited: should draw 4-dim, but I can't sorry.
   But it should be clear that the light cone is a cone, a 2-dim hypersurface.
   We can at least draw a 2-dim hypersurface in a 3-dim space-time:

   ![](image)

   This should make it clear that the spacelike region is connected, while
   future and past light cones are not (they "touch" at the origin).

Now, if \(\Lambda : a^0 \to -a^0, a^i \to -a^i\) and \(a^0 > 0\), it maps

future light cone \(\to\) past light cone. Cannot go from one to the other continuously
(going through origin is not allowed, since \(\Lambda = 0\) is not invertible, and does not leave \(a^0\) invariant).
Explicit form of Lorentz Transformations.

Find explicit solutions to 
\[ M_{\mu \nu} = M_{\mu 
u} \quad \text{and} \quad A_{\mu} = A_{\mu} \]

That is,
\[ \begin{align*}
\mu = 0 & \quad \Rightarrow (A^{\mu}) = \frac{1}{2} \sum_{\nu} (A^{\nu})_{\nu} = 1 \\
\mu = 1 & \quad \Rightarrow \sum_{\nu} A_{\nu} = 0 \\
\mu = 2 & \quad \Rightarrow \sum_{\nu} A_{\nu} = -\delta_{ij} \\
\end{align*} \]

Observations, including some solutions:

(a) Notation: 
\[ R^{\mu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \end{pmatrix} \]
\[ \begin{align*}
\Lambda_{\mu}^{\nu} &= \delta_{\mu \nu} \\
\end{align*} \]

(b) Boosts: 
\[ \begin{pmatrix} c \cos \theta & c \sin \theta & 0 & 0 \\
c \sin \theta & c \cos \theta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \end{pmatrix} \]

(c) Lorentz transformations in the x-direction \( x = x' \):
\[ \Lambda = \begin{pmatrix} \cosh \beta & \sinh \beta & 0 & 0 \\
\sinh \beta & \cosh \beta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \end{pmatrix} \]

Saying \( x = 0 \), the origin of the primed system is moving with velocity \( v = \beta c = c \tanh \theta \) as measured in the primed system 
\[ \begin{align*}
x' &= (c \sinh \theta)c - (\sinh \theta)c \\
0 & = \cosh \beta \cosh \beta + \sinh \beta \sinh \beta \\
\end{align*} \]

(d) Lorentz: how many independent parameters for LT's?

Wait-up: for rotations first, \( R^2 = I \), \( R \) is 3 x 3 matrix. \( \theta \) is angle. \( \sin \theta \) is \( \text{symmetric matrix} \), \( 9 - \frac{3 \times 3}{2} = 6 \) rotations \( \Rightarrow 9 - 6 = 3 \) independent parameters + Euler Angles / \( \mathbf{V} \)

Simpler: \( N^{\mu} = 0 \), \( \mu = 2 \) \( \frac{\pi}{4} = 10 \) arbitrary, \( \mu = 4 \) \( \text{metric matrix} = 10 \rightarrow 4 \rightarrow 6 \) independent parameters \( \Rightarrow 3 \text{Euler angles} + 3 \text{boosts} \)
(w) Product of two transformations is a transformation.

While obvious physically, this can be expressed and shown mathematically:

Exercise: If \( \Lambda_1, \Lambda_2 \) are LT's, show that \( \Lambda_1 \Lambda_2 \) is an LT (i.e., satisfies \( \Lambda^{\mu}\Lambda_\nu = \delta^{\mu}_{\nu} \)).

This makes LTs a group (called \( O(3,1) \)).

One can use this to build LTs out of infinitesimal ones

\[ \Lambda = 1 + \epsilon \quad \text{with} \quad \epsilon \ll 1 \]

\[ \Lambda^\gamma_\eta \Lambda = \eta \Rightarrow (1 + \epsilon)^\gamma_\eta (1 + \epsilon) = \eta \Rightarrow \epsilon^\gamma_\eta + \epsilon \epsilon = 0 \]

Of course any vector \( \epsilon \alpha = \eta \alpha \beta \epsilon \beta \) gives an antisymmetric \( \alpha \beta \eta \Rightarrow \epsilon \alpha + \epsilon \beta = 0 \)

A \( \epsilon \) is antisymmetric under \( \eta \) means \( \frac{\delta^{\gamma\beta} \epsilon^\alpha_\gamma}{\epsilon^\alpha_\gamma} \) is independent parameters - same result as other

Now \( \Lambda^\alpha_\gamma = (1 + \epsilon)(1 + \epsilon)(1 + \epsilon) \) is a transformation

and we can have

\[ \Lambda = \exp \left( \epsilon^\gamma_\eta \gamma_\eta \right) = \exp(\epsilon) \]

is a general expression for LT's. See e.g. Jackson.

Exercise: What is the analog in Lorentz rotations?

Tensors vs. pseudo-tensors.

Tensors are pseudo-tensors transform the same way under proper LT's:

\[ T^\mu_\nu \Rightarrow \Lambda^\mu^\gamma \Lambda^\gamma_\nu \Rightarrow T^\mu^\gamma \Lambda^\gamma_\nu \]

Inverses still transform this way under any (not necessarily proper) LT. But for a pseudo-tensor under space inversion

\[ T^\mu_\nu \Rightarrow - \Lambda^\mu^\gamma \Lambda^\gamma_\nu \Rightarrow T^\mu^\gamma \Lambda^\gamma_\nu \]

where \( \Lambda^\mu^\gamma = \text{diag}(1, -1, -1, -1) \) and a pseudo-tensor under time-reversal

\[ T^\mu_\nu \Rightarrow - \Lambda^\mu^\gamma \Lambda^\gamma_\nu \Rightarrow T^\mu_\nu \]

where \( \Lambda = \text{diag}(-1, -1, 1, 1) \).

Ex: \( \epsilon^{\mu\nu\lambda} = \epsilon^{\nu\lambda\mu} \) is an invariant pseudo-tensor (under \( \Lambda^\gamma_\eta \)).
Particular examples:
A vector \( V^\mu = (v^0, v^1) \)

\( u^0 \) \( V^0 = u^0 \) and \( V' = -V \) under \( P \)

\( V' = -V^0 \) and \( V = V' \) under \( T \)

An "axial" vector is a pseudo-vector under parity: \( A^\mu = (A^0, A^i) \) has

\( A^0 = -A^0 \) and \( A^i = A^i \) under \( P \)