Electric and magnetic phenomena are described by Maxwell's equations

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \qquad (1)$$

$$\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{j} \qquad (2)$$

$$\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0 \qquad (3)$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad (4)$$

(given in Gaussian units - more on this later) plus the Lorentz force equation

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right) \qquad (5)$$

In these $\vec{E} = \vec{E}(\vec{x},t)$ and $\vec{B}=\vec{B}(\vec{x},t)$ are fields - which I am displaying explicitly.
Just to make sure we agree on notation,

$\rho = \rho(\vec{x},t)$ = electric charge density

$\vec{j} = \vec{j}(\vec{x},t)$ = electric current density

$c$ = speed of light $\simeq 2.99\times10^8$ m/s

$\vec{v}$ = velocity of charge-$q$ particle

$\vec{F}$ = force due $\vec{E}$ & $\vec{B}$ fields on said particle

And $\vec{E} = (E_x, E_y, E_z)$ etc are 3-vectors (we'll distinguish 3-vectors from other d-vectors)

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = (\partial_x, \partial_y, \partial_z)$$

so that on a function $f(x,y,z)$, $\vec{\nabla}f = \vec{grad}(f)$ is a vector $(\partial_x f, \partial_y f, \partial_z f)$ ("gradient")

on a vector $\vec{E}$, $\vec{\nabla}\cdot\vec{E} = \text{div}(\vec{E})$ is a scalar (ie, pure number) $= \partial_x E_x + \partial_y E_y + \partial_z E_z$ (divergence)

and on a vector $\vec{E}$, $\vec{\nabla}\times\vec{E} = \vec{curl}(\vec{E})$ is a vector $= (\partial_y E_z - \partial_z E_y,\ \partial_z E_x - \partial_x E_z,\ \partial_x E_y - \partial_y E_x)$ (curl)

We start our discussion of Electrodynamics by exploring two key aspects of eqs (1)-(5):

(i) They are invariant under Lorentz transformations.

(ii) The fundamental dynamical variables are fields.

We will look at these together, moving back and forth between them. We will make contact with the more familiar aspects of special relativity (eg, boosts on point particles) only at the end, for completeness.

## Space-time

Fields are functions of space and time. This in itself does not require we think of space and time as part of the same continuum "space-time". It is the invariance of Eqs (1)-(5) under Lorentz transformations, and that these mix space and time, that lead us to consider space and time on an almost equal footing.

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Warm-up: Rotations and space.

Points in space are accounted for with coordinates $\vec{x} = (x^1, x^2, x^3) \in \mathbb{R}^3$

There is much arbitrariness in how this is done. Given a coordinate system $\vec{x}$ can define a new one $\vec{x}' = \vec{x}'(\vec{x})$ (3 functions of 3 variables), with the obvious constraint that the map be 1-to-1, invertible (if you know about manifolds and differential geometry, this can be done in patches). But we'd like to focus on coordinates we can assign with a meter stick; call them "Cartesian".

Given one such coordinate system, others are obtained by

- translations $\vec{x}' = \vec{x} + \vec{a}$ $\quad \vec{a}$ = fixed vector
- rotations $\vec{x}' = R\,\vec{x}$ $\quad R$ = orthogonal matrix

Here is another look at this. In our 'ruled' space distance is given by

$$ds^2 = dx^2 + dy^2 + dz^2 = \sum_{i=1}^{3} (dx^i)^2 \tag{6}$$

OK, recall distance between 2-points, $P_1$ and $P_2$, is given by

$\int_{P_1}^{P} ds$ along a straight line $\left(\text{or, equivalently, } \min_{\text{all paths}} \int_{P_1}^{P_2} ds\right)$.

Question: what is the set of transformations $\vec{x} \to \vec{y}(\vec{x})$ that preserve the form of (6),

$$ds^2 = \sum_{i=1}^{3} (dx^i)^2 = \sum_{i=1}^{3} (dy^i)^2 \quad ? \qquad (7)$$

Since $\vec{y} = \vec{y}(\vec{x})$ we have

$$\sum_{i=1}^{3} (dy^i)^2 = \sum_{i=1}^{3} \left( \sum_{j=1}^{3} \frac{\partial y^i}{\partial x^j} dx^j \right)^2 = \sum_{j,k} \left( \sum_{i=1}^{3} \frac{\partial y^i}{\partial x^j} \frac{\partial y^i}{\partial x^k} \right) dx^j dx^k$$

One can show $\displaystyle\sum_{i=1}^{3} \frac{\partial y^i}{\partial x^j} \frac{\partial y^i}{\partial x^k} = \delta_{jk} = \begin{cases} 1 & \text{if } j=k \\ 0 & j \neq k \end{cases}$ ("Kronecker delta") (7')

only if $\vec{y} = \vec{y}(\vec{x})$ is a linear transformation. $y^i = \displaystyle\sum_{j=1}^{3} R^i{}_j x^j + a^i$

with $R^i{}_j$ a 3x3 matrix of numbers and $a^i$ a 3-vector of numbers (by "numbers" we mean constants, independent of $\vec{x}$).

Moreover it is required that

$$\sum_{i=1}^{3} R^i{}_j R^i{}_k = \delta_{jk} \qquad (8)$$

which follows directly from (7').

Note: the reason for the peculiar upper/lower indices will become clear soon.

Condition (8) defines "orthogonal" matrices.

It is convenient to introduce a <u>metric tensor</u> $\eta_{ij}$ so that

$$ds^2 = \sum_{i=1}^{3} \sum_{j=1}^{3} \eta_{ij} dx^i dx^j \qquad (9)$$

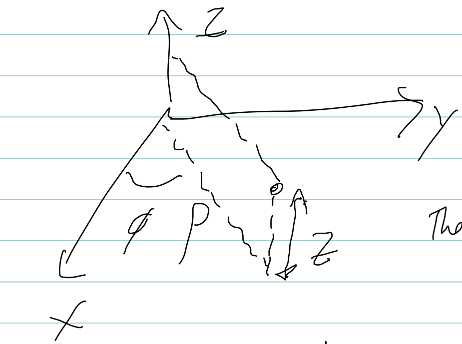
Of course, $\eta_{ij} = \delta_{ij}$ in our Cartesian system. But we have already seen that if we go nutty with coordinate choices, $y^i = y^i(x^j)$ then

$$ds^2 = \sum_{k,l}^{3} g_{kl} dy^k dy^l \quad \text{with } g_{kl} = \sum_{i,j}^{3} \frac{\partial x^i}{\partial y^k} \frac{\partial x^j}{\partial y^l} \eta_{ij} \qquad (10)$$

(we used $x^i = x^i(\vec{y})$ as the inverse of $\vec{y} = \vec{y}(\vec{x})$). This can be convenient! We can relate the metric tensors in coordinate systems that are not Cartesian.

Examples follow:

★ Cylindrical coordinates.



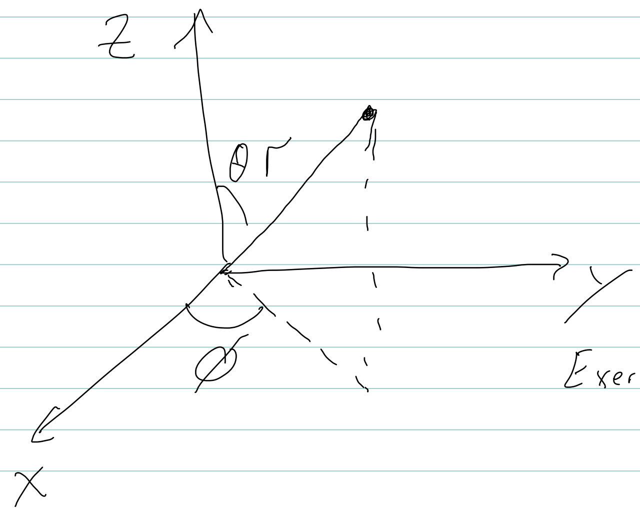

$x = \rho\cos\phi$
$y = \rho\sin\phi$
$z = z$ ← ok, I should give them different labels, but why bother....

Then $ds^2 = (d\rho\cos\phi - \rho\sin\phi\, d\phi)^2 + (d\rho\sin\phi + \rho\cos\phi\, d\phi)^2 + (dz)^2$

$$= d\rho^2 + \rho^2 d\phi^2 + dz^2 \qquad (11)$$

and $g_{k\ell} = \mathrm{diag}(1, \rho^2, 1)$ (or $g_{\rho\rho} = 1$, $g_{\phi\phi} = \rho^2$, $g_{zz} = 1$, all others vanish).

★ Spherical coordinates




$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

Exercise: verify $ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$ (12)

We will make this useful momentarily.

But lets go back to rotations: consider $x^i \to R^i{}_j\, x^j$ (this is often going to be shorthand, or shortspeak, for let $y^i = R^i{}_j x^j$ in a function of $\vec{y}$):

$$ds^2 = \sum_{i,j=1}^{3} \eta_{ij}\, dx^i dx^j \longrightarrow \sum_{i,j=1}^{3} \eta_{ij} \Big(\sum_{k=1}^{3} R^i{}_k\, dx^k\Big)\Big(\sum_{\ell=1}^{3} R^j{}_\ell\, dx^\ell\Big) = \sum_{k,\ell=1}^{3} \eta'_{k\ell}\, dx^k dx^\ell$$

where $\eta'_{k\ell} = \sum_{i,j=1}^{3} R^i{}_k R^j{}_\ell\, \eta_{ij}$ (13)

The invariance condition (7) is now $\eta'_{ij} = \eta_{ij}$

Notice that since $\eta_{ij} = \delta_{ij}$, condition (13) is just the same as (8). But it tells us something interesting: orthogonal transformations are those that leave the metric tensor invariant under transformations $\eta \to \eta'$ given in (13)

Einstein convention:

Repeated indices are presumed summed over their understood range unless otherwise stated.

So $y^i = R^i{}_j x^j$ stands for $y^i = \sum_{j=1}^{3} R^i{}_j x^j$

and $ds^2 = \eta_{ij}\, dx^i dx^j$ stands for $ds^2 = \sum_{i,j=1}^{3} \eta_{ij}\, dx^i dx^j$

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Sometimes we even imply the indices, $Y = Rx$ means $y^i = R^i{}_j x^j$

And

$R^T \eta R = \eta$ characterizes transformations that leave metric invariant

and $\eta = \mathbb{1}$ means $R^T R = \mathbb{1}$, a more familiar condition for orthogonal matrix.

Physics:

Let's consider a rotation on Maxwell's equations. Start from Gauss's law (1):

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho$$

or rather

$$\partial_i E^i = 4\pi\rho$$

where $\partial_i = \frac{\partial}{\partial x^i}$, and the lower index on $\partial_i$, upper on $x^i$, will be explained later.

Consider a coordinate change $y^i = y^i(x^j) = R^i{}_j x^j$

We want to show that there is a matrix function of $R$, say $D(R)^i{}_j$ such that (1) is form invariant if

$$E'^i(\vec{y},t) = D^i{}_j E^j(\vec{x},t) = D^i{}_j E^j(R^{-1}\vec{y},t)$$

Since $t$ is going along for a ride, I will omit below. "Form invariant" or plainly "invariant" means

$$\partial'_i E'^i = \frac{\partial}{\partial y^i} E'^i(\vec{y}) = 4\pi\rho(\vec{y})$$

To see this is the case, and infer $D(R)$, compute the divergence:

$$\partial'_i E'^i = D^i{}_j\, \partial'_i E^j(R^{-1}y) = D^i{}_j\, \partial'_i (R^{-1}y)^k \left(\partial_k E^j(x)\right)\Big|_{x=R^{-1}y} = D^i{}_j (R^{-1})^k{}_i \left(\partial_k E^j(x)\right)\Big|_{x=R^{-1}y}$$

We want this to equal $4\pi\rho(\vec{x})\big|_{\vec{x}=R^{-1}\vec{y}} = \partial_i E^i(\vec{x})\big|_{\vec{x}=R^{-1}\vec{y}}$. Comparing we see that we need

$$(R^{-1})^k{}_i D^i{}_j = \delta^k_j \Rightarrow D^i{}_j = R^i{}_j \quad \text{or in matrix notation} \quad D = R$$

(So indeed, $D$ is a function of $R$, namely, $D(R) = R$).

Re-cap: Eq (1) is invariant under the change of coordinates ("rotations")
$$\vec{y} = R\vec{x}$$
if in the new coordinate system $\vec{E}'(\vec{y}) = R\vec{E}(\vec{x})$ and $\rho'(\vec{y}) = \rho(\vec{x})$.

Or simply, (1) is invariant under $\vec{E}'(\vec{y}) = R\vec{E}(R^{-1}y)$ and $\rho'(\vec{y}) = \rho(R^{-1}\vec{y})$.

We say that $\vec{E}$ is a $\boxed{\text{vector}}$ <u>because</u> it transforms under rotations just like $\vec{x}$ does
$$\left(\text{namely } \vec{x} \to R\vec{x},\quad \vec{E} \to R\vec{E}\right).$$

We say that $\rho$ is a $\boxed{\text{scalar}}$: it transforms under rotations just as $ds^2$
$$\left(\text{namely} \quad \rho \to \rho\right).$$

<u>Exercise</u>: show that with $c$ a scalar, and $\vec{B}$, $\vec{j}$, $\vec{v}$ and $\vec{F}$ vectors eqs (2)-(5) are also invariant.

Maxwell equations are invariant under rotations. This may seem trivial, particularly since we have written them in an explicitly covariant notation. That is, once we know that

(i) Dot products are scalars (ie, do not transform)

(ii) $\vec{\nabla}$ is a vector

(iii) cross products of vectors are vectors

we can "see" that each equation is invariant because both sides of the equality transform the same way, eg
$$\underbrace{\underbrace{\underset{\text{vector}}{\vec{\nabla}} \underset{\times\text{ prod}}{\times} \underset{\text{vector}}{\vec{B}}}_{\text{vector}} - \frac{1}{c}\underset{\text{vector}}{\frac{\partial \vec{E}}{\partial t}}}_{\text{vector}} = \frac{4\pi}{c}\underset{\text{vector}}{\vec{j}}$$

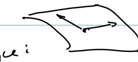
Our aim is to show that Eqs (1)-(5) are invariant under a bigger set of transformations, namely Lorentz transformations (plus translations in space-time, but these are already explicit). The problem is much simpler if we can make the symmetry explicit, as we just showed for rotations in Ampère's law (Eq (2)) rather than going through explicit computations as we did with Gauss's law (and I proposed as an Exercise for Eqs (2)-(5)).

Before leaving rotations, let's use the technology we developped to derive a couple of useful equations.

When going to curvilinear coordinates (eg, spherical) we have to be more careful in defining vectors. It is not necessarily true that new coordinates $y^i = y^i(\vec{x})$ transform as vectors. What is always true is that the infinitesimal displacement between two points is a vector, and the vector transformation is given by

$$dy^i = \frac{\partial y^i}{\partial x^j} dx^j$$

$dx^i$ defines a tangent vector to a curved surface:



which corresponds to $dy^i = R^i{}_j\, dx^j$ for transformations $\vec{y} = \vec{y}(\vec{x})$ of the form $\vec{y} = R\vec{x}$.

The dot-product of two vectors $\vec{a}$ & $\vec{b}$ is $\vec{a}\cdot\vec{b} = \eta_{ij}\, a^i b^j$

Recall $ds^2 = g_{ij}\, dy^i dy^j$ has

$$g_{ij} = \frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}\eta_{kl} \qquad (14)$$

So if $a'^i = \frac{\partial y^i}{\partial x^k} a^k$ for vectors, then

$$\vec{a}'\cdot\vec{b}' = g_{ij}\, a'^i b'^j = \underline{\left(\frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}\eta_{kl}\right)}\left(\frac{\partial y^i}{\partial x^m} a^m\right)\left(\frac{\partial y^j}{\partial x^n} b^n\right) = \eta_{kl}\, a^k b^l = \vec{a}\cdot\vec{b}$$

Consider the gradient $\partial_i \phi$ of a scalar function, ie, $\phi'(\vec{y}) = \phi(\vec{x})$.

$$\partial'_i \phi' = \frac{\partial \phi'(y)}{\partial y^i} = \frac{\partial}{\partial y^i}\phi(x) = \frac{\partial x^j}{\partial y^i}\frac{\partial \phi}{\partial x^j}$$

Aha! This is NOT a vector! $\partial_i' \phi' = \frac{\partial x^j}{\partial y^i}\partial_j \phi$ instead of $a'^i = \frac{\partial y^i}{\partial x^j} a^j$

In the language of differential geometry these are "1-forms" (spoken "one forms").
In old physics language

$a^i$ = contra-variant vector

$\partial_i \phi$ = co-variant vector

Any $\omega_i$ that transforms as $\omega'_i = \frac{\partial x^j}{\partial y^i}\omega_j$ is a 1-form (or a co-variant vector)

Note that $\omega'_i a'^i = \omega_i a^i$, ie, the "contraction" of a 1-form and a vector is a scalar (sometimes this is used to define 1-forms).

Note furthermore that $a_i \equiv g_{ij} a^j$ transforms as (and therefore IS) a 1-form

THIS IS WHY WE HAVE DIFFERENTIATED BETWEEN UPPER AN LOWER INDICES.

Given a 1-form $a_i$ can I make a vector $a^i$? Yes! Let $g^{ij}$ denote the inverse matrix to $g_{ij}$, so that

$$g^{ij} g_{jk} = \delta^i_k \qquad (g^{-1}g = \mathbb{1})$$

Then

$$a^i = g^{ij} a_j$$

is a vector.

Exercise: Prove the above assertion.

Invariant integrals.

Consider $\int_V d^3x \equiv \int_V dx\,dy\,dz = \int_V \prod_{i=1}^{3} dx^i$

In changing variables to curvilinear coordinates $\vec{y} = \vec{y}(\vec{x})$

$$\int_V d^3x = \int_V d^3y \; J$$

$$J = \text{Jacobian} = \left| \det\left(\frac{\partial x^i}{\partial y^j}\right)\right|$$

Recall, if $ds = g_{ij}\, dy^i dy^j = \eta_{ij}\, dx^i dx^j$ then (eq (14)): $g_{ij} = \frac{\partial x^k}{\partial y^i} \frac{\partial x^l}{\partial y^j} \eta_{kl}$

That is $g \equiv \det g_{ij} = J^2 \det \eta$ (and $\det \eta = 1$, but let's keep it explicit, for now)

So $\int_V d^3x \sqrt{\eta} = \int_V d^3y \sqrt{g}$ is the invariant integration volume.

Examples:

(i) Cylindrical: $g = \det\begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \rho^2 \Rightarrow$ volume $= d\rho\, d\theta\, dz \cdot \rho = \rho\, d\rho\, d\theta\, dz$

(ii) Spherical: $g = \det\begin{pmatrix} 1 & & \\ & r^2 & \\ & & r^2\sin^2\theta \end{pmatrix} = r^4 \sin^2\theta$ vol $= dr\, d\theta\, d\phi\; r^2 \sin\theta = r^2 dr\; d(\cos\theta)\; d\phi$

Divergence in curvilinear coordinates

Looks a priori messy, but neat trick:

Consider $\int_V d^3y \sqrt{g}\, a^i \partial_i \phi$ clearly invariant under coordinate transformations

Take $\phi$ to vanish at spatial $\infty$ and integrate by parts (in fact we will want $\phi$ to have local support):

$$\underbrace{\int_V d^3y \sqrt{g}\, a^i \partial_i \phi}_{\text{invariant}} = -\int_V d^3y\, \phi\, \partial_i(\sqrt{g}\, a^i) = -\int \underbrace{d^3y \sqrt{g}\,\phi}_{\text{invariant}} \underbrace{\left[\frac{1}{\sqrt{g}} \partial_i (\sqrt{g}\, a^i)\right]}_{\text{must be invariant}}$$

invariant $\Rightarrow$ must be invariant

Moreover, comparing to Cartesian coordinates, this invariant is what we called $\mathrm{div}(\vec{a})$

So

$$\mathrm{div}(\vec{a}) = \frac{1}{\sqrt{g}}\, \partial_i \left(\sqrt{g}\, a^i\right)$$

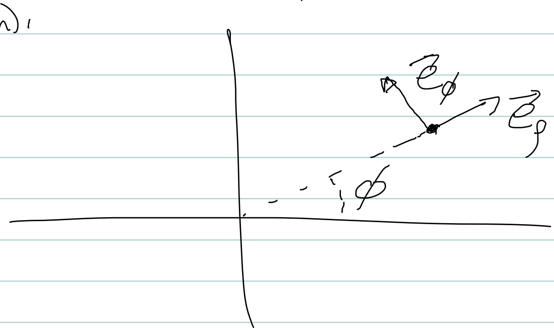
Examples:

(i) Cylindrical: $\mathrm{div}(\vec{a}) = \frac{1}{\rho} \partial_\rho (\rho a^\rho) + \partial_\phi a^\phi + \partial_z a^z$

(ii) Spherical $\mathrm{div}(\vec{a}) = \frac{1}{r^2} \partial_r (r^2 a^r) + \frac{1}{\sin\theta} \partial_\theta (\sin\theta\, a^\theta) + \partial_\phi a^\phi$

Note: This result differs from many textbooks (eg, Jackson or Garg). The reason is their meaning for $a^i$ is different – mine is better :)

Suppose you want to write $\vec{a} = A^i \vec{e}_i$, where $\vec{e}_i$ are orthonormal vectors in the Carthesian sense. For example, for cylindrical coordinates (omitting the z-direction):




$$|\vec{e}_\rho| = |\vec{e}_\phi| = 1$$

$$\vec{e}_\phi \cdot \vec{e}_\rho = 0$$

Then, for example $\vec{a} \cdot \vec{a} = g_{ij} a^i a^j = A^i A^j \vec{e}_i \cdot \vec{e}_j = A^i A^j \delta_{ij}$. Since $g_{ij}$ is diagonal (a condition for this to work) we find $A^i = \sqrt{g_{ii}}\, a^i$ (no sum on $i$) and

$$\mathrm{div}(a) = \frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g} \frac{1}{\sqrt{g_{ii}}} A^i\right)$$

cyl: $\frac{1}{\rho} \partial_\rho (\rho A^\rho) + \frac{1}{\rho} \partial_\phi A^\phi + \partial_z A^z$

spher: $\frac{1}{r^2} \partial_r (r^2 A^r) + \frac{1}{r\sin\theta} \partial_\theta (\sin\theta\, A^\theta) + \frac{1}{r\sin\theta} \partial_\phi A^\phi$

Lesson: make sure you know what your symbols mean (especially when you use formulas from the back flap of a textbook).

## Laplacian

A simple extension of the previous exercise: use $a^i = g^{ij}\partial_j f$ where $f$ is a scalar. Then

$$\nabla^2(f) = \vec{\nabla}\cdot\vec{\nabla}f = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\,g^{ij}\partial_j f)$$

In Carthesian coordinates $\nabla^2 f = \eta^{ij}\partial_i\partial_j f = (\partial_1^2+\partial_2^2+\partial_3^2)f$

— — —

Exercise:

Cylindrical: $\nabla^2 f = \frac{1}{\rho}\partial_\rho(\rho\,\partial_\rho f) + \frac{1}{\rho^2}\partial_\phi^2 f + \partial_z^2 f$

Spherical: $\nabla^2 f = \frac{1}{r^2}\partial_r(r^2\partial_r f) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta f) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2 f$

— — —

## Tensors, invariants and the peculiar cross product.

Restrict attention to rotations, $y^i = R^i{}_j x^j$.

Vectors $a'^i = R^i{}_j a^j$ $\quad$ (so $b'_i a'^i = R_i{}^n R^i{}_j b_n a^j = \delta^n_j b_n a^j = b_j a^j$)

1-forms $b'_i = R_i{}^n b_n$

2-index tensors: $T'^{ij} = R^i{}_m R^j{}_n T^{mn}$

$$Q'_{ij} = R_i{}^m R_j{}^n Q_{mn}$$

Note that $g_{im} g_{jn} T^{mn}$ transforms just like $Q_{ij}$

Tensors are defined by their transformation properties, not by having indices.

We can form a 2-index tensor from two vectors by a "tensor product"

$$T^{ij} = a^i b^j$$

Similarly $T_{ij} = a_i b_j$ and $T^i{}_j = a^i b_j$ are tensors. The latter, obviously has $T'^i{}_j = R^i{}_m R_j{}^n T^m{}_n$.

3-index tensors: $T'^{ijk} = R^i{}_l R^j{}_m R^k{}_n T^{lmn}$, etc. The generalization is obvious: $T'^{i_1\cdots i_p}{}_{j_1\cdots j_q} = R^{i_1}{}_{m_1}\cdots R^{i_p}{}_{m_p} R_{j_1}{}^{n_1}\cdots R_{j_q}{}^{n_q} T^{m_1\cdots m_p}{}_{n_1\cdots n_q}$

Definition: $s^{lm} = s^{ml}$ is a symmetric tensor.

$a^{lm} = -a^{ml}$ is an anti-symmetric tensor

with generalizations to higher index tensors, eg $s^{i_1\cdots i_n}$ is completely symmetric if it is invariant under permutations of the indices.

Useful: If $s^{lm} = s^{ml}$ and $a_{lm} = -a_{ml}$ $\Rightarrow$ $s^{lm}a_{lm} = 0$.

Proof: $s^{lm}a_{lm} = -s^{lm}a_{ml}$ (anti-symmetry of $a_{ml}$)

$= -s^{ml}a_{lm}$ (dummy variables, change labels)

$= -s^{lm}a_{lm}$ (symmetry of $s^{lm}$)

$\Rightarrow 2s^{lm}a_{lm} = 0 \Rightarrow s^{lm}a_{lm} = 0$

<u>Invariant tensor</u>: Def. $T'^{i_1\cdots i_p}{}_{j_1\cdots j_q} = T^{i_1\cdots i_p}{}_{j_1\cdots j_q}$

We have already encountered one:

$$\eta'_{ij} = R_i{}^m R_j{}^n \eta_{mn} = \eta_{mn}$$

(where, of course, $\eta_{mn} = \delta_{mn}$).

If you like math, this generalizes to spaces in any number of dimensions: $\delta_{ij}$ is always an invariant tensor.

In 3-dimensions, there is another interesting tensor. Let

$$\epsilon_{ijk} = \begin{cases} +1 & (ijk) \text{ an even permutation of } (123) \\ -1 & (ijk) \text{ an odd permutation of } (123) \\ 0 & \text{otherwise} \end{cases}$$

This is the completely anti-symmetric 3-index tensor, or Levi-Civita tensor.

That is $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$ $\epsilon_{213} = \epsilon_{132} = \epsilon_{321} = -1$ $\epsilon_{111} = \epsilon_{112} = \cdots = \epsilon_{333} = 0$.

Consider $T_{ijk} \equiv R_i{}^l R_j{}^m R_k{}^n \epsilon_{lmn}$

First note that if any two indices in $T_{ijk}$ are equal then it vanishes, eg

$$T_{11k} = \underbrace{R_1{}^l R_1{}^m}_{\text{symmetric under } l\leftrightarrow m} R_k{}^n \underbrace{\epsilon_{lmn}}_{\text{anti-symmetric under } l\leftrightarrow m} \Rightarrow 0$$

It is then easy to see $T_{ijk}$ is completely antisymmetric

Now $T_{123} = R_1{}^\ell R_2{}^m R_3{}^n \epsilon_{\ell m n} = \det(R)$

or, since $T_{ijk}$ is completely antisymmetric, like $\epsilon_{ijk}$, and $\epsilon_{123} = +1$,

$$T_{ijk} = \det(R)\epsilon_{ijk}$$

Furthermore $\eta_{ij} = R_i{}^k R_j{}^\ell \eta_{k\ell} = R_i{}^k \eta_{k\ell} (R^T)^\ell{}_j = (R\eta R^T)_{ij}$

$$\Rightarrow \det\eta = \det(R\eta R^T) = \det R \det\eta \det R^T = (\det R)^2 \det\eta$$

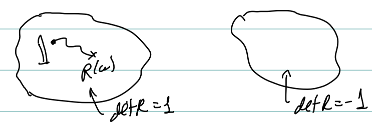
$$\Rightarrow \det R = \pm 1$$

$\Rightarrow$ $\boxed{\text{Under rotations + } \det(R) = +1 \quad \epsilon_{ijk} \text{ is an invariant tensor.}}$

Importantly, it flips sign under rotations with $\det(R) = -1$.

Let $R(\omega)$ be a continuous function $[0,1] \to \{3\times3 \text{ orthogonal matrices}\}$ such that $R(0) = \mathbb{1}$. Then $\det(R(\omega))$ is a continuous function $[0,1] \to \mathbb{R}$ which can only take values $-1$ and $+1$, and has $\det(R(0)) = \det \mathbb{1} = +1 \Rightarrow \det(R(\omega)) = +1$

In words rotations that can be reached from $\mathbb{1}$ continuously all have $\det R = +1$.




If $R_1$ & $R_2$ both have $\det R = 1$ then so does $R_3 = R_1 R_2$: $\det(R_3) = \det(R_1 R_2) = \det R_1 \cdot \det R_2 = (+1)^2 = +1$

but if both have $\det R = -1$ then $R_3 = R_1 R_2$ has $\det R_3 = +1$.

In fact every rotation $R'$ with $\det(R') = -1$ can be written as $R' = (-\mathbb{1})\cdot R$ where $\det(R) = +1$.

$(-\mathbb{1})$, of course, is a "spatial inversion", or "reflection", or "parity transformation"

Mathy stuff: The set of rotations $\{3\times3 \text{ real matrices} \mid R^T R = \mathbb{1} \ (\text{or } R\eta R^T = \eta)\}$ form a group, called $O(3)$, for "orthogonal group in 3 dimensions".

Obvious extension: $O(N)$ ... $N$ dimensions.

The subset of matrices with $\det R = +1$ form a subgroup, $SO(3)$, for special orthogonal.

The subset with $\det R = -1$ does not form a group. (Question: why?)

## Cross Product:

For proper rotations (ie, with $\det(R)=+1$) we have $\epsilon_{ijk}$ is an invariant tensor.

$\Rightarrow$ $\omega_i = \epsilon_{ijk} a^j b^k$ transforms as a 1-form and $\omega^i = \eta^{ij}\omega_j$ transforms as a vector

Also,

(i) $a_{ij} = \epsilon_{ijk} b^k$ is a 2-index antisymmetric tensor

(ii) If $a^{ij}$ is a 2-index anti-symmetric tensor, then $\omega_k = \frac{1}{2}\epsilon_{kij}a^{ij}$ is a 1-form

and $a^{ij} = \epsilon^{ijk}\omega_k$

That is, there is a 1-to-1, invertible, correspondance between vectors and anti-symmetric 2-index tensors

Exercise: Show $\epsilon^{ijk}\epsilon_{mjk} = 2\delta^i_m$ and then $a^{ij} = \epsilon^{ijk}\left(\frac{1}{2}\epsilon_{kmn}a^{mn}\right)$.

In Carthesian coordinates we do not distinguish $\omega^i$ from $\omega_i$ since $\omega^i = \eta^{ij}\omega_j = \delta^{ij}\omega_j$

So $\omega_i = \epsilon_{ijk} a^j b^k$ is a vector $\vec{\omega}$ made out of $\vec{a}$ & $\vec{b}$ with components

$$\omega_1 = a_2 b_3 - a_3 b_2 \,, \quad \omega_2 = a_3 b_1 - a_1 b_3 \,, \quad \omega_3 = a_1 b_2 - a_2 b_1$$

denoted by $\vec{\omega} = \vec{a}\times\vec{b}$

Space inversions: vectors vs pseudo-vectors (also called "axial" vectors).

Let $P = -\mathbb{1}$ be a space inversion ("P" is for parity)

Vectors $\vec{a}\to R\vec{a}$ transform as $\vec{a}\to P\vec{a} = -\vec{a}$ under space inversions.

Pseudovectors, however, transform as $\vec{\omega}\to +\vec{\omega}$ under space inversions.

This is the statement that if $\vec{a}\to R\vec{a}$ & $\vec{b}\to R\vec{b}$

then

$$\vec{a}\times\vec{b} \longrightarrow \det(R)\, R\,(\vec{a}\times\vec{b})$$

Exercise: show this

For "improper" rotations (ie, those with $\det(R) = -1$) this means $\vec{a}\times\vec{b} \to -R(\vec{a}\times\vec{b})$

In particular, under space inversions, $\vec{a}\times\vec{b} \to +(\vec{a}\times\vec{b})$

The cross product
- vector $\times$ vector is pseudovector
- vector $\times$ pseudovector is vector
- pseudo $\times$ pseudo is pseudovector.

## Tensors in curvilinear coordinates

The generalization is straightforward.
Recall if $y^i = y^i(x)$ then

$$a'^i = \frac{\partial y^i}{\partial x^j} a_j \qquad \left(\text{just as } dy^i = \frac{\partial y^i}{\partial x^j} dx^j\right)$$

and

$$\omega'_i = \frac{\partial x^j}{\partial y^i} \omega_j$$

A 2-index tensor has $T'^{ij} = \frac{\partial y^i}{\partial x^m}\frac{\partial y^j}{\partial x^n} T^{mn}$

$$T'^i{}_j = \frac{\partial y^i}{\partial x^m}\frac{\partial x^n}{\partial y^j} T^m{}_n$$

$$T'_{ij} = \frac{\partial x^m}{\partial y^i}\frac{\partial x^n}{\partial y^j} T_{mn}$$

etc.

Exercise: check that these are consistent with $T^m{}_n = g_{nk} T^{mk} = g^{ml} T_{ln}$
That is, we can "raise" and "lower" indices using the metric and its inverse to consistently make other tensors.

Recall that if $ds^2 = g_{mn} dy^m dy^n = \eta_{ij} dx^i dx^j$ then $g_{mn} = \frac{\partial x^i}{\partial y^m}\frac{\partial x^j}{\partial y^n}\eta_{ij}$

$\Rightarrow$ the metric tensor $\underline{\underline{is}}$ a tensor (not just in name).

It is not generally invariant. That $\eta_{ij} = \delta_{ij}$ is invariant under rotations is the statement that there is a special set of coordinate transformations (rotations + translations) that leave $\eta_{ij}$ invariant.

What about $\epsilon_{ijk}$?
By the same calculation as above

$$\frac{\partial x^l}{\partial y^i}\frac{\partial x^m}{\partial y^j}\frac{\partial x^n}{\partial y^k}\epsilon_{lmn} = \det\left(\frac{\partial x}{\partial y}\right)\epsilon_{ijk} = \sqrt{g}\ \epsilon_{ijk}$$

That is, $\epsilon_{ijk} T'^{ijk} = \epsilon_{ijk}\frac{\partial y^i}{\partial x^l}\frac{\partial y^j}{\partial x^m}\frac{\partial y^k}{\partial x^n} T^{lmn} = \det\left(\frac{\partial y}{\partial x}\right)\epsilon_{lmn} T^{lmn} = \frac{1}{\sqrt{g}}\epsilon_{lmn} T^{lmn}$

so that $\sqrt{g}\,\epsilon_{ijk} T'^{ijk} = \sqrt{\eta}\ \epsilon_{lmn} T^{lmn}$

$\Rightarrow$ In any frame there is a 3-form, completely anti-symmetric, given by $\omega_{ijk} = \sqrt{g}\ \epsilon_{ijk}$

This is called a (metric) volume form

# Curl and Stoke's Theorem

We first review in Cartesian coordinates

$\vec{\text{curl}}(\vec{A}) = \vec{\nabla}\times\vec{A}$ has $(\vec{\nabla}\times\vec{A})^i = \epsilon^{ijk}\,\partial_j A_k$ (Balancing upper lower indices in an ad-hoc way).

eg $(\vec{\nabla}\times\vec{A})_x = \partial_y A_z - \partial_z A_y$

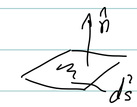
which we prefer to write as $(\vec{\nabla}\times\vec{A})_1 = \partial_2 A_3 - \partial_3 A_2$

If $R^i{}_j$ is a rotation matrix (rigid, in the sense that it's $\vec{x}$ independent) then $\text{curl}(\vec{A})$ is a vector (actually if $\vec{A}$ is a vector, $\text{curl}(\vec{A})$ is a pseudovector).

Exercise: Show this. (Hint $A_i'(\vec{x}) = R_i{}^j A_j(R^{-1}\vec{x})$).

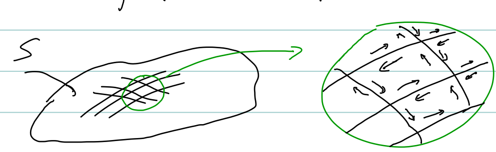
Stoke's theorem states that

$$\int_S (\vec{\nabla}\times\vec{A})\cdot\hat{n}\, d^2s = \oint_{\partial S} \vec{A}\cdot d\vec{l}$$



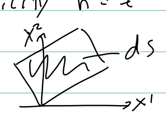

when $\int_S$ is a surface integral over $S$, a sum over infinitesimal surfaces normal (locally) to $\hat{n}$ with area $d^2s$, and $\oint_{\partial S}$ is a line integral over the boundary $\partial S$ of the whole surface $S$, with tangent line element $d\vec{l}$.

Sketch of Proof:



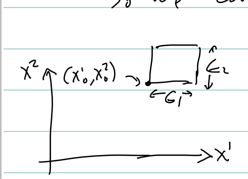

Note that if it is true over an infinitesimal area element then it is true over the whole of $S$ bounded by $\partial S$ (the $\Sigma$ elements of $\sum_N (\vec{\nabla}\times\vec{A})\cdot\hat{n}\,\Delta s \to \int_S (\vec{\nabla}\times\vec{A})\cdot\hat{n}\,d^2s$ while the line integrals cancel except at the boundry $\partial S$, so $\sum_{\text{paths}} \vec{A}\cdot\Delta\vec{l} \to \oint_{\partial S}\vec{A}\cdot d\vec{l}$

So need to show infinitesimally: now an infinitesimal surface element is flat and has normal $\hat{n}$ everywhere. Take for simplicity $\hat{n} = \hat{z}$ (ie, $3^{rd}$ direction). Then $d^2s = dx\,dy$. If the boundaries are not aligned




make a rotation to align them: since $\vec{\nabla}\times\vec{A}$ is a vector $(\vec{\nabla}\times\vec{A})\cdot\hat{n}$ won't change.

So we consider




$$\int_{x_0^1}^{x_0^1+\epsilon^1} dx^1 \int_{x_0^2}^{x_0^2+\epsilon^2} dx^2\, (\partial_1 A_2 - \partial_2 A_1) = \int_{x_0^2}^{x_0^2+\epsilon^2} dx^2 \left(A_2(x_0^1+\epsilon^1, x^2) - A_2(x_0^1, x^2)\right) - \int_{x_0^1}^{x_0^1+\epsilon^1} dx^1 \left(A_1(x^1, x_0^2+\epsilon^2) - A_1(x^1, x_0^2)\right)$$

$$= \oint_{\square} \vec{A}\cdot d\vec{l}$$

While this proof may not seem quite general, the fact that we can always find a rotation to put $\hat{n}$ in the $\hat{z}$ direction, and that this just corresponds to a change of variables makes it a truly general argument!

Note: I was a bit careless about direction of $\hat{n}$ vs orientation of loop (but I did it right).

The curl in curvilinear coordinates is tricky. The reason is that when we see that it involves $\epsilon_{ijk}$ we may think it involves the volume form $\sqrt{g}\,\epsilon_{ijk}$ when going curvilinear.

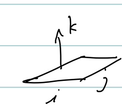
But since Stoke's theorem holds and should be generalized, and this involves a surface integral of the (normal component of) the curl, it is actually the "volume" form of the 2-dimensional space $S$ that should be involved. If the metric restricted to the surface $S$ (at a point on the surface element) is $h_{ij}$ then we want $\frac{1}{\sqrt{h}}\epsilon^{ij}\partial_i a_j$ for the component of $\mathrm{curl}(\vec{a})$ along the normal.

This is simple in orthogonal coordinates (with $g_{ij}=0$ if $i\neq j$), because the surface element $dx^i dx^j$ $(i\neq j)$ has a normal along $\epsilon^{kij}$ (ie, the 3rd direction).

So the 3 components of $\mathrm{curl}(\vec{a})$ are $\epsilon^{kij}\frac{1}{\sqrt{h}}\partial_i a_j$. And as before, expressing this in terms of components of normalized metric, so that $(A^i)^2 = g_{ii}(a^i)^2 = g^{ii}(a_i)^2 = \frac{1}{g_{ii}}(a_i)^2$

and noting that in the $ij$ plane $\sqrt{h}=\sqrt{g_{ii}g_{jj}}$ we have

$$\mathrm{curl}(\vec{A})^k = \epsilon^{kij}\frac{1}{\sqrt{g_{ii}g_{jj}}}\partial_i\left(\sqrt{g_{jj}}\,A^j\right)$$



Examples

Cylindrical: $\epsilon^{\rho\theta z}\frac{1}{\sqrt{g_{\theta\theta}g_{zz}}}\left(\partial_\theta(\sqrt{g_{zz}}A^z)-\partial_z(\sqrt{g_{\theta\theta}}A^\theta)\right) = (+1)\frac{1}{\sqrt{\rho^2\cdot 1}}\left(\partial_\theta(\sqrt{1}A^z)-\partial_z(\sqrt{\rho^2}A^\theta)\right) = \frac{1}{\rho}\partial_\theta A^z - \partial_z A^\theta$

a bit faster... $\epsilon^{\theta z\rho}\frac{1}{\sqrt{1\cdot 1}}\left(\partial_z\sqrt{1}A^\rho - \partial_\rho\sqrt{1}A^z\right) = \left(\partial_z A^\rho - \partial_\rho A^z\right)$, and $\epsilon^{z\rho\theta}\frac{1}{\rho}\left(\partial_\rho(\rho A^\theta)-\partial_\theta A^\rho\right)$

or $\mathrm{curl}(\vec{A}) = \left(\frac{1}{\rho}\partial_\theta A^z - \partial_z A^\theta,\ \partial_z A^\rho - \partial_\rho A^z,\ \frac{1}{\rho}\partial_\rho(\rho A^\theta) - \frac{1}{\rho}\partial_\theta A^\rho\right)$

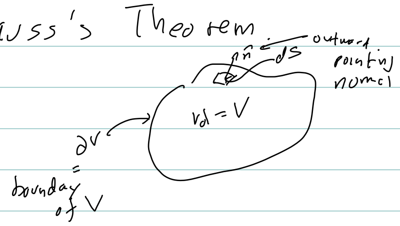
Spherical: $[\mathrm{curl}(\vec{A})]^r = \frac{1}{r^2\sin\theta}\left[\partial_\theta(r\sin\theta A^\phi) - \partial_\phi(rA^\theta)\right] = \frac{1}{r\sin\theta}\left[\partial_\theta(\sin\theta A^\phi) - \partial_\phi A^\theta\right]$

Exercise: Compute remaining components

ANS $[\mathrm{curl}(\vec{A})]^\theta = \frac{1}{r\sin\theta}\partial_\phi A^r - \frac{1}{r}\partial_r(rA^\phi)$

$[\mathrm{curl}(\vec{A})]^\phi = \frac{1}{r}\partial_r(rA^\theta) - \frac{1}{r}\partial_\theta A^r$

## Gauss's Theorem




$$\int_{\partial V} \vec{A}\cdot\hat{n}\, ds = \int_V \vec{\nabla}\cdot\vec{A}\, dV$$

This is easy to prove for an infinitesimal cube, and then extended to finite volumes by summation, as in Stoke's case

$\Rightarrow$ left as exercise (but countless textbooks have it; still you should be able to construct the proof).

Explicit sample calculations using Stoke's & Gauss's theorem are given as part of Homework #1, and in problem session.

## Back to Space-Time (it's about time).

Coordinates in space-time: $(\underbrace{x^0 = ct, x^1, x^2, x^3}_{\vec{x}\ (\text{or } \vec{r})})$

A point in space-time is called "an event"

Basic property of space-time: invariance of the interval.

Interval between $(x^0, \vec{x})$ and $(x^0+dx^0, \vec{x}+d\vec{x})$ $\quad ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$

Notation: greek indices $\mu, \nu, \ldots$ range 0–3 (while latin indices $i, j, \ldots$ range 1–3)

Einstein summation convention as before, applies to any type of index.

Metric $\eta_{\mu\nu}$ is 4x4, diagonal with $\eta_{\mu\nu} = \begin{cases} +1 & \mu=\nu=0 \\ -1 & \mu=\nu=1,2,3 \end{cases}$

So $ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$ ie, the more familiar looking $c^2(dt)^2 - (d\vec{r})^2$

Lorentz transformations: $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, are defined to be those that leave $ds^2$ form invariant. As with rotations this means

$\Lambda^\mu{}_\nu$ is a Lorentz transformation $\iff \eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$ or $\Lambda^T \eta \Lambda = \eta$ for short

ie, leave the metric invariant.

Vectors & tensors

A (contravariant) vector transforms as $a^\mu \to a'^\mu = \Lambda^\mu{}_\nu a^\nu$

Indices can be lowered with $\eta_{\mu\nu}$, eg $a_\mu = \eta_{\mu\nu} a^\nu$

The inverse metric $\eta^{-1}$ is denoted $\eta^{\mu\nu}$, ie, $\eta^{\mu\lambda}\eta_{\lambda\nu} = \delta^\mu_\nu$

A low index vector (or covariant vector, or really, a 1-form) transforms so that $\omega'_\mu a'^\mu = \omega_\mu a^\mu$ (invariant)

$\omega'_\mu = \omega_\nu \Lambda^{-1\,\nu}{}_\mu$ $\quad$ Since $\eta^{-1}\Lambda^T\eta\Lambda = 1 \to \eta^{\kappa\rho}\Lambda^\mu{}_\rho \eta_{\mu\nu} \Lambda^\nu{}_\sigma = \delta^\kappa_\rho \Rightarrow (\Lambda^{-1})^\kappa{}_\nu = \Lambda_\nu{}^\kappa$

or $\omega'_\mu = \Lambda_\nu{}^\mu \omega_\nu$

Indices can be raised with $\eta^{-1}$: $a^\mu = \eta^{\mu\nu} a_\nu$

Shorthand: $a^2 = \eta_{\mu\nu} a^\mu a^\nu = a^\mu a_\mu = \eta^{\mu\nu} a_\mu a_\nu$ ; $a\cdot b = \eta_{\mu\nu} a^\mu b^\nu = a_\mu b^\mu = a^\mu b_\mu = \eta^{\mu\nu} a_\mu b_\nu$

Tensors: $T'^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_q} = \Lambda^{\mu_1}{}_{\kappa_1} \cdots \Lambda^{\mu_p}{}_{\kappa_p} \Lambda_{\nu_1}{}^{\sigma_1} \cdots \Lambda_{\nu_q}{}^{\sigma_q}\, T^{\kappa_1 \cdots \kappa_p}{}_{\sigma_1 \cdots \sigma_q}$

eg $T'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma T^{\rho\sigma}$. Again $\eta, \eta^{-1}$ lower, raise indices, eg $T^\mu{}_\nu = \eta_{\nu\rho} T^{\mu\rho}$

Many generalizations from the discussion of rotations are straightforward, e.g.:

- A field is a function of spacetime. A scalar field $\phi(x^\mu)$ (or simply $\phi(x)$) satisfies

$$\phi'(x') = \phi(x) \qquad \text{under} \quad x' = \Lambda x$$

This is often written as

$$\phi'(x) = \phi(\Lambda^{-1}x)$$

A vector field satisfies

$$A'^\mu(x) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x)$$

and

$$B'_\mu(x) = \Lambda_\mu{}^\nu B_\nu(\Lambda^{-1}x)$$

- The gradient is a (co-variant) vector: $\partial_\mu\phi$ transforms as $B_\mu$ above.

Some things are a little different.
- $\epsilon_{ijk}$ is not an invariant tensor. Neither is $\epsilon_{\mu\nu\sigma}$.
  But $\epsilon_{\mu\nu\lambda\sigma}$ is invariant under transformations with $\det(\Lambda)=+1$.

As before $\det(\Lambda)=\pm 1$

Exercise: show this.

But now there are 4 connected components of the group of Lorentz transformations:
- Space inversion: as before $\vec{x}' = -\vec{x}$ (and $t'=t$) gives $\det(\Lambda) = -1$

but now
- Time reflection: $t'=-t$ and $\vec{x}'=\vec{x}$ also gives $\det(\Lambda) = -1$.

Are these disconnected? To understand the meaning of the question go back to rotations for a moment. Are

$$P = \begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \quad \text{and} \quad P_z = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

in disconnected components, both with $\det(R)=-1$? The answer is no:

$$P_z = R_z P = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \quad \text{with } R_z \text{ a } 180^\circ \text{ rotation about the } z \text{ axis}$$

Back to 4D & Lorentz Transformations:

Since $\eta_{\mu\nu} = \eta_{\rho\sigma} \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu$, $\quad (\Lambda^0{}_0)^2 - \Sigma(\Lambda^i{}_0)^2 = 1 \Rightarrow \Lambda^0{}_0 = \pm\sqrt{1+\Sigma(\Lambda^i{}_0)^2}$

$\Rightarrow$ cannot smoothly connect $\Lambda^0{}_0 \le -1$ to $\Lambda^0{}_0 \ge +1$

And for fixed sign of $\Lambda^0{}_0$, a space-inversion gives a flip in sign of $\det\Lambda$

So there are 4-connected components

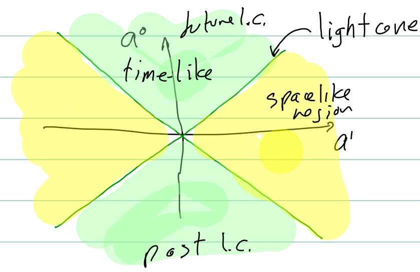
$$\begin{array}{ccc} \boxed{\begin{matrix}\det\Lambda=+1\\ \Lambda^0{}_0>0\end{matrix}} & \xrightarrow{P} & \boxed{\begin{matrix}\det\Lambda=-1\\ \Lambda^0{}_0>0\end{matrix}} \\ \downarrow T & & \downarrow T \\ \boxed{\begin{matrix}\det\Lambda=-1\\ \Lambda^0{}_0<0\end{matrix}} & \xrightarrow{P} & \boxed{\begin{matrix}\det\Lambda=+1\\ \Lambda^0{}_0<0\end{matrix}} \end{array}$$

a subgroup of Lorentz, "Proper LTs"

T = time reflection
P = space inversion

Since $a^2$ is invariant, if $a^2>0$ then $a^0>|\vec{a}|$ or $a^0<-|\vec{a}|$

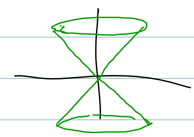
Light-cone diagram




Remarks (i) This diagram is for an arbitrary vector, not necesarily coordinates

(ii) We still call the regions by their coordinate analogs (ie, spacelike and timelike regions). We also say $a^\mu$ is timelike/lightlike/spacelike according to whether $a^2>0$, $a^2=0$, $a^2<0$.

(iii) The 2dim image above is limited: should draw 4dim, but I can't. Sorry :)
But it should be clear that the light cone is a cone, a 3dim hypersurface.
We can at least draw a 2dim hypersurface in a 3dim spacetime:



This should make it clear that the spacelike region is connected, while future and past light cones are not (they "touch" at the origin).

Now, if $\Lambda: a^0 \to -a^0$, $a^i \to a^i$ and $a^2>0$, it maps future light cone $\longleftrightarrow$ past light cone. Cannot go from one to the other continuously (going through origin is not allowed, since $\Lambda=0$ is not invertible, and does not leave $\eta$ invariant).

# Explicit form of Lorentz Transformations.

Find explicit solutions to $\quad \eta_{\mu\nu} = \eta_{\rho\sigma} \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu$

That is

$\mu\nu = 00 \qquad (\Lambda^0{}_0)^2 - \sum_{i=1}^{3} (\Lambda^i{}_0)^2 = 1 \qquad (1)$

$\mu\nu = 0i \qquad \Lambda^0{}_0 \Lambda^0{}_i - \sum_k \Lambda^k{}_0 \Lambda^k{}_i = 0 \qquad (2)$

$\mu\nu = ij \qquad \Lambda^0{}_i \Lambda^0{}_j - \sum_k \Lambda^k{}_i \Lambda^k{}_j = -\delta_{ij} \qquad (3)$

Observations, including some solutions:

(i) Rotations: $\Lambda^0{}_0 = 1 \quad \Lambda^0{}_i = \Lambda^i{}_0 = 0 \quad , \Lambda^i{}_j = R^i{}_j$ with $R$ a rotation ($R \in O(3)$)

ex, rotations about z axis ($z = x^3$)

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & -s & 0 \\ 0 & s & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad c = \cos\theta \quad s = \sin\theta$$

$x'^\mu = \Lambda^\mu{}_\nu x^\nu$ is

$ct' = ct \quad z' = z$

$x' = \cos\theta\, x - \sin\theta\, y$

$y' = \sin\theta\, x + \cos\theta\, y$

(ii) Boosts: $\Lambda^0{}_0 = \cosh\eta \quad \sum_{\mu=1}^{3} (\Lambda^i{}_0)^2 = \sinh^2\eta$ solves (1)

eg, in $x$ direction ($x = x^1$)

$$\Lambda = \begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & \\ 0 & 0 & & 1 \end{pmatrix}$$

or

$ct' = \cosh\eta\, ct + \sinh\eta\, x$

$x' = \sinh\eta\, ct + \cosh\eta\, x$

$y' = y$

$z' = z$

Setting $x = 0$, the origin of un-primed system is moving with velocity $v = \beta c = c \tanh\eta$ as measured in primed system $x' = (c\sinh\eta) t = \left(\frac{c\sinh\eta}{\cosh\eta}\right) t'$

$1 - \tanh^2\eta = \frac{1}{\cosh^2\eta} \Rightarrow \gamma \equiv \cosh\eta = \frac{1}{\sqrt{1-\beta^2}}$ and $\sinh\eta = \tanh\eta \cdot \cosh\eta = \gamma\beta$

(iii) Counting: how many independent parameters for LT's?

Warm-up: for rotations first $R^T R = \mathbb{1}$ $R$ is $3\times 3 = 9$ entries, the condition is on a symmetric matrix, ie on $\frac{3\times 4}{2} = 6$ components $\Rightarrow 9 - 6 = 3$ independent parameters $\rightarrow$ Euler Angles! ✓✓

Similarly $\Lambda^T \eta \Lambda = \eta$ puts $\frac{4\times 5}{2} = 10$ constraints on $4\times 4 = 16$ matrix $\Rightarrow 16 - 10 = 6$ independent parameters $\Rightarrow$ 3 Euler angles + 3 boosts

(iv) Product of two transformations is a transformation

While obvious physically, this can be expressed and shown mathematically:

Exercise: If $\Lambda_1$ & $\Lambda_2$ are LT's show that $\Lambda_1\Lambda_2$ is an LT. (ie, satisfies $\Lambda^T\eta\Lambda=\eta$).

This makes LTs a group (called $O(3,1)$).

One can use this to build LTs out of infinitesimal ones

$$\tilde{\Lambda} = 1 + \epsilon \qquad \text{with } \epsilon \text{ infinitesimal}$$

$$\tilde{\Lambda}^T\eta\tilde{\Lambda} = \eta \Rightarrow (1+\epsilon)^T\eta(1+\epsilon) = \eta \Rightarrow \epsilon^T\eta + \eta\epsilon = 0$$

Or lowering indices $\epsilon_{\mu\nu} = \eta_{\mu\lambda}\epsilon^\lambda{}_\nu$ this is $\epsilon_{\mu\nu} + \epsilon_{\nu\mu} = 0$ $\Rightarrow$ anti-symmetric

A $4\times4$ anti-symmetric matrix has $\frac{4\times3}{2}=6$ independent parameters $\to$ same counting as above!

Now $\tilde{\Lambda}^N = (1+\epsilon)(1+\epsilon)\cdots(1+\epsilon)$ is a transformation

and one can show

$$\Lambda = \lim_{N\to\infty}\left(1+\frac{1}{N}\omega\right)^N = \exp(\omega)$$

is a general expression for LT's. See eg Jackson.

Exercise: What is the analog in the case of rotations?

__Tensors & pseudo tensors.__

Tensors & pseudo tensors transform the same way under proper LTs:

$$T'^{\mu_1\cdots}{}_{\nu_1\cdots} = \Lambda^{\mu_1}{}_{\kappa_1}\cdots\Lambda_{\nu_1}{}^{\rho_1}\cdots T^{\kappa_1\cdots}{}_{\rho_1\cdots}$$

Tensors still transform this way under any (not neccesarily proper) LT. But for a pseudo-tensor under space inversion

$$T'^{\mu_1\cdots}{}_{\nu_1\cdots} = -\mathcal{P}^{\mu_1}{}_{\kappa_1}\cdots\mathcal{P}_{\nu_1}{}^{\rho_1}\, T^{\kappa_1\cdots}{}_{\rho_1\cdots}$$

where $\mathcal{P}^{\mu_1}{}_{\kappa_1} = \mathrm{diag}(1,-1,-1,-1)$, and a pseudo-tensor under time-reversal

$$T'^{\mu_1\cdots}{}_{\nu_1\cdots} = -\mathcal{T}^{\mu_1}{}_{\kappa_1}\cdots\mathcal{T}_{\nu_1}{}^{\rho_1}\, T^{\kappa_1\cdots}{}_{\rho_1\cdots}$$

where $\mathcal{T} = \mathrm{diag}(-1,1,1,1)$.

Ex: $\epsilon'^{\mu\nu\lambda\sigma} = \epsilon^{\mu\nu\lambda\sigma}$ is an invariant pseudo-tensor (under both P & T).

Particular examples:

A vector $V^\mu = (V^0, V^i) = (V^0, \vec{V})$

has $V'^0 = V^0$ and $\vec{V}' = -\vec{V}$ under $P$

$V'^0 = -V^0$ and $\vec{V}' = \vec{V}$ under $T$

An "axial" vector is a pseudo-vector under parity: $A^\mu = (A^0, \vec{A})$ has

$A'^0 = -A^0$ and $\vec{A}' = \vec{A}$ under $P$

## Electrodynamics:

### Field-strength tensor $F^{\mu\nu} = -F^{\nu\mu}$

$$F_{0i} = -F^{0i} = E^i \qquad F_{ij} = F^{ij} = -\epsilon^{ijk} B^k$$

I do not think much is gained by writing this as a 4x4 matrix but most textbooks do:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ & 0 & -B^3 & B^2 \\ \text{anti-} & & 0 & -B^1 \\ \text{sym} & & & 0 \end{pmatrix} \qquad F_{\mu\nu} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ & 0 & -B^3 & B^2 \\ \text{anti-} & & 0 & -B^1 \\ \text{sym} & & & 0 \end{pmatrix}$$

Charge-current (or 4-current - much like 4-momentum for $p^\mu = \left(\frac{1}{c}E, p^i\right)$).

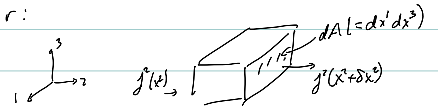
$$j^\mu = (c\rho, j^i)$$

Ex: Check the units (ie, engineering dimensions) $[c\rho] = [j^i]$.

### Charge conservation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0$$

Reminder:




In this infinitesimal volume in time $\Delta t$ change in charge $\Delta q$ = charge in − charge out

$= j^2(x^2) dA\,\Delta t - j^2(x^2+dx^2)\, dA\, \Delta t$

+ similar for other 2 pairs of faces

$= -\Delta t \left( \frac{\partial j^2}{\partial x^2} dx^2 dx^1 dx^3 + 2 \text{ terms} \right) = -\Delta t\, dV\, \vec{\nabla}\cdot\vec{j}$

This is $\boxed{\partial_\mu j^\mu = 0}$

### Maxwell's Equations

$$\partial_\nu F^{\nu\mu} = \frac{4\pi}{c} j^\mu \qquad \to \text{non-homogeneous} \quad \begin{cases} \text{Gauss} \\ \text{Ampere} \end{cases}$$

$$\epsilon^{\mu\nu\lambda\sigma} \partial_\nu F_{\lambda\sigma} = 0 \qquad \to \text{homogeneous} \quad \begin{cases} \text{Faraday} \\ \text{Gauss for (absence of) magnetic "charge"} \end{cases}$$

Exercise: Verify this by writing the 4 equations corresponding to the 2 cases $\mu=0$ and $\mu=i$ (x 2 equations).

Current conservation is required for consistency of Maxwell's equations:

$$\partial_\mu j^\mu = \partial_\mu (\partial_\nu F^{\nu\mu}) = \underbrace{\partial_\mu \partial_\nu}_{\text{symmetric}} \underbrace{F^{\nu\mu}}_{\text{anti-symmetric}} = 0$$

Note: Hom equation is, equivalently, $\partial_\nu F_{\lambda\sigma} + \partial_\lambda F_{\sigma\nu} + \partial_\sigma F_{\nu\lambda} = 0$

We are finally in a position to say something interesting in a trivial sort of way:

Maxwell equations are form-invariant Lorentz transformations:

$$F'^{\mu\nu}(x) = \Lambda^\mu{}_\rho \Lambda^\nu{}_\lambda F^{\rho\lambda}(\Lambda^{-1}x)$$

$$J'^\mu(x) = \Lambda^\mu{}_\rho J^\rho(\Lambda^{-1}x)$$

Exercise: although fairly explicit, you should know how to verify this statement!

We can explore some consequences by looking at components for specific transformations. We know everything (?) about rotations, so concentrate on boosts. Take boosts in $x^1 = x$-dir

$$-E'^i = F'^{0i} = \Lambda^0{}_\mu \Lambda^i{}_\nu F^{\mu\nu} \quad \text{with } \Lambda = \begin{pmatrix} c & -s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$c = \cosh\eta$
$s = \sinh\eta$
($x' = cx - sx^0 = \gamma(x - vt)$
so $x'=0$ is $x = vt \Rightarrow \beta = \frac{v}{c}$ is vel of $K'$ in $K$).

Skip the following tedious calculation in class:

$$-E'^1 = \Lambda^0{}_\mu \Lambda^1{}_\nu F^{\mu\nu} = \Lambda^0{}_0 \Lambda^1{}_k F^{0k} + \Lambda^0{}_1 \Lambda^1{}_0 F^{10} = c^2(-E^1) + s^2(E^1) = -E^1$$

$$-E'^2 = \Lambda^0{}_\mu \Lambda^2{}_\nu F^{\mu\nu} = \Lambda^0{}_0 \delta^2_\nu F^{0\nu} + \Lambda^0{}_1 \delta^2_\nu F^{1\nu} = c(-E^2) + (-s)(-B^3) = -(cE^2 - sB^3)$$

$$-E'^3 = \text{idem, note that } F^{13} = +B^2 \qquad = -(cE^3 + sB^2)$$

$$-B'^3 = F'^{12} = \Lambda^1{}_\mu \Lambda^2{}_\nu F^{\mu\nu} = \Lambda^1{}_0 F^{02} + \Lambda^1{}_1 F^{12} = (-s)(-E^2) + c(-B^3) = -(cB^3 - sE^2)$$

$$B'^2 = F'^{13} = \text{idem} \qquad = cB^2 + sE^3$$

$$-B'^1 = F'^{23} = \Lambda^2{}_\mu \Lambda^3{}_\nu F^{\mu\nu} = F^{23} = -B^1$$

or

$$\vec{E}'_\parallel = \vec{E}_\parallel \qquad \vec{E}'_\perp = \gamma(\vec{E}_\perp + \vec{\beta} \times \vec{B})$$

$$\vec{B}'_\parallel = \vec{B}_\parallel \qquad \vec{B}'_\perp = \gamma(\vec{B}_\perp - \vec{\beta} \times \vec{E})$$

where $\vec{\beta} = \frac{1}{c}$(velocity of unprimed system in primed one) $= -\frac{1}{c}$(vel. of primed system as seen from unprimed).

Note: $\vec{E}$ & $\vec{B}$ are clearly NOT components of some "$E^\mu$" & "$B^\mu$" 4-vectors.

Properties under discrete transformations P & T.

To determine transformation properties of $\vec{E}$ & $\vec{B}$, start from $\rho = j^0$. We know $\rho' = \rho$ under both P & T

So $j^\mu = (j^0 = \rho c, \vec{j})$ is a pseudo-tensor under T but not under P.

So we should have $\vec{j}' = -\vec{j}$, and this is physically sensible: P reverses current direction, as does T (running movie backwards).

Now $\partial_\mu$ is a vector (not-pseudo). So in

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\nu$$

we'll better have that $F'^{\mu\nu} = \underline{\underline{(+1)}} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma F^{\rho\sigma}$ for any LT: $F^{\mu\nu}$ is also not-pseudo.

So what does this mean for $\vec{E}$ & $\vec{B}$?

$$E'^i = F'^{i0} = -F^{i0} = -E^i \text{ under P}, \qquad E'^i = E^i \text{ under T}$$

$$B'^i = \underset{\text{(cyclic)}}{F'^{jk}} = F^{jk} = B^i \text{ under P}, \qquad B'^i = -B^i \text{ under T}$$

In accord with intuition.

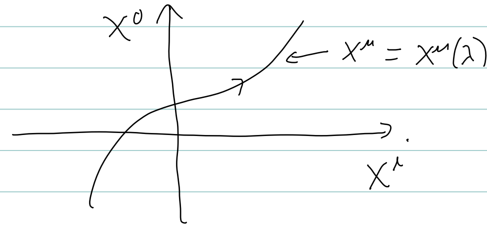
Exercise: verify that the Lorentz Force equation is co-variant under P & T

That Laws of Nature are invariant under any symmetries is an empirically determined fact. Could it be that they are only invariant under <u>proper</u> LT's but not under P nor T? YES, weak interactions are do not respect either.

But electromagnetic, strong, and gravitational interactions do respect both P & T. This will come handy in solving problems, modeling systems, etc.

# Kinematics of Relativistic Particle

World-line: or particle trajectory

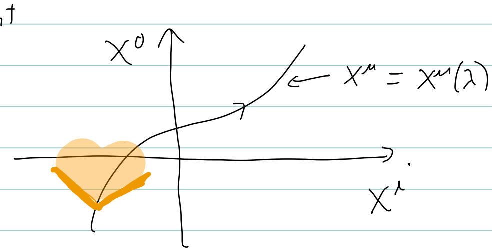



Not every collection of 4 functions $(x^0(\lambda), x^i(\lambda))$ is acceptable as describing the motion of a point particle:

(i) Move forward in time, or $\frac{dx^0}{d\lambda} > 0$

(ii) Move no faster than speed of light: $|\vec{v}| \le c \iff \frac{\left|\frac{d\vec{x}}{d\lambda}\right|}{\frac{dt}{d\lambda}} \le c \iff \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\eta_{\mu\nu} \geq 0$

(i)+(ii) ie, it lies in <u>future light cone</u> at each point of world line




Above we used condition (i) to implicitly invert $x^0 = x^0(\lambda)$ to give $\lambda = \lambda(x^0)$ (or $\lambda = \lambda(t)$) to write $\frac{d\vec{x}}{dt}(\lambda(t)) = \frac{d\vec{x}}{d\lambda}\cdot\frac{d\lambda}{dt}$.

We could have equally chosen a different parametrization of the same physical trajectory in the same coordinate frame:

Say the functions $x_1^\mu(\lambda_1)$ & $x_2^\mu(\lambda_2)$ are physically the same.

It must be that if we write $\lambda_2$ in terms of $\lambda_1$ then $x_2(\lambda_2)$ agrees with $x_1(\lambda_1)$:

$$x_2^\mu(\lambda_2(\lambda_1)) = x_1^\mu(\lambda_1)$$

That is, they both give the same trajectory $\vec{x}(t)$ as function of time $t = \frac{x^0}{c}$.

Two common preferred parameters:

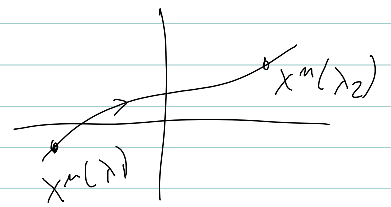
(i) $\lambda = x^0$, so $x^\mu(x^0) = (x^0, x^i(x^0))$

(ii) $\lambda = \tau$ (proper time) $x^\mu(\tau)$, so that $\frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau} = 1$

(Note, I use $\tau$ in units of space; often elsewhere $d\tau^2 = \frac{1}{c^2}dx^\mu dx_\mu$).

To be clear $d\tau^2 = \eta_{\mu\nu} dx^\mu dx^\nu$. We often use the symbol $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ for general intervals, and reserve $d\tau$ for intervals that satisfy $d\tau^2 > 0$.

Given a parametrization $x^\mu = x^\mu(\lambda)$ we can give the proper distance between two events $x^\mu(\lambda_1)$ & $x^\mu(\lambda_2)$ along $x^\mu(\lambda)$ by




$$S(\lambda_1, \lambda_2) = \int_{\lambda_1}^{\lambda_2} ds$$

$$\text{or} \quad S(\lambda_1, \lambda_2) = \int_{\lambda_1}^{\lambda_2} \sqrt{\frac{dx^\mu}{d\lambda} \frac{dx_\mu}{d\lambda}}\, d\lambda$$

$\frac{dx^\mu}{d\lambda}$ is a tangent vector to $x^\mu(\lambda)$

When $x^\mu$ is parametrized by $\tau$ we call this a 4-velocity

$$u^\mu \equiv \frac{dx^\mu}{d\tau}$$

It satisfies $u^2 = 1$

It's $\tau$ derivative is the 4-acceleration $a^\mu \equiv \frac{du^\mu}{d\tau}$

Since

$$\frac{d(u^2)}{d\tau} = \frac{d1}{d\tau} = 0 \quad \text{we have} \quad a^\mu u_\mu = 0$$

A particle of mass $m$ has 4-momentum $p^\mu \equiv mcu^\mu$

It satisfies $p^2 = m^2c^2$. $m$ is a Lorentz invariant (since $u^\mu$ is a 4-vector and we want our definition of $p^\mu$ to also give a 4-vector). The zeroth component is called energy $p^\mu = (p^0, p^i) = \left(\frac{E}{c}, p^i\right)$

We often reserve the symbol $v^i$ for a 3-vector that corresponds to the velocity of the particle: $v^i = \frac{dx^i}{dt}$

$$U^i = \frac{dx^i}{d\tau} = \frac{dx^i}{dt}\frac{dt}{d\tau} \qquad U^0 = \frac{dx^0}{d\tau} = c\frac{dt}{d\tau} \quad \Rightarrow \quad U^i = \frac{v^i}{c}\, U^0$$

Since $U^2 = 1$ $\quad U^{0\,2} - \vec{U}^2 = U^{0\,2}\left(1 - \frac{\vec{v}^2}{c^2}\right) = 1 \quad \Rightarrow \quad U^0 = \frac{1}{\sqrt{1 - v^2/c^2}}$

It is customary (and sometimes confusing, beware!) to use the same symbols here as for Lorentz transformations: $\vec{\beta} = \vec{v}/c$, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$
so that

$$U^\mu = (\gamma, \gamma\beta^i) \quad \text{and} \quad p^\mu = (\gamma mc, \gamma\beta^i mc)$$

$$\text{or} \quad E = \frac{mc^2}{\sqrt{1 - \vec{v}^2/c^2}} \quad \text{and} \quad \vec{p} = \frac{m\vec{v}}{\sqrt{1 - v^2/c^2}}.$$

This Lorentz-boost notation is natural in that one can obtain these results from boosting to an arbitrary frame with velocity $-\vec{v}$ from the (possibly instantaneous) rest frame

$$U^\mu = (1, 0)$$

In the instantaneous rest frame, $\vec{p} = 0$, $E = mc^2$ and $a^\mu = (0, a^i)$

## Lorentz Force

We are ready to write $\vec{F} = q(\vec{E} + \frac{\vec{v}}{c} \times \vec{B})$ in covariant form.
It is easy to "derive" the covariant form.
Since $\frac{d\vec{p}}{dt} = \vec{F}$, we look for an equation with

$$c\frac{dp^\mu}{d\tau} = \text{4 vector made of } \vec{E}, \vec{B} \text{ and } \vec{v}$$

Now $\vec{E}, \vec{B}$ are encoded in the 2-index tensor $F^{\mu\nu}$, and $\vec{v}$ in the 4 vector $U^\mu$. The 4-vector we search for must be linear in $F^{\mu\nu}$ (since $\vec{F}$ is linear in $\vec{E}, \vec{B}$); it is not necessarily linear in $U^\mu$ since $U^2 = 1$. Possibilities

$$F^{\mu\nu}U_\nu, \qquad F^{\sigma\nu}U_\sigma U_\nu U^\mu = 0, \qquad F^{\sigma\nu}M_{\sigma\nu}U^\mu = 0$$

(the last two vanish by $F^{\sigma\nu} = -F^{\nu\sigma}$, $U^\sigma U^\nu = U^\nu U^\sigma$, $M^{\sigma\nu} = M^{\nu\sigma}$).

So $\frac{dp^\mu}{d\tau} \propto F^{\mu\nu}U_\nu$. The proportionality constant is $q$: $\boxed{\frac{dp^\mu}{d\tau} = \frac{q}{c}F^{\mu\nu}U_\nu}$

It is understood that the field $F^{\mu\nu}(x)$ is at $x^\mu = x^\mu(\tau)$.

Note $\quad c\dfrac{dp^i}{d\tau} = q\left(F^{i0}U^0 - F^{ij}U^j\right)$

$$= q\left(E^i U^0 + \epsilon^{ijk}B^k U^j\right)$$

$$= qU^0\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)^i$$

and since $c\dfrac{dp^i}{d\tau} = \dfrac{dp^i}{dt}U^0 \quad \Rightarrow \quad \dfrac{d\vec{p}}{dt} = q\left(\vec{E} + \dfrac{\vec{v}}{c}\times\vec{B}\right)$

---

What about $c\dfrac{dp^0}{d\tau} = \dfrac{1}{c}U^0\dfrac{dE}{dt}$? It is not really an independent equation

since $U^\mu a_\mu = 0 \Rightarrow \quad U^0\dfrac{dp^0}{d\tau} = U^i\dfrac{dp^i}{d\tau} \quad \Rightarrow \quad \dfrac{dE}{dt} = c\dfrac{U^i}{U^0}\dfrac{dp^i}{dt} = \vec{v}\cdot\dfrac{d\vec{v}}{dt} = \dfrac{d}{dt}\left(\dfrac{m}{2}v^2\right)$

The right hand side is $\quad q\left(F^{0i}U_i\right) = q\left(-F^{0i}U^i\right) = q\vec{E}\cdot\left(\gamma\dfrac{\vec{v}}{c}\right)$

so $\quad \dfrac{dE}{dt} = q\,\vec{v}\cdot\vec{E} \qquad$ ( $E$ on LHS is energy, $\vec{E}$ on RHS is Electric field).

This follows trivially from $\dfrac{d}{dt}\left(\frac{1}{2}m\vec{v}^2\right) = \vec{v}\cdot\vec{F}$ with $\vec{F} = \dfrac{d}{dt}(m\vec{v})$.

Note that $\quad \dfrac{d\vec{p}}{dt} = q\left(\vec{E} + \dfrac{\vec{v}}{c}\times\vec{B}\right) \quad$ is NOT $\quad m\dfrac{d^2\vec{x}}{dt^2} = q\left(\vec{E} + \dfrac{\vec{v}}{c}\times\vec{B}\right)$

because $\quad \dfrac{d\vec{p}}{dt} = m\dfrac{d}{dt}\left(\dfrac{\vec{v}}{\sqrt{1-v^2/c^2}}\right)$

Exercise: Show

$$m\ddot{\vec{x}} = q\sqrt{1-\vec{v}^2/c^2}\left[\vec{E} - \left(\frac{\vec{v}}{c}\cdot\vec{E}\right)\frac{\vec{v}}{c} + \frac{\vec{v}}{c}\times\vec{B}\right]$$

## Vector Potential & 4-vectorpotential

Recall from electrostatics $\vec{\nabla}\times\vec{E}=0 \Rightarrow \vec{E}=-\vec{\nabla}\phi$ for some electrostatic potential $\phi=\phi(\vec{x})$.

We'd like to generalize this to include $\vec{B}$ field & time dependence.

$$\vec{\nabla}\times\vec{E}+\frac{1}{c}\frac{\partial\vec{B}}{\partial t}=0 \qquad (3)$$

We can also use the other homogeneous equation

$$\vec{\nabla}\cdot\vec{B}=0 \qquad (4)$$

Start from (4): since $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})=0$ for any $\vec{A}$, we propose $\vec{B}=\vec{\nabla}\times\vec{A}$
Can this hold for any $\vec{B}$? Let (I am not balancing indices, using $\delta_{\mu\nu}$ for metric):

$$A^i=-\frac{1}{\nabla^2}(\vec{\nabla}\times\vec{B})^i \quad\Rightarrow\quad (\vec{\nabla}\times\vec{A})^i=-\epsilon^{ijk}\partial_j\left(\frac{1}{\nabla^2}\epsilon^{kmn}\partial_m B^n\right)$$
$$=-\frac{1}{\nabla^2}\left(\partial_i\vec{\nabla}\cdot\vec{B}-\nabla^2 B^i\right)$$

So, provided $\vec{\nabla}\cdot\vec{B}=0$ and $\nabla^2$ is invertible (ie $\vec{B}$ is well behaved) then $\vec{B}=\vec{\nabla}\times\vec{A}$ is always possible. It is <u>not</u> unique:

$$\vec{B}=\vec{\nabla}\times\vec{A}'=\vec{\nabla}\times\vec{A} \iff \vec{\nabla}\times(\vec{A}'-\vec{A})=0$$

. We know the solution to this!
It is just like electrostatics: $\vec{A}'-\vec{A}=-\vec{\nabla}\omega$ for some scalar $\omega$.

Now for (3): using $\vec{B}=\vec{\nabla}\times\vec{A}$

$$\vec{\nabla}\times\vec{E}+\frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A})=0 \quad\text{or}\quad \vec{\nabla}\times\left(\vec{E}+\frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right)=0 \quad\Rightarrow\quad \vec{E}+\frac{1}{c}\frac{\partial\vec{A}}{\partial t}=-\vec{\nabla}\phi \quad \text{as before.}$$

Summary: (3) & (4) insure we can write

$$\vec{E}=-\vec{\nabla}\phi-\frac{1}{c}\frac{\partial\vec{A}}{\partial t} \qquad \vec{B}=\vec{\nabla}\times\vec{A}$$

Names:
$\vec{A}$ "vector potential".

Note: if $\vec{A}\to\vec{A}'=\vec{A}-\vec{\nabla}\omega$ and $\phi\to\phi'=\phi+\frac{1}{c}\frac{\partial\omega}{\partial t}$ then $E\to E$ & $B\to B$

This is called "gauge invariance" of $\vec{E}$ & $\vec{B}$.

Relativistic extension

We can combine the above expressions as follows:

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu}$$

Check: $F_{0i} = E^i$, $F_{0i} = \partial_0 A_i - \partial_i A_0 = -\partial_0 A^i - \partial_i A^0 = -\frac{1}{c}\frac{\partial A^i}{\partial t} - \partial_i \phi$

works! provided $\phi = A^0 = A_0$ ; and $\vec{A}$ is $A^i$

$F_{ij} = -\epsilon_{ijk} B^k = \partial_i A_j - \partial_j A_i = -(\partial_i A^j - \partial_j A^i)$

or $B^3 = \partial_1 A^2 - \partial_2 A^1$ etc ✓

Alternatively, $\epsilon^{\mu\nu\lambda\sigma}\partial_\nu F_{\lambda\sigma} = 0 \iff F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

(This is the 4D generalization of $\nabla \times \vec{B} = 0 \iff \vec{B} = \vec{\nabla}\times\vec{A}$).

$A_\mu$ is also called "vector potential", often also called a gauge field.

Gauge invariance: $A_\mu \to A_\mu + \partial_\mu \omega$

Trivially seen directly: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \Rightarrow \partial_\mu(A_\nu + \partial_\nu\omega) - \partial_\nu(A_\mu + \partial_\mu\omega) = \partial_\mu A_\nu - \partial_\nu A_\mu$

since $\partial_\mu\partial_\nu\omega - \partial_\nu\partial_\mu\omega = 0$.

Non-homogeneous equations in terms of $A_\mu$:

Now $\partial_\mu F^{\mu\nu} = \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial^2 A^\nu - \partial^\nu \partial\cdot A$

Here $\partial^2 = \eta^{\mu\nu}\partial_\mu\partial_\nu = \frac{1}{c^2}\frac{\partial}{\partial t^2} - \vec{\nabla}^2$ is also "$\Box$" or "$\Box^2$" in textbooks

is the "d'Alembert operator", or "d'Alembertian" which appears prominently in the wave equation.

(Also $\partial\cdot A = \eta^{\mu\nu}\partial_\mu A_\nu$). So Maxwell (1) & (2) are

$$\partial^2 A_\mu - \partial_\mu(\partial A) = \frac{4\pi}{c} j_\mu \qquad (X)$$

An application of gauge invariance: since $F_{\mu\nu}$ is invariant under $A_\mu \to A'_\mu = A_\mu + \partial_\mu \omega$, given a solution $A_\mu$ of (X) above we can find another one, $A'_\mu$, such that $\partial \cdot A' = 0$, since

in $\partial \cdot A' = \partial \cdot A + \partial^2 \omega = 0$ provided the solution to

$$\omega = -\frac{1}{\partial^2} \partial \cdot A \quad \text{exists.}$$

This is generally the case. So we can look for solutions to (X) that satisfy both (X) and $\partial \cdot A = 0$. But the latter means the former is

$$\partial^2 A_\mu = \frac{4\pi}{c} j_\mu \quad \Rightarrow \quad \boxed{A_\mu = \frac{4\pi}{c} \frac{1}{\partial^2} j_\mu}$$

The procedure above is called "gauge fixing", and the field $A_\mu$ that satisfies $\partial \cdot A = 0$ is said to be in "covariant" or "Lorentz" gauge.

# Inverting differential operators. Green functions. Distributions.

We have seen a need for inverting $\nabla^2$ & $\partial^2$. Let's!
For $\nabla^2$ we may as well put it in the important context of solving the Poisson Equation. This arises in electrostatics (and in Newtonian gravitation):

since $\vec{E} = -\vec{\nabla}\phi$ for static fields, we have then

$$\text{Gauss's law } \vec{\nabla}\cdot\vec{E} = 4\pi\rho \Rightarrow -\nabla^2\phi = 4\pi\rho \Rightarrow \phi = -4\pi\frac{1}{\nabla^2}\rho$$

Solve $\nabla^2\phi = -4\pi\rho$

"Solve" means, given $\rho = \rho(\vec{x})$, determine $\phi(\vec{x})$; there are other related problems, in which we specify $\rho = \rho(\vec{x})$ and a boundary to the volume under consideration, and specify a boundary condition. In our case, we want $\rho(\vec{x})$ with compact support and $\phi = \phi(\vec{x})$ everywhere in space, up to solutions to $\nabla^2\phi = 0$.

$\rho$ has compact support, and is sufficiently well behaved, so $\int d^3x\, \rho^2 < \infty \Rightarrow$ we can find its Fourier transform

$$\tilde{\rho}(\vec{k}) \equiv \int d^3x\, e^{i\vec{k}\cdot\vec{x}}\rho(\vec{x}) \quad \Rightarrow \quad \rho(\vec{x}) = \frac{1}{(2\pi)^3}\int d^3k\, e^{-i\vec{k}\cdot x}\tilde{\rho}(\vec{k})$$

Do this for $\phi$ as well. Then

$$\nabla^2\phi = \frac{1}{(2\pi)^3}\vec{\nabla}^2\int d^3k\, e^{-i\vec{k}\cdot\vec{x}}\tilde{\phi}(\vec{k}) = \frac{1}{(2\pi)^3}\int d^3k\,(-k^2)\tilde{\phi}(\vec{k})$$

$$\Rightarrow \quad -k^2\tilde{\phi}(\vec{k}) = -4\pi\tilde{\rho}(\vec{k}) \quad \Rightarrow \quad \tilde{\phi}(\vec{k}) = -\frac{1}{k^2}\left(-4\pi\tilde{\rho}(\vec{k})\right) = \frac{4\pi}{k^2}\tilde{\rho}(k)$$

and $$\phi(\vec{x}) = \frac{4\pi}{(2\pi)^3}\int d^3k\, e^{-i\vec{k}\cdot\vec{x}}\frac{1}{k^2}\tilde{\rho}(\vec{k})$$

We can express this back in terms of $\rho(\vec{x})$:

$$\phi(\vec{x}) = \frac{4\pi}{(2\pi)^3}\int d^3k\, e^{-i\vec{k}\cdot\vec{x}}\frac{1}{k^2}\int d^3y\, e^{i\vec{k}\cdot\vec{y}}\rho(\vec{y}) = \int d^3y\, G(\vec{x},\vec{y})\,\rho(\vec{y})$$

Here $G(\vec{x},\vec{y})$ is a Green function, satisfying $\nabla^2_{(x)}G(\vec{x},\vec{y}) = -4\pi\delta^{(3)}(\vec{x}-\vec{y})$

where $\nabla^2_{(x)}$ means it is the Laplacian with respect to $\vec{x}$ (to distinguish from $\vec{y}$), and $\delta^{(3)}(\vec{x})$ is the 3D Dirac-delta function.

$\delta^{(3)}(\vec{x}) = \delta(x)\,\delta(y)\,\delta(z)$, with $\delta(x)=0$ for $x\neq 0$ and $\int_{-\infty}^{\infty} dx\, \delta(x) = 1$

so that $\int d^3x\, \delta^{(3)}(\vec{x}) = 1$

Just like $\int_{-\infty}^{\infty} d\omega\, e^{i\omega t} = 2\pi\delta(t)$, we have $\int d^3k\, e^{i\vec{k}\cdot\vec{x}} = (2\pi)^3\,\delta^{(3)}(\vec{x})$

Now, we read off from above

$$G(\vec{x},\vec{y}) = \frac{4\pi}{(2\pi)^3}\int d^3k\; e^{i\vec{k}\cdot(\vec{y}-\vec{x})}\,\frac{1}{k^2} \qquad (G)$$

from which we learn that $G(\vec{x},\vec{y}) = G(\vec{x}-\vec{y})$ reflecting invariance under $\vec{x}\to\vec{x}+\vec{a}$ $\vec{a}$ = constant vector of Poisson equation.

Exercise: verify that $G$ satisfies $\nabla^2 G = -4\pi\delta^{(3)}$ by direct computation

Note that we can add to this any solution to the associated homogeneous equation. if $\nabla^2 F = 0$ then $\nabla^2(G+F) = -4\pi\delta^{(3)}$ (provided $\nabla^2 G = -4\pi\delta^{(3)}$).

We'll take a brief look at $\nabla^2\phi = 0$ and boundary value problems in electrostatics later.

Physical interpretation: compare $\nabla_x^2\, G(x,y) = -4\pi\delta^{(3)}(\vec{x}-\vec{y})$ with $\nabla^2\phi = -4\pi\rho$

$\Rightarrow$ $G$ is the electric potential due to a charge distribution $\rho(\vec{x}) = \delta^{(3)}(\vec{x}-\vec{y})$

That is $\rho = 0$ everywhere except at $\vec{x}=\vec{y}$, and $\int d^3x\,\rho = 1$

$\Rightarrow$ a unit charge at $\vec{y}$.

But we know $\phi = \frac{q}{r}$ for charge at origin: the $\propto \frac{1}{r}$ is from Coulomb's law

the constant of proportionality from $\int_V \vec{\nabla}\cdot\vec{E}\, dV = \int_{\partial V} \vec{E}\cdot\hat{n}\, da$ using $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$

so that LHS $= 4\pi q$ and $\vec{E} = -\nabla\phi$ with $V$ = a sphere so RHS $= \int R^2 d\Omega \left(\frac{\partial}{\partial r}\left(-\frac{K}{r}\right)\right)\Big|_{r=R} = 4\pi K$.

$\Rightarrow$ $\phi = \frac{q}{|\vec{x}-\vec{y}|}$ for charge $q$ at $\vec{y}$. Setting $q=1$, $G(\vec{x}-\vec{y}) = \frac{1}{|\vec{x}-\vec{y}|}$

Exercise:

(i) Check $\nabla^2 \frac{1}{|x|} = 0$ for $\vec{x}\neq 0$, and $\int \nabla^2 \frac{1}{|x|}\, d^3x = -4\pi$

(ii) Verify by direct integration of $(G)$ that $G(\vec{x}) = \frac{1}{|x|}$.

Now, let's try to do for $\partial^2$ what we did for $\nabla^2$:

Let $-\partial^2 \phi = 4\pi\rho$ (which is Gauss's Law in Lorentz gauge).

We try the same technique

$$\tilde{\rho}(k) = \int d^4x \, e^{-ik\cdot x} \rho(x) \qquad \text{where } k^\mu \text{ is a 4-vector, } k\cdot x = k^\mu x_\mu$$

Incidentally, $k\cdot x = k^0 x^0 - \vec{k}\cdot\vec{x} = \omega t - \vec{k}\cdot\vec{x}$ so $\omega = k^0 c$ has the interpretation of frequency (while $\vec{k}$ is still a wave-vector).

and $$\rho(x) = \frac{1}{(2\pi)^4} \int d^4k \, e^{ik\cdot x} \tilde{\rho}(k)$$

As before $$\partial^2 e^{ik\cdot x} = -k^2 e^{ik\cdot x}$$

and $$\phi(x) = \int d^4y \, G(x,y) \rho(y)$$

with $G(x,y) = G(x-y)$ and $\partial^2 G(x) = -4\pi \delta^{(4)}(x)$.

But there is one interesting twist: while formally

$$G(x) = \frac{4\pi}{(2\pi)^4} \int d^4k \, e^{ik\cdot x} \frac{1}{k^2} \Rightarrow \partial^2 G = \frac{4\pi}{(2\pi)^4} \int d^4k \, e^{ik\cdot x} (-k^2) \frac{1}{k^2} = -4\pi \delta^4(x)$$

the actual integral is ill-defined:

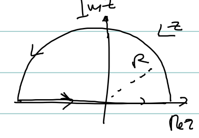
$$\frac{1}{k^2} = \frac{1}{(k^0)^2 - (\vec{k})^2} \qquad \text{diverges for } (k^0)^2 = (\vec{k})^2$$

Consider the $k^0$ integration:

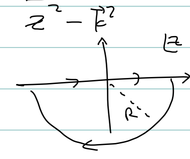
$$\int_{-\infty}^{\infty} dk^0 \frac{e^{ik^0 x^0}}{(k^0)^2 - (\vec{k})^2}$$

Consider the complex variable integration $\displaystyle\int_{C} dz \frac{e^{izx^0}}{z^2 - \vec{k}^2}$

where $C$ is




for $x^0 > 0$ and




for $x^0 < 0$

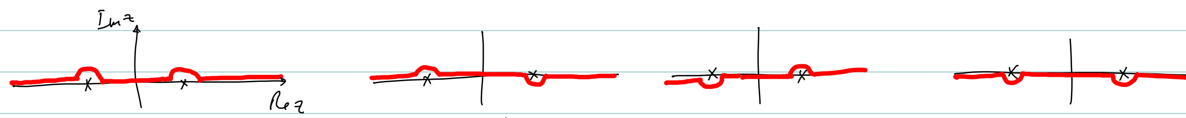
These choices guarantee $e^{izx^0} = e^{-R|x^0|}$ on the circle and vanishes exponentially fast as $R \to \infty$.

This allows us to analyze our integral using Cauchy's theorem. The only problem is that there are poles at

$$z = +|\vec{k}| \quad \text{and} \quad z = -|\vec{k}|$$

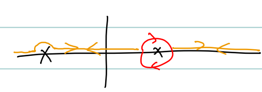
and the contour of integration goes through them.

So let's propose the following: $G$ is defined by a choice of deformation of the contour to go just above or just below the poles. There are 4 possibilities:




(1) or $G_{++}$ (2) $G_{+-}$ (3) $G_{-+}$ (4) $G_{--}$

By Cauchy's theorem the size of the deformation does not matter. So it can be infinitesimal, and the resulting integrals should all give Green functions $\Rightarrow$ the difference between any two should be a solution to the homogeneous equation. Easy to see. Take (1) − (2). Same as (1) followed by reversing direction of (2): cancels everywhere except at $z = +|\vec{k}|$



But this is an easy integral: $\displaystyle\int_{\substack{\text{little}\\\text{circle}}} dz \frac{e^{izx^0}}{(z-|k|)(z+|k|)} = 2\pi i \left.\frac{e^{izx^0}}{z+|k|}\right|_{z=|k|} = \frac{2\pi i\, e^{i|k|x^0}}{2|k|}$

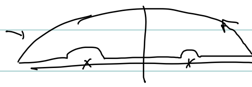
and $\displaystyle \Delta G = \frac{4\pi}{(2\pi)^3} \int d^3k \, \frac{e^{-i\vec{k}\cdot\vec{x} + i|\vec{k}|x^0}}{2|\vec{k}|}$

This is a well defined integral and has $\partial^2(\Delta G) = 0$. In fact

Exercise: if $F(x) = \int d^3k \, f(\vec{k}) \, e^{i\omega x^0 - i\vec{k}\cdot\vec{x}}$ is well defined, show that

$\partial^2 F = 0$ provided $\omega = \pm|\vec{k}|$

So all 4 Green functions are good candidates. But they have interesting physically distinguishing properties.

(1) $G_{++}$. Since there are no poles inside $C$ for $C \to$  corresponding to $x^0>0$

we have $G_{++}=0$ for $x^0>0$. This is called the "Advanced" Green fct.

(4) $G_{--}$. Similarly $G_{--}=0$ for $x^0<0$. "Retarded" Green functions

(3) Appears naturally in Quantum field theory, "Feynman" Green function and has the peculiar property that it can be deformed into an integral along imaginary axes. Not of interest to us for now.

Before we compute $G_{Adv}$ and $G_{ret}$ explicitly, note physical interpretation:

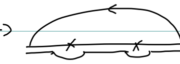
Since
$$\phi(x) = \int d^4y\, G(x-y)\, \rho(y)$$

we have $\phi(\vec{x},x^0) = \int d^4y\, G_{ret}(x,y)\,\rho(\vec{y},y^0) = \int d^3y \int_{-\infty}^{x^0} dy^0\, G_{ret}(\vec{x}-\vec{y}, x^0-y^0)\,\rho(\vec{y},y^0)$

$\Rightarrow \phi(\vec{x},t)$ depends on $\rho(\vec{x},t')$ only for times $t'$ before $t$, ie, $t'<t$. This makes sense for future evolution: one must know the history of the charge distribution in the past to predict the future.

Similarly $G_{Adv}$ gives $\phi(\vec{x},t)$ from knowledge of the future charge distribution which can be of interest in, say, scattering if we know the future outcome.

Explicit computation.

We have $G_{ret} = \frac{4\pi}{(2\pi)^4}\int d^3k\, e^{-i\vec{k}\cdot\vec{x}} \int dz\, e^{izx^0} \frac{1}{z^2-\vec{k}^2} \longrightarrow$ 

$$= \frac{4\pi}{(2\pi)^3} i \int d^3k\, e^{-i\vec{k}\cdot\vec{x}} \left( \frac{e^{i|k|x^0}}{2|k|} + \frac{e^{-i|k|x^0}}{-2|k|} \right)$$

$$= \frac{i}{(2\pi)^2} \int d^3k\, \frac{1}{|k|}\, e^{-i\vec{k}\cdot x} \left( e^{i|k|x^0} - e^{-i|k|x^0} \right)$$

Now $\int d^3k \frac{e^{-i\vec{k}\cdot\vec{x}}}{|\vec{k}|} e^{\mp ikx^0} = 2\pi \int_0^\infty k^2 dk \int_{-1}^{1} d\cos\theta \ \frac{1}{k} e^{\mp ikx^0} e^{-ikx\cos\theta}$ $\quad x = |\vec{x}|$ for short

$$= 2\pi \frac{1}{(-ix)} \int_0^\infty dk \ e^{\mp ikx^0} \left( e^{-ikx} - e^{ikx} \right)$$

Combining

$$G_{ret} = \frac{i}{(2\pi)^2} \frac{2\pi}{i|x|} \int_0^\infty dk \ (e^{ikx} - e^{-ikx})(e^{ikx^0} - e^{-ikx^0})$$

$$= \frac{1}{2\pi} \frac{1}{|x|} \cdot \frac{1}{2} \int_{-\infty}^\infty dk \ (e^{ikx} - e^{-ikx})(e^{ikx^0} - e^{-ikx^0})$$

since integrand is even under $k \to -k$.

$$= \frac{1}{2\pi} \frac{1}{|x|} \frac{1}{2} 2\pi \left[ 2\delta(|\vec{x}| - x^0) + 2\delta(|\vec{x}| + x^0) \right]$$

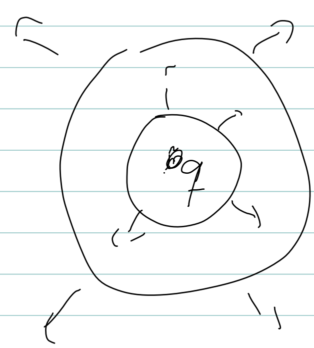
and since this is only for $x^0 > 0$, $\delta(|\vec{x}| + x^0) = 0$ so

$$\boxed{G_{ret}(\vec{x}, x^0) = \frac{\delta(x^0 - |\vec{x}|)}{|\vec{x}|}}$$

Ex: show that

$$G_{adv}(\vec{x}, x^0) = -\frac{\delta(x^0 + |\vec{x}|)}{|x|}$$

A charge that appears for an instant and then dissapears (not physical?), at $t=0$, at the origin, produces a potential field that propagates at the speed of light, spherically outward from the origin, with amplitude $\frac{1}{|x|}$ concentrated on that shell at $|\vec{x}| = ct$:




Cool! ◡̈ Moreover, easy to see real physical applications: start from charge distribution at rest and then have it start moving.

## Action Integral, Lagrangian, Conservation Laws

Obtaining the equations of motion (EOM) as extrema of an action integral $\mathcal{S}$, is useful in various ways: exploring symmetries, quantization, extension to other models of reality, and so on.

One of the nice things about it (not emphasized in kindergarten) is that covariance of EOM under some transformation corresponds to invariance of $\mathcal{S}$. This is powerful stuff: if you have a good reason to suspect nature is symmetric under a group $G$ of transformations, then make sure $\mathcal{S}$ is invariant under $G$.

(Digression: for those of you who know group theory, the set of physical transformations on a system forms a group, $G$, with multiplication given as one transformation followed by the other:

(i) there is an identity element $e \in G$, the "do nothing" transformation

(ii) for every transformation $g \in G$ there is an inverse $g^{-1} \in G \ni g^{-1}g = gg^{-1} = e$, the "undo" what you "did"

(iii) if $g_1 \in G$, $g_2 \in G \Rightarrow g_1 g_2 \in G$ ).

An example is the EOM for a relativistic free particle (we'll add EM interactions later). It has world-line $x^\mu(\lambda)$. We want a functional $\mathcal{S}[x^\mu(\lambda)]$ that is invariant under Lorentz transformations. It should also be independent of the way in which we parametrize the world-line, ie, it should be reparametrization invariant. We know the interval $ds$ is invariant, $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ and written this way it is independent of $\lambda$

We therefore have as simplest invariant action integral

$$\mathcal{S}[x^\mu(\lambda)] = \alpha \int ds \quad , \text{with } \alpha \text{ some constant.}$$

If we want to be more careful we can specify that this is for paths between $x_1^\mu$ and $x_2^\mu$ so that $x^\mu(0) = x_1^\mu$ and $x^\mu(1) = x_2^\mu$ are fixed end-points and write

$$\mathcal{S}[x(\lambda); x_1, x_2] = \alpha \int_{x_1}^{x_2} ds$$

but often we leave $x_1$ & $x_2$ as implicitly understood. Note that, eg, $\int ds^2$ makes no sense (the square of an infinitesimal integrates to zero).

For computations use $ds^2 = \eta_{\mu\nu} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} d\lambda^2 > 0$

$$S[x(\lambda)] = \kappa \int_0^1 d\lambda \sqrt{\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}$$

Now $S[x+\delta x] - S[x] = \kappa \int_0^1 d\lambda \left( \sqrt{\eta_{\mu\nu}(\dot{x}+\delta\dot{x})^\mu(\dot{x}+\delta\dot{x})^\nu} - \sqrt{\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} \right)$ where $\dot{x} = \frac{dx^\mu}{d\lambda}$

$$= \kappa \int_0^1 dx\, \frac{\eta_{\mu\nu} \frac{dx^\mu}{d\lambda}\frac{d\delta x^\nu}{d\lambda}}{\sqrt{\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}$$

$$= -\kappa \int_0^1 d\lambda\, \delta x^\nu \frac{d}{d\lambda}\left[\frac{\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}}{\sqrt{\eta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}}\right] + \text{boundary terms}$$

So that the EOM is $\quad \frac{d}{d\lambda}\left[\frac{\kappa}{\sqrt{\eta_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}}}\frac{dx^\mu}{d\lambda}\right] = 0 \qquad (*)$

If we choose $\lambda = x^0$, then this is

$$\frac{d}{dt}\left(\frac{\kappa}{\sqrt{1-v^2/c^2}}\right) = 0 \qquad \frac{d}{dt}\left[\frac{\kappa}{\sqrt{1-\vec{v}^2/c^2}}\vec{v}\right] = 0 \qquad (@)$$

but no $\mu=0$ equation because there is no variation $\delta x^0$ of the trajectory (since $x^0(\lambda)=\lambda$ is fixed). Using $\lambda = x^0$ we can connect to Lagrangian mechanics; generally

$$S[q(t); t_1, t_2] = \int_{t_1}^{t_2} dt\, L(q,\dot{q})$$ ← Lagrangian, a function of two variables

and EOM's are $\quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \quad$ and momentum $p = \frac{\partial L}{\partial \dot{q}}$

The relativistic particle has, from above

$$S = \kappa c \int_{t_1}^{t_2} \sqrt{1-\vec{v}^2/c^2}\, dt \quad . \text{ So } \quad L(x^i, v^i) = \kappa c\sqrt{1-\vec{v}^2/c^2}, \quad p^i = \frac{\partial L}{\partial v^i} = \frac{1}{\sqrt{1-v^2/c^2}}\left(\frac{-\kappa v^i}{c}\right)$$

We see that with $\kappa = -mc$ this corresponds to momentum as found earlier (plus $S$ then has the proper dimensions). For EOM we have

$$\frac{\partial L}{\partial v^i} = \frac{1}{\sqrt{1-v^2/c^2}} m v^i \qquad \frac{\partial L}{\partial x^i} = 0 \Rightarrow \frac{d}{dt}\left(\frac{m}{\sqrt{1-v^2/c^2}} v^i\right) = 0 \quad \text{as above.}$$

Note that this is just $\frac{d\vec{p}}{dt} = 0$

The non-relativistic limit $L = -mc^2\sqrt{1-v^2/c^2} = -mc^2\left[1 - \frac{1}{2}\frac{v^2}{c^2} + \dots\right] = -mc^2 + \frac{m}{2}v^2 + \dots$

$-mc^2$ is an irrelevant, BUT INTERESTING, constant: if we write $L = T - V$ where $T = \frac{1}{2}mv^2$ is kinetic and $V$ is potential energies, then $V = mc^2$, which the student should recognize as the rest energy of a particle in special relativity.

The Hamiltonian (or energy, since there is no explicit time dependence) is

$$H(q,p) = \dot{q}p - L$$

where $\dot{q} = \dot{q}(q,p)$ is obtained from $p = \frac{\partial L}{\partial \dot{q}}$.

In the present case ($H = H(q,p)$, so rewrite by solving for $\vec{v}$ in terms of $\vec{p}$)

$$H = v^i \frac{mv^i}{\sqrt{1-\vec{v}^2/c^2}} - \left(-mc^2\sqrt{1-\vec{v}^2/c^2}\right)$$

$$= \frac{mc^2}{\sqrt{1-v^2/c^2}}\left[\left(1-\frac{\vec{v}^2}{c^2}\right) + \frac{v^2}{c^2}\right] = \frac{mc^2}{\sqrt{1-v^2/c^2}} = E$$

The Hamiltonian should be written in terms of $p$,

$$\frac{\vec{p}^2}{m^2c^2} = \frac{\vec{v}^2/c^2}{1-\vec{v}^2/c^2} \Rightarrow \frac{\vec{v}^2}{c^2}\left(1+\frac{\vec{p}^2}{m^2c^2}\right) = \frac{\vec{p}^2}{m^2c^2} \Rightarrow \frac{\vec{v}^2}{c^2} = \frac{\vec{p}^2}{m^2c^2+\vec{p}^2} \Rightarrow 1-\frac{\vec{v}^2}{c^2} = \frac{m^2c^2}{(mc)^2+\vec{p}^2}$$

So $E = \sqrt{(mc^2)^2 + \vec{p}^2c^2}$. No surprise!

Note that (@) above is $\frac{dE}{dt} = 0$.

The variation (functional derivative) defines the 4-momentum

$$\frac{\delta S}{\delta x^\mu(s)} = -p_\mu \frac{d}{d\lambda}\delta x^\mu$$

This makes it clear that $p_\mu$ transforms as a lower index 4vector. Using $\lambda = t$ the components of $p^\mu$ are what we saw above

$$p^\mu = \left(\frac{E}{c}, \vec{p}\right)$$

so they form a 4-vector. Using the expression for $E$ above, $p^2 = E^2/c^2 - \vec{p}^2 = (mc)^2$

# Conservation Laws & Continuous Symmetries

Let us write generically

$$S[x^\mu] = \int_0^1 d\lambda \; \mathcal{L}\left(x^\mu, \frac{dx^\mu}{d\lambda}\right)$$

where $\mathcal{L}(x^\mu, v^\mu)$ is a function of these two variables. If the parameter where time $(\lambda = t)$ then $\mathcal{L}$ would correspond with the Lagrangian.

We can recover equations of motion by extremizing $S$:

$$\delta S = 0 \Rightarrow \int_0^1 d\lambda \left[ \frac{\partial \mathcal{L}}{\partial x^\mu} \delta x^\mu + \frac{\partial \mathcal{L}}{\partial v^\mu} \frac{d\delta x^\mu}{d\lambda} \right] = 0$$

Here $\frac{\partial \mathcal{L}}{\partial x^\mu}$ and $\frac{\partial \mathcal{L}}{\partial v^\mu}$ are understood to be evaluated on $v^\mu = \frac{dx^\mu}{d\lambda}$. Integrating by parts:

$$\frac{\partial \mathcal{L}}{\partial v^\mu} \delta x^\mu \Big|_0^1 + \int_0^1 d\lambda \left[ \frac{\partial \mathcal{L}}{\partial x^\mu} - \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial v^\mu}\right)\right] \delta x^\mu = 0$$

The first term vanishes because $\delta x^\mu = 0$ at $\lambda = 0, 1$. The second term must then vanish, for any $\delta x^\mu(\lambda) \Rightarrow$ the $[\,..\,]$ must vanishes at each $\lambda$. This is a version of the Euler-Lagrange equation:

$$\boxed{\frac{\partial \mathcal{L}}{\partial x^\mu} - \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial v^\mu}\right) = 0} \qquad (EOM)$$

We define 4-momentum by

$$p_\mu = -\frac{\partial \mathcal{L}}{\partial v^\mu}$$

Symmetries: suppose that $x^\mu \to x'^\mu + \epsilon^\mu(x) + \mathcal{O}(\epsilon^2)$ is a symmetry of $S$, that is $S[x'] = S[x]$
Then
infinitesimal

$$0 = S[x'] - S[x] = \int d\lambda \left[ \frac{\partial \mathcal{L}}{\partial x^\mu} \epsilon^\mu(x) + \frac{\partial \mathcal{L}}{\partial v^\mu} \frac{d\epsilon^\mu(x)}{d\lambda} \right]$$

Now, evaluating this for solutions of the EOM, $\frac{\partial \mathcal{L}}{\partial x^\mu} = \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial v^\mu}\right)$ we have

$$0 = \int d\lambda \left[ \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial v^\mu}\right)\epsilon^\mu + \frac{\partial \mathcal{L}}{\partial v^\mu}\frac{d\epsilon^\mu}{d\lambda}\right] = \int d\lambda \frac{d}{d\lambda}\left[\frac{\partial \mathcal{L}}{\partial v^\mu}\epsilon^\mu(x)\right] = \frac{\partial \mathcal{L}}{\partial v^\mu}\epsilon^\mu \Big|_{in}^{fin} = -p_\mu \epsilon^\mu \Big|_{ini}^{fin}$$

or $p_\mu \epsilon^\mu(\text{final}) = p^\mu \epsilon_\mu(\text{initial})$

But "final" is arbitrary along world line $x^\mu(\lambda)$: this just means $\boxed{p_\mu \epsilon^\mu(x) = \text{constant along } x^\mu(\lambda)}$

Translations: suppose $x^\mu \to x^\mu + \epsilon^\mu$ with $\epsilon^\mu$ = constant is a symmetry.

Then $p_\mu \epsilon^\mu$ = constant. Since $\epsilon_\mu$ is an arbitrary fixed vector this implies:

$p_\mu$ = constant : Conservation of momentum (as a result of translational invariance in space-time)

Angular-Momentum

Assume invariance under Lorentz transformations $x'^\mu = \Lambda^\mu{}_\nu x^\nu$. The infinitesimal version is $x'^\mu = x^\mu + \epsilon^\mu{}_\nu x^\nu$ with $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$ ( $\epsilon_{\mu\nu} = \eta_{\mu\lambda} \epsilon^\lambda{}_\nu$ )

$\Rightarrow$ $p_\mu \epsilon^\mu{}_\nu x^\nu$ = constant

$\Rightarrow$ $\epsilon_{\mu\nu} x^\mu p^\nu$ = constant

Since $\epsilon_{\mu\nu}$ is an arbitrary anti-symmetric tensor, it has 6 independent components that allow us to impose 6 independent conditions on $x^\mu p^\nu$. Since

$$x^\mu p^\nu = \tfrac{1}{2}(x^\mu p^\nu + x^\nu p^\mu) + \tfrac{1}{2}(x^\mu p^\nu - x^\nu p^\mu)$$

and $\epsilon_{\mu\nu} \cdot S^{\mu\nu} = 0$ for any $S^{\mu\nu} = S^{\nu\mu}$, it follows that the 6 components of

$$M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$$

are constrained by the above, $M^{\mu\nu}$ = constant. This 2-index antisymmetric tensor contains angular momentum: $M^{ij} = x^i p^j - x^j p^i = \epsilon^{ijk} L_k$, where $L_k = \epsilon_{ijk} x^i p^k$ or $\vec{L} = \vec{r} \times \vec{p}$, as usual. But what is $M^{0i}$?

$$M^{0i} = x^0 p^i - x^i p^0 = \text{constant} \Rightarrow x^i = \left(\frac{p^i}{p^0}\right) x^0 + \text{constant}$$

In the non-relativistic limit: $p^0 = mc$ and $x^i = \frac{p^i}{m} t$, an elementary equation.

For many particles this is interesting:

$$M^{0i}_{tot} = \sum_n M^{0i}_n = x^0 \sum_n p^i_n - \sum_n x^i_n p^0_n = x^0 P^i_{TOT} - \sum x^i_n p^0_n$$

Define the center of energy (mass) as

$$X^i_{cm} = \frac{\sum_n x^i_n p^0_n}{\sum_n p^0_n} \quad \text{and total energy } P^0 = \sum_n p^0_n$$

Then $x^0 P^i = X^i_{cm} P^0$ or

$$X^i_{CM} = \left(\frac{P^i}{P^0}\right) x^0 = \frac{P^i}{E/c^2} t$$

This is the relativistic generalization of center of mass.

## Interactions with E&M fields.

The non-relativistic limit of a point charge interacting with (i.e., moving under the effect of a) electric field described by a potential $\phi = A^0$ is described by a Lagrangian

$$L = \frac{1}{2} m \vec{v}^2 - q\,\phi(\vec{x}) \qquad (\text{recall } L = T - V)$$

Relativistic generalization: $\frac{1}{2}mv^2$, we have seen, is from $-mc\sqrt{1-\vec{v}^2/c^2}$.
Now, we want $S$ to be invariant under Lorentz transformations. The kinetic term is explicitly invariant if we write again

$$S = -mc\int ds = -mc\int d\lambda\,\sqrt{\eta_{\mu\nu} U^\mu U^\nu}$$

with $U^\mu = \frac{dx^\mu}{d\lambda}$.

But $q\phi = qA^0$ is the 0-th component of a 4-vector. Now

$$U^\mu = \frac{dx^\mu}{ds} \quad \text{has} \quad U^0 = \frac{1}{\sqrt{1-\vec{v}^2/c^2}}, \quad U^i = \frac{v^i/c}{\sqrt{1-v^2/c^2}}$$

In the NR limit $U^0 \simeq 1$ and $U^i \simeq v^i$, so we can write

$$-q\,U^\mu A_\mu \simeq -q\phi + \mathcal{O}(\vec{v})$$

The relativistic generalization is then

$$\int dt\,(-q\,\phi(\vec{x})) = \int dt\,\left(-q\,\frac{1}{c}\frac{dx^0}{dt}\phi\right) \to \int dt\left(-\frac{q}{c}\,\frac{dx^\mu}{dt}A_\mu\right)$$

or simply

$$S = \int -mc\,ds - \frac{q}{c}\,dx^\mu A_\mu$$

$$= \int d\lambda\left[-mc\sqrt{\eta_{\mu\nu}U^\mu U^\nu} - \frac{q}{c}U^\mu A_\mu\right]$$

Momentum conjugate; equation of motion (EOM)

Writing

$$S = \int dt\left[-mc^2\sqrt{1-v^2/c^2} - \frac{q}{c}\,\frac{dx^\mu}{dt}A_\mu\right] \Rightarrow L = -mc^2\sqrt{1-\frac{\vec{v}^2}{c^2}} + \frac{q}{c}\vec{v}\cdot\vec{A} - qA^0$$

we find the canonical momentum $\vec{P}$:

$$P^i = \frac{\partial L}{\partial v^i} = \underbrace{p^i}_{\text{as before}} + \frac{q}{c}A^i \qquad \text{or} \quad \boxed{\vec{P} = \vec{p} + \frac{q}{c}\vec{A}}$$

Now $H = \vec{P}\cdot\vec{v} - L = \vec{P}\cdot\vec{v} + mc^2\sqrt{1-v^2/c^2} - \frac{q}{c}\,\vec{v}\cdot\vec{A} + qA^0$

Writing $\vec{p} = \vec{P} - \frac{q}{c}\vec{A}$ this is

$$H = \vec{p}\cdot\vec{v} + mc^2\sqrt{1-\vec{v}^2/c^2} - qA^0(\vec{x},t)$$

The first two terms are as in the free case, so we can write immediately

$$H = \sqrt{\vec{p}^2c^2 + (mc^2)^2} + qA^0$$

or

$$\boxed{H = \sqrt{(\vec{P} - \frac{q}{c}\vec{A})^2 + (mc^2)^2} + qA^0}$$

At low velocity

$$L = \frac{1}{2}m\vec{v}^2 + \frac{q}{c}\vec{v}\cdot\vec{A} - qA^0$$

$$\text{and } H = \frac{1}{2m}(\vec{P} - \frac{q}{c}\vec{A})^2 + qA^0$$

For the EOM, we already have $\frac{\partial L}{\partial \vec{v}} = \vec{P} = \vec{p} + \frac{q}{c}\vec{A}$

but now we also need $\frac{\partial L}{\partial x^i}$. Recall that $A^0 = A^0(\vec{x})$ and $A^i = A^i(\vec{x})$, so we have

$$\frac{\partial L}{\partial x^i} = \frac{q}{c}v^j\frac{\partial A^j}{\partial x^i} - q\frac{\partial A^0}{\partial x^i}$$

So we have $\frac{d}{dt}(p^i + \frac{q}{c}A^i) = \frac{q}{c}v^j\partial_i A^j - q\partial_i A^0$

We want an equation for $\frac{dp^i}{dt}$ ($p^i$ is shorthand for $\frac{mv^i}{\sqrt{1-v^2/c^2}}$). Now $\vec{A} = \vec{A}(\vec{x}(t),t)$ so

$$\frac{d\vec{A}}{dt} = \frac{\partial\vec{A}}{\partial t} + \frac{dx^j}{dt}\frac{\partial\vec{A}}{\partial x^j} = \partial_t\vec{A} + v^j\partial_j\vec{A}$$

$$\Rightarrow \frac{dp^i}{dt} = \frac{q}{c}\Big(\underbrace{v^j(\partial_i A^j - \partial_j A^i)}_{[\vec{v}\times(\nabla\times\vec{A})]^i = (\vec{v}\times\vec{B})^i}\Big) \underbrace{- \frac{q}{c}\partial_t A^i - q\,\partial_i A^0}_{qE^i}$$

$$\text{or } \frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)$$

(Check of curl: $\partial_i A_j - \partial_j A_i = \epsilon_{ijk}\epsilon_{kmn}\partial_m A_n = \epsilon_{ijk}(\nabla\times\vec{A})_k$. Then this times $v^j \Rightarrow \epsilon_{ijk}v^j(\nabla\times\vec{A})^k$)

One can check by direct computation $\frac{dE_{kin}}{dt} = q\vec{v}\cdot\vec{E}$ (we know from $V^\mu\frac{dp^\mu}{d\tau} = 0$) where $E_{kin} = \frac{mc^2}{\sqrt{1-v^2/c^2}}$

$$\left(\frac{dE_{kin}}{dt} = mc^2\frac{1}{(1-v^2/c^2)^{3/2}}\frac{1}{c^2}\vec{v}\cdot\frac{d\vec{v}}{dt}, \text{ and } \vec{v}\cdot\frac{d\vec{p}}{dt} = \frac{m}{\sqrt{1-v^2/c^2}}\vec{v}\cdot\frac{d\vec{v}}{dt} + \frac{v^2}{c^2}\frac{d}{dt}\frac{mc^2}{\sqrt{1-v^2/c^2}}\right.$$

$$\left.\text{so } \vec{v}\cdot\frac{d\vec{p}}{dt} = \left(1-\frac{v^2}{c^2}\right)\frac{dE_{kin}}{dt} + \frac{v^2}{c^2}\frac{dE_{kin}}{dt} \Rightarrow \frac{dE_{kin}}{dt} = \vec{v}\cdot\frac{d\vec{p}}{dt} = q\vec{v}\cdot\vec{E}\right).$$

Exercise: Obtain the covariant form of the EOM from using

$$S = \int d\lambda \left[ -mc\sqrt{v^2} - \frac{q}{c} v^\alpha A_\alpha \right]$$

in Euler-Lagrange.

Solution:

$$-p_\alpha = \frac{\partial \mathcal{L}}{\partial v^\alpha} = -mc\frac{v_\alpha}{\sqrt{v^2}} - \frac{q}{c} A_\alpha$$

$$\frac{\partial \mathcal{L}}{\partial x^\alpha} = -\frac{q}{c} v^\beta \partial_\alpha A_\beta$$

$$\Rightarrow \quad \frac{d}{d\lambda}\left( \frac{mc\, v_\alpha}{\sqrt{v^2}} + \frac{q}{c} A_\alpha \right) = \frac{q}{c} v^\beta \partial_\alpha A_\beta$$

or

$$\frac{dp_\alpha}{d\lambda} = \frac{q}{c}\Big(v^\beta \partial_\alpha A_\beta - \underbrace{\frac{d}{d\lambda} A_\alpha}_{= \frac{dx^\beta}{d\lambda}\partial_\beta A_\alpha = v^\beta \partial_\beta A_\alpha}\Big)$$

$$\Rightarrow \quad \frac{dp_\alpha}{d\lambda} = \frac{q}{c} v^\beta (\partial_\alpha A_\beta - \partial_\beta A_\alpha) = \frac{q}{c} v^\beta F_{\alpha\beta}$$

as expected.

## Motion in constant crossed $\vec{E},\vec{B}$ fields

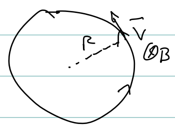
Warm-up: $\vec{E}=0$, $\vec{B}=$ constant

We know this from introductory courses: $\frac{d\vec{p}}{dt} = \frac{q}{c}\vec{v}\times\vec{B}$

except now $\vec{p} = \frac{m\vec{v}}{\sqrt{1-v^2/c^2}}$. However $\frac{dE_{kin}}{dt} = q\vec{v}\cdot\vec{E}=0 \Rightarrow v^2 = $ constant

so $m\frac{d\vec{v}}{dt} = \sqrt{1-\frac{v^2}{c^2}}\; q\frac{\vec{v}}{c}\times\vec{B}$

$\Rightarrow$ $\vec{v}$ has a constant component along $\vec{B}$, plus it has a circular trajectory in the plane $\perp$ to $\vec{B}$.




$\frac{v_\perp^2}{R} = \sqrt{1-\frac{v^2}{c^2}}\;\frac{q}{mc}v_\perp B$ $\qquad v_\perp = \sqrt{1-\frac{v_\perp^2+v_\parallel^2}{c^2}}\;\frac{qRB}{mc}$

(or it gives $R$ given initial $\vec{v}$).

— — — —

Now consider $\vec{E}$ & $\vec{B}$, with $\vec{E}\perp\vec{B}$, both uniform and constant. It is useful to 1st consider this in a frame moving in direction of $\vec{E}\times\vec{B}$. Let $\vec{E}=(0,E^2,0)$ & $\vec{B}=(0,0,B^3)$, and consider boost in $\vec{\beta}=(\beta,0,0)$ direction $\Lambda = \begin{pmatrix} c & -s \\ -s & c\end{pmatrix}$ with $s=-\gamma\beta$. We had computed

$E'^1 = E^1 = 0$, $E'^2 = cE^2 - sB^3$, $E'^3 = cE^3 + sB^2 = 0$

$B'^1 = B^1 = 0$, $B'^2 = cB^2 + sE^3 = 0$ $\quad B'^3 = cB^3 - sE^2$

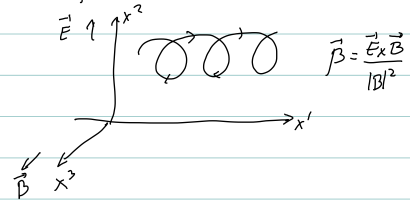
So $\vec{E}'$ & $\vec{B}'$ are still $\perp$, and along $x^2$ & $x^3$ axes, respectively. Now note that we can choose a frame

$E'^2 = 0$ provided $\beta = \frac{s}{c} = \frac{E^2}{B^3}$ which must satisfy $\beta<1$, ie, provided $|\vec{E}|<|\vec{B}|$

Similarly, $|\vec{B}'|=0$ in frame $\beta = \frac{B^3}{E^2}$, provided $|\vec{B}|<|\vec{E}|$.

Motion

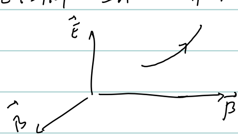
(i) $|\vec{E}|<|\vec{B}|$. In $K'$ frame, $\vec{E}'=0$, $\vec{B}'=$ constant $\Rightarrow$ circular motion in plane $\perp$ to $\vec{B}$; in $K$ frame we have the boost of this motion



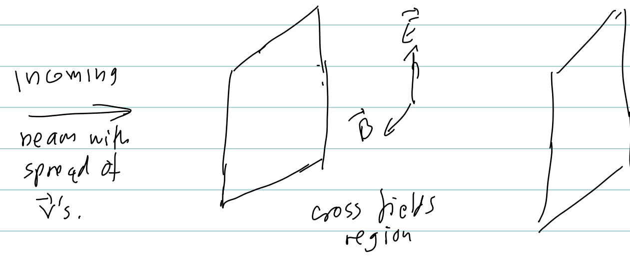

$\vec{\beta} = \frac{\vec{E}\times\vec{B}}{|\vec{B}|^2}$ $\qquad$ "$\vec{E}\times\vec{B}$ drift"

($\vec{\beta}$: "$E\times B$ drift velocity").

(ii) $|\vec{E}|>|\vec{B}|$. In $K'$ frame, $\vec{B}'=0$, $\vec{E}'=$ constant. Rectilinear acceleration in $K'$



Velocity selector.




Consider one particle going into $E \times B$ region with velocity $\vec{v}$.
Let $\vec{U}/c = \frac{\vec{E} \times \vec{B}}{|B|^2}$ (to differentiate from $\vec{v}$).

In $K'$, where $\vec{E}' = 0$, $\vec{v}' = \frac{\vec{v} - \vec{U}}{1 - \frac{\vec{v} \cdot \vec{U}}{c^2}}$ or since all are in $x$-direction $v' = \frac{v - U}{1 - vU/c^2}$.

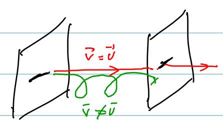
In $K'$ this is circular motion with radius

$$R = \frac{mcv'}{\sqrt{1 - v'^2/c^2}\, qB}$$

For $v' = 0$, ie, $\vec{v} = \vec{U}$ the trajectory in $K$ is a straight line ✓

For $v' \neq 0$ the trajectory is not straight, but

So use slits:




only $v = \frac{Ec}{B}$ goes through: velocity selector.

This followed by a magnetic spectrometer (pure $\vec{B}$ = constant), that selects $\vec{p}$ can be used for simultaneous $\vec{v}$ and $\vec{p}$ measurement $\Rightarrow$ measure mass and charge.

(The $B$ = constant region selects $\frac{mv}{\sqrt{1 - v^2/c^2}} = p = \frac{qB}{R}$ by measuring $R$.)

## Elements of Classical Relativistic Field Theory

Before giving an action principle for $\vec{E}$ & $\vec{B}$, let's do a more general analysis — and simpler: consider first the continuum mechanics of a scalar field $\phi(x^\mu)$ ($\phi(x)$ for short).

$$S[\phi] \quad \text{is} \quad \text{a functional} \quad \{\text{space of } \phi(x)\} \to \mathbb{R}.$$

Lagrangian density $\mathcal{L}$:

$$S[\phi] = \int dt\, L = \int d^4x\, \mathcal{L}$$

with $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu\phi)$ a function of (5) variables, $\phi$ and $\partial_\mu\phi$.

(Again, think of this as a function $\mathcal{L}(\phi, a_\mu)$ which we evaluate at $a_\mu = \partial_\mu\phi$).

EOM (Euler-Lagrange equations):

$$\delta S[\phi] = 0 \Rightarrow \int d^4x \left[ \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} \partial_\mu \delta\phi \right] = 0$$

(we have computed $S[\phi + \delta\phi] - S[\phi]$ for arbitrary function $\delta\phi(x)$).
There are implicit boundary conditions:

* initial/final: $\phi(\vec{x}, t_1) = \phi_1(\vec{x})$, $\phi(\vec{x}, t_2) = \phi_2(\vec{x})$ are initial and final field values.

* spatial infinity: assume $\phi(\vec{x}, t)$ is localized: $\phi(x^\mu) \to 0$ as $|\vec{x}| \to \infty$.

Integrate by parts and ignore boundary terms since $\delta\phi(x) = 0$ on boundaries (either $t = t_{1,2}$ or $|\vec{x}| = \infty$):

$$\int d^4x\, \delta\phi(x) \left[ \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\right)\right] = 0$$

$\delta\phi(x)$ is arbitrary:

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} = 0} \qquad \text{(EOM)}$$

Example:

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi = \frac{1}{c^2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \vec{\nabla}\phi\cdot\vec{\nabla}\phi$$

Then $\dfrac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} = \eta^{\mu\nu}\partial_\nu\phi$ and $\dfrac{\partial \mathcal{L}}{\partial\phi} = 0$

$\Rightarrow$ EOM is $\partial_\mu(\eta^{\mu\nu}\partial_\nu\phi) = 0$ or $\partial^2\phi = 0$ ie $\phi$ satisfies wave equation.

Exercise: $\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{\mu^2}{2}\phi^2$

gives EOM $\partial^2\phi + m^2\phi = 0$ "Klein-Gordon" equation.

(often written $(\partial^2 + m^2)\phi = 0$).

Canonical momentum, Hamiltonian density

Define $\pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}$ ($\partial_0$ insted of $\partial_t$ just rescales how we measure $t$).

and $\mathcal{H}(\pi, \phi) = \pi\partial_0\phi - \mathcal{L}$

Hamilton's equations follow from extrema of $S = \int d^4x\,(\pi\partial_0\phi - \mathcal{H})$

Ex: $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\mu^2}{2}\phi^2 = \frac{1}{2}(\partial_0\phi)^2 - \frac{1}{2}(\vec\nabla\phi)^2 - \frac{\mu^2}{2}\phi^2$

$\pi = \partial_0\phi$

$\mathcal{H} = \pi\partial_0\phi - \left(\frac{1}{2}(\partial_0\phi)^2 - \frac{1}{2}(\vec\nabla\phi)^2 - \frac{\mu^2}{2}\phi^2\right)$

or $\mathcal{H} = \underbrace{\frac{1}{2}\pi^2}_{\text{"K"}} + \underbrace{\frac{1}{2}(\vec\nabla\phi)^2 + \frac{\mu^2}{2}\phi^2}_{\text{"V"}}$

Note that $(\nabla\phi)^2$ is potential energy. The derivative is different in character that $\partial_0$: it is just the difference of two variables, $\phi(x+\epsilon)$ & $\phi(x)$, for $\epsilon = (0, \vec\epsilon)$, as $\vec\epsilon \to 0$. So $(\nabla\phi)^2$ belongs with $\mu^2\phi^2$!

Now $S = \int d^4x\,(\pi\partial_0\phi - \mathcal{H})$

$\delta S = 0 \Rightarrow \partial_0\phi - \pi = 0$ & $\underset{\substack{\uparrow \\ \text{int by parts}}}{-\partial_0\pi} + \nabla^2\phi - \mu^2\phi = 0$ $\Big\}$ using $\pi = \partial_0\phi$ from 1st in 2nd $\quad -\partial_0^2\phi + \nabla^2\phi - \mu^2\phi = 0$ or $-(\partial^2 + \mu^2)\phi = 0$

Klein-Gordon ✓

## Continuous symmetries and Noether's theorem.

We'll do this by first looking at explicit symmetries and then generalizing.

Consider an action $S = \int d^4x \mathcal{L}$ with $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu\phi)$ that does not depend explicitly on $x^\mu$. Then this is invariant under $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu$. In this case this invariance holds for any $\epsilon^\mu$ independent of $x$. But we will consider $\epsilon$ infinitesimal, so that $\delta\phi = \phi(x+\epsilon) - \phi(x) = \epsilon^\mu \partial_\mu \phi$

It is useful to consider $\epsilon^\mu = \epsilon^\mu(x)$, and only specialize to the case that $\epsilon$ is $x$-independent at the end. We still have

$$\delta\phi = \epsilon^\mu \partial_\mu \phi$$

In addition, since $\partial_\mu\phi$ is a function of $x$

$$\delta \partial_\mu \phi = \epsilon^\nu \partial_\nu(\partial_\mu\phi)$$

Similarly since $\mathcal{L}$ is implicitly a function of $x$, we have $\delta\mathcal{L} = \epsilon^\mu \partial_\mu \mathcal{L}$. So

$$\delta S = \int d^4x \, \epsilon^\mu \partial_\mu \mathcal{L} \qquad (*)$$

Now, for any variation $\delta\phi$, we have

$$\delta S = \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi)}\delta\partial_\lambda\phi\right]$$

and using the EOM

$$\delta S = \int d^4x \left[\partial_\lambda\left(\frac{\partial\mathcal{L}}{\partial\partial_\lambda\phi}\right)\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi)}\delta\partial_\lambda\phi\right]$$

$$= \int d^4x \left[\partial_\lambda\left(\frac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi)}\delta\phi\right) + \frac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi)}\left(\delta\partial_\lambda\phi - \partial_\lambda\delta\phi\right)\right]$$

Using $x \to x + \epsilon$

$$= \int d^4x \left[\partial_\lambda\left(\frac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi)}\epsilon^\mu\partial_\mu\phi\right) + \frac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi)}\left[\epsilon^\mu\partial_\mu\partial_\lambda\phi - \partial_\lambda(\epsilon^\mu\partial_\mu\phi)\right]\right]$$

$$= \int d^4x \, \epsilon^\mu \partial_\lambda\left(\frac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi)}\partial_\mu\phi\right)$$

Since this is equal to $(*)$ we have

$$\int d^4x \, \epsilon^\mu\left[\partial_\lambda\left(\frac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi)}\partial_\mu\phi\right) - \partial_\mu\mathcal{L}\right] = 0 \qquad \text{for arbitrary } \epsilon^\mu, \text{ or}$$

$$\partial_\lambda\left(\frac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi)}\partial_\mu\phi\right) - \partial_\mu\mathcal{L} = 0 \Rightarrow \partial_\lambda\left[\frac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi)}\partial_\mu\phi - \delta^\lambda_\mu\mathcal{L}\right] = 0$$

This means the tensor $T^{\lambda}{}_{\mu} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\lambda \phi)} \partial_\mu \phi - \delta^{\lambda}_{\mu} \mathcal{L}$

defines 4 conserved currents: $\partial_\lambda T^{\lambda\mu} = 0$

We have raised the second index $T^{\lambda\mu} = \eta^{\mu\nu} T^{\lambda}{}_{\nu} = \eta^{\mu\nu} \frac{\partial \mathcal{L}}{\partial(\partial_\lambda \phi)} \partial_\nu \phi - \eta^{\lambda\mu} \mathcal{L}$

Example: With $\mathcal{L} = \frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \eta_{\mu\nu} \eta^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi$$

Now

$$\partial_\mu T^{\mu}{}_{\nu} = \underbrace{\partial^2 \phi}_{\substack{=0 \\ \text{by EOM}}} \partial_\nu \phi + \eta^{\mu\alpha} \partial_\mu \phi \, \partial_\alpha \partial_\nu \phi - \eta^{\alpha\beta} \partial_\alpha \phi \, \partial_\alpha \partial_\nu \phi = 0 \checkmark\checkmark$$

Exercise: Find $T_{\mu\nu}$ for $\mathcal{L} = \frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{\mu^2}{2} \phi^2$. Check explicitly, by use of EOM, that $\partial_\mu T^{\mu}{}_{\nu} = 0$.

The conserved "charges" are

$$P^\mu = \frac{1}{c} \int d^3x \, T^{\mu 0}$$

are interpreted as $P^0 = E/c$ = energy/c and $P^i$ = momentum.

$\left(\text{Units } [S] = E \cdot T = P \cdot X\right.$, $\int d^4x \, \mathcal{L}$ $\quad [\mathcal{L}] = \frac{PX}{X^4} = \frac{P}{X^3}$ same as $[T^{\mu\nu}]$

Digression: (i) another approach

For spacetime symmetries, $S[\phi'(x')] = S[\phi(x)]$ involves $\int d^4x'$ vs $\int d^4x$

In comparing $\int d^4x' \mathcal{L}(\phi'(x'))$ to $\int d^4x \mathcal{L}(\phi(x))$ we want the same integration region, so we use $\int d^4x \left[ \left| \frac{\partial x'}{\partial x} \right| \mathcal{L}(\phi'(x'(x))) - \mathcal{L}(\phi(x)) \right] = 0$

Now the Jacobian $\det\left(\frac{\partial x'^\mu}{\partial x^\nu}\right) = \det\left|\frac{\partial}{\partial x^\nu}(x^\mu + \epsilon^\mu(x))\right| = \det(\delta^\mu_\nu + \partial_\nu \epsilon^\mu)$

Using $\det A = e^{\mathrm{Tr}\ln A}$ (for any matrix $\det A = \prod \lambda_n$ with $\lambda_n$ the eigenvalues, so taking the log, $\ln \det A = \ln \prod_n \lambda_n = \sum_n \ln \lambda_n = \mathrm{Tr}\ln A$), we have

$$\det(\delta^\mu_\nu + \partial_\nu \epsilon^\mu) = e^{\mathrm{Tr}\ln(\delta^\mu_\nu + \partial_\nu \epsilon^\mu)} \approx e^{\partial_\mu \epsilon^\mu} = 1 + \partial_\mu \epsilon^\mu$$

So

$$\int d^4x \left[ \partial_\mu \epsilon^\mu \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \phi} \epsilon^\mu \partial_\mu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \epsilon^\mu \partial_\mu \partial_\nu \phi \right] = 0$$

Since $\epsilon^\mu(x)$ is arbitrary, we will make it localized, and in particular $\epsilon^\mu = 0$ at the boundaries. So we integrate by parts the first term. Use the EOM for the second, and

$$\int d^4x \, \epsilon^\mu(x) \left[ -\partial_\mu \mathcal{L} + \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial_\mu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial_\mu \partial_\nu \phi \right] = 0$$

$$\int d^4x \, \epsilon^\mu(x) \, \partial_\nu \left[ -\delta^\nu_\mu \mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial_\mu \phi \right] = 0$$

$$\Rightarrow \quad T^\nu{}_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial_\mu \phi - \delta^\nu_\mu \mathcal{L} \quad \text{are } (\mu = 0, \ldots 3) \text{ conserved.}$$

(ii) Since the EOM is derived from arbitrary variations, why do we get conservation laws from specific variations only <u>after</u> using the EOM?

ANS: For spacetime transformations (eg, translations $x \to x' = x + \epsilon$, or Lorentz Ts $x' = \Lambda x$) the transformation is not just $\phi'(x) = \phi(x) + \delta\phi(x)$; there is at least the Jacobian factor and there may also be a difference between $\delta \partial_\mu \phi$ and $\partial_\mu \delta\phi$, eg, for Lorentz trans's, $\partial_\mu \phi \to \Lambda_\mu{}^\nu \partial_\nu \phi(\Lambda^{-1}x)$ but $\phi \to \phi(\Lambda^{-1}x)$ so $\delta \partial_\mu \phi$ includes $\epsilon_\mu{}^\nu \partial_\nu \phi$ but not so $\partial_\mu(\delta\phi)$.

For internal transformations we can work with constant $\epsilon$, at the Lagrangian level, so clearly different. Also, internal transformations do not necessarily satisfy boundary conditions. For example, if at $t = t_{initial}$ $\phi_I(\vec{x}, t_{in}) = \phi_{I0}(\vec{x})$ then the transformed field $\phi'_I(\vec{x}, t) = R_{IJ}\phi_J(\vec{x}, t)$ does <u>not</u> have $\phi'_I(\vec{x}, t_{ini}) = \phi_{I0}(\vec{x})$.

Next consider Lorentz invariance: $\delta x^\mu = \epsilon^\mu{}_\nu x^\nu$. We have

$$\phi'(x) = \phi(\Lambda^{-1}x) = \phi(x^\mu - \epsilon^\mu{}_\nu x^\nu) = \phi(x) - \epsilon^\mu{}_\nu x^\nu \partial_\mu \phi(x)$$

Similarly $\delta \mathcal{L} = -\epsilon^\mu{}_\nu x^\nu \partial_\mu \mathcal{L}$

For $\partial_\mu \phi$ we must be careful that it is a vector, $a'_\mu(x) = \Lambda_\mu{}^\nu a_\nu(\Lambda^{-1}x)$ so that

$$\delta a_\mu = \epsilon_\mu{}^\nu a_\nu(x) - \epsilon^\lambda{}_\nu x^\nu \partial_\lambda a_\mu$$

Hence

$$\delta S = \int d^4x \left[-\epsilon^\mu{}_\nu x^\nu \partial_\mu \mathcal{L}\right]$$

but also

$$= \int d^4x \left[\partial_\lambda\left(\frac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi)}(-\epsilon^\mu{}_\nu x^\nu \partial_\mu\phi)\right) + \frac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi)}\left[(\epsilon_\lambda{}^\nu\partial_\nu\phi - \epsilon^\alpha{}_\nu x^\nu \partial_\alpha\partial_\lambda\phi) - \partial_\lambda(-\epsilon^\mu{}_\nu x^\nu \partial_\mu\phi)\right]\right]$$

$$= \int d^4x \left[-\epsilon^\mu{}_\nu\, \partial_\lambda\left(\frac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi)} x^\nu \partial_\mu\phi\right)\right]$$

With $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$ arbitrary we have (use $\partial^\mu\phi = \eta^{\mu\nu}\partial_\nu\phi$ for short):

$$M^{\lambda\mu\nu} \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi)} x^\nu \partial^\mu\phi - \eta^{\mu\lambda} x^\nu \mathcal{L} - (\mu \leftrightarrow \nu) = T^{\lambda\mu}x^\nu - T^{\lambda\nu}x^\mu$$

satisfy $\partial_\lambda M^{\lambda\mu\nu} = 0$ (6 conserved currents: 3 for rotations + 3 for boosts).

The conserved charges are

$$M^{\mu\nu} = \int d^3x\, M^{0\mu\nu}$$

with

$$M^{ij} = \int d^3x\, M^{0ij} = \int d^3x\,(T^{0i}x^j - T^{0j}x^i)$$

These have the form of $\vec{r}\times\vec{p}$, where $\vec{p}$ is a density then integrated over the density $\Rightarrow$ this is angular momentum.

$$M^{0i} = \int d^3x\, M^{00i} = \int d^3x\,(T^{00}x^i - T^{0i}x^0)$$

$$= \int d^3x\,(T^{00}x^i) - x^0 P^i$$

is analogous to particle case, giving

$$P^i = \frac{\int d^3x\,(T^{00}x^i)}{\int d^3x\,(T^{00})} \cdot \frac{\int d^3x\, T^{00}}{x^0} + C^i = \frac{x^i_{CM}}{x^0}\frac{E_{TOT}}{c} + C^i$$

Exercise: Compute $M^{\mu\nu\lambda}$ for $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi$, and check $\partial_\mu M^{\mu\nu\lambda} = 0$.

Sol: $T^{\mu\nu} = \partial^\mu\phi\,\partial^\nu\phi - \frac{1}{2}\eta^{\mu\nu}\partial_\alpha\phi\,\partial^\alpha\phi$

so
$$M^{\mu\lambda\nu} = T^{\mu\lambda}x^\nu - T^{\mu\nu}x^\lambda$$
$$= \partial^\mu\phi(\partial^\lambda\phi\, x^\nu - \partial^\nu\phi\, x^\lambda) - \frac{1}{2}(\eta^{\mu\lambda}x^\nu - \eta^{\mu\nu}x^\lambda)\partial_\alpha\phi\,\partial^\alpha\phi$$

We have already checked $\partial_\mu T^{\mu\nu} = 0$. So

$$\partial_\mu M^{\mu\lambda\nu} = \partial_\mu(T^{\mu\lambda}x^\nu - T^{\mu\nu}x^\lambda)$$
$$= (\partial_\mu T^{\mu\lambda})x^\nu + T^{\mu\lambda}\partial_\mu x^\nu - \partial_\mu T^{\mu\nu}x^\lambda - T^{\mu\nu}\partial_\mu x^\lambda$$
$$= T^{\mu\lambda}\delta^\nu_\mu - T^{\mu\nu}\delta^\lambda_\mu$$
$$= T^{\nu\lambda} - T^{\lambda\nu}$$
$$= 0 \quad \text{since } T^{\mu\nu} \text{ is symmetric.}$$

Note: in some cases $T^{\mu\nu}$ is not symmetric, and additional care is needed. One can define an improved $T^{\mu\nu}$ which is symmetric. Rather than doing the general case, we'll see this explicitly in the case of E&M.

"Internal" symmetries. Consider the case of two scalar fields

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2$$ where both $\phi_1(x)$ & $\phi_2(x)$ are scalar fields.

This is form invariant under the transformations:

$$\phi_1'(x) = \cos\alpha\,\phi_1(x) + \sin\alpha\,\phi_2(x)$$
$$\phi_2'(x) = -\sin\alpha\,\phi_1(x) + \cos\alpha\,\phi_2(x)$$

This is a continuous set of transformations since $\alpha$ can take on any value in $\mathbb{R}$.
Note that this transformation has $x^\mu$ unchanged ($x^\mu \to x'^\mu = x^\mu$). It only "mixes" the fields $\phi_1$ & $\phi_2$; such symmetries are generically called "internal".

More generally, consider a set of fields $\phi_I$, $I=1,\dots,N$, with a Lagrangian density $\mathcal{L}(\phi_I, \partial_\mu\phi_I)$ that is invariant under

$$\delta\phi_I(x) = \epsilon(D\phi)_I(x)$$

where $\epsilon$ is an infinitesimal parameter, $(D\Phi)_I$ is a linear transformation on $\Phi$ and we indicate explicitly that $x$ is unaffected. That $\mathcal{L}$ is invariant means

$$\sum_I \frac{\partial\mathcal{L}}{\partial\phi_I}\epsilon D\phi_I + \frac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi_I)}\epsilon\partial_\lambda D\phi_I = 0$$ (we are taking $\epsilon$ to be $x$ independent)

$\sum_I$ understood throughout

EOM: $\dfrac{\partial\mathcal{L}}{\partial\phi_I} = \partial_\lambda\left(\dfrac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi_I)}\right) \Rightarrow \epsilon\left[\partial_\lambda\left(\dfrac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi_I)}\right)D\phi_I + \dfrac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi_I)}\partial_\lambda D\phi_I\right] = 0$

Hence $\partial_\lambda\left[\dfrac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi_I)}D\phi_I\right] = 0$ or $\boxed{\partial_\lambda J^\lambda = 0 \text{ for } J^\lambda = \dfrac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi_I)}D\phi_I}$

"Noether's Theorem"

Example: The 2 field case above has ($\alpha = \epsilon$, infinitesimal)

$$\delta\phi_1 = \epsilon\phi_2 \quad \delta\phi_2 = -\epsilon\phi_1 \quad \text{So } (D\phi)_1 = +\phi_2 \quad (D\phi)_2 = -\phi_1$$

Also $\dfrac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi_I)} = \eta^{\lambda\nu}\partial_\nu\phi_I = \partial^\lambda\phi_I$. So $J^\lambda = \partial^\lambda\phi_1\phi_2 + \partial^\lambda\phi_2(-\phi_1) = \phi_2\partial^\lambda\phi_1 - (\partial^\lambda\phi_2)\phi_1$

Note that $\partial_\lambda J^\lambda = \phi_2\partial^2\phi_1 - \partial^2\phi_2\,\phi_1$ vanishes by EOM.

Exercise: Consider $\mathcal{L} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - V(\phi_1,\phi_2)$ for some function $V$.
(i) Give examples of $V$ that are polynomial in $\phi_I$ and invariant under $\delta\phi_1 = \phi_2$, $\delta\phi_2 = -\phi_1$.
(ii) Including $V$, compute $J^\lambda$.

# Action integral for Electromagnetic Field

The Euler–Lagrange equations for the $\vec{E}$ & $\vec{B}$ fields should correspond to Maxwell's equations, the source-less form if the action integral does not include the interaction with particles (which we'll ignore at first — easy to include later).

The problem is that Maxwell's equations are first order in derivatives. But this is not what follows normally from Euler-Lagrange. To see this, consider what can be written as possible forms for the Lagrangian. It should be quadratic in the field variables $\vec{E}$ & $\vec{B}$, invariant under rotations (assuming Lorentz invariance is even more constraining — see further below). So we can have terms

2nd pass:

$$\vec{E}^2,\ \vec{B}^2,\ \underbrace{\vec{E}\cdot\vec{B}}_{P\ \&\ T\ \text{odd}}$$

and derivatives on this

$$\underbrace{\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}}_{\frac{1}{2}\frac{\partial\vec{E}^2}{\partial t},\ T\ \text{odd}} \quad \underbrace{\vec{B}\cdot\frac{\partial\vec{B}}{\partial t}}_{\text{idem}} \quad \underbrace{\vec{E}\cdot\frac{\partial\vec{B}}{\partial t}}_{P\ \text{odd}} \quad \underbrace{\frac{\partial\vec{E}}{\partial t}\cdot\vec{B}}_{\text{idem}}$$

$$(\vec{\nabla}\cdot\vec{E})^2,\ (\vec{\nabla}\times\vec{E})^2,\ \text{etc...}$$

The terms in the last line is quadratic in derivatives $\rightarrow$ will give 2-derivative EOM's.

The first line has $E^2$ & $B^2$. In fact energy density $\propto \vec{E}^2+\vec{B}^2$ and it would make sense that, since $L=T-V$, that it involves this. But then the EOMs would have no derivatives.

There is an easy workaround. If we use the potentials $\phi$ & $\vec{A}$ (or $A^\mu=(A^0,\vec{A})$) as the fundamental fields (fundamental in the sense that then $\vec{E}$ & $\vec{B}$ are derived) then if $\mathcal{L}\propto(\partial A)^2$ the EOM will involve $\partial\cdot\partial A\sim\partial(E \text{ or } \vec{B})$ which is precisely what we need.

Additionally, we want $\mathcal{L}$ to be gauge invariant (so that Maxwell equations are too).
$\Rightarrow$ Write $\mathcal{L}$ in terms of $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$.
As a bonus, the homogeneous (sourceless) equations follow automatically.

Possible terms for $\mathcal{L}$

(i) Made of $F_{\mu\nu}$

(ii) Quadratic in fields

(iii) Lorentz invariant

(iv) P and T invariant

Possibilities are limited:

$$F_{\mu\nu}F^{\mu\nu} \quad , \quad \underbrace{\epsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma}}_{P,T \text{ odd}}$$

So we conclude

$$\mathcal{L} = \frac{1}{4}\kappa F_{\mu\nu}F^{\mu\nu}$$

with some constant $\kappa$ to be determined.

Euler-Lagrange

Write $\mathcal{L} = \frac{\kappa}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = \kappa(\partial^\mu A^\nu - \partial^\nu A^\mu) \quad \frac{\partial \mathcal{L}}{\partial A^\mu} = 0 \quad \Rightarrow \quad \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = \kappa(\partial^2 A_\nu - \partial_\nu \partial\cdot A) = 0$$

Compare with Maxwell's equations: $\partial^2 A_\nu - \partial_\nu(\partial\cdot A) = \frac{4\pi}{c} j_\nu$ ; since we have $j_\nu = 0$ (for now), these agree! ✓✓

To determine $\kappa$, we now add a charge particle and insist in getting the correct Maxwell's equations

$$S = \int d^4x \, \kappa F_{\mu\nu}F^{\mu\nu} + \int d\lambda\left(-mc\sqrt{U_\alpha U^\alpha} - \frac{q}{c} U^\alpha A_\alpha\right)$$

Then $\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}$ is as above, but now $\frac{\partial \mathcal{L}}{\partial A_\mu}$ no longer vanishes. To make the coupling term a lagrangian density, choose $\lambda = x^0$, so $U^\alpha \to \frac{dx^\alpha}{dx^0} = (1, \vec{v}/c)$ and include $\int d^3x' \, \delta^3(x' - \vec{x}(\lambda))$ to replace $A_\mu(x^0, \vec{x}(x^0))$ by $A_\mu(x)$. Then

dummy ↗ world-line

$$-\frac{q}{c}\int d\lambda \, U^\alpha A_\alpha = -\frac{q}{c}\int d^4x' \, \delta^{(3)}(\vec{x}' - \vec{x}(x^0)) \frac{dx^\mu}{dx^0} A_\mu(x')$$

$$\Rightarrow \mathcal{L}_q = -\frac{q}{c}\delta^3(x' - \vec{x}(x^0)) \frac{dx^\alpha}{dx^0} A_\alpha = -\frac{1}{c^2} j^\alpha A_\alpha \quad \text{since } j^\alpha = cq\frac{dx^\alpha}{dx^0}\delta^3(x' - \vec{x}(x^0)) \quad \text{and } \frac{\partial \mathcal{L}_q}{\partial A_\mu} = -\frac{1}{c^2} j^\mu$$

$$\text{So} \quad \kappa(\partial^2 A_\nu - \partial_\nu \partial\cdot A) = -\frac{1}{c^2} j^\nu \quad \Rightarrow \kappa = -\frac{1}{4\pi c} \quad \Rightarrow \quad \boxed{S_{E\&M} = \int d^4x \left(-\frac{1}{16\pi c} F_{\mu\nu}F^{\mu\nu}\right)}$$

Leaving out particle dynamics, it is convenient to factor out $4\pi c$:

$$S = \frac{1}{4\pi c}\int d^4x \left[ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{4\pi}{c} J^\mu A_\mu \right]$$

Note that $F^{\mu\nu} F_{\mu\nu} = 2F^{0i}F_{0i} + F^{ij}F_{ij} \Rightarrow \mathcal{L}_{free} = \frac{1}{2}(E^2 - B^2) \cdot \frac{1}{4\pi c}$

<u>Canonical Momentum and Hamiltonian</u>

$\pi^\nu$ the momentum conjugate to $A_\nu$: $\pi^\nu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\nu)} = \frac{-1}{4\pi c}(\partial^0 A^\nu - \partial^\nu A^0) = -\frac{1}{4\pi c}F^{0\nu} = \begin{cases} 0 & \nu = 0 \\ \frac{E^i}{4\pi c} & \nu = i \end{cases}$

Note $\pi^0 = 0$. There is no canonical momentum associated with $A_0$. This is because $\mathcal{L}$ contains no time derivative of $A_0$. $A_0$ is not dynamical, it's "EOM" is an equation of constraint.

Now $(4\pi c)\pi^i = -\partial^0 A^i + \partial^i A^0$ so $\partial^0 A^i = \partial^i A^0 - (4\pi c)\pi^i$

So $\mathcal{H} = \pi^i \partial_0 A_i - \mathcal{L}$

$$= \pi^i \left(-(4\pi c)\pi_i + \partial_i A_0\right) - \frac{1}{2}\frac{((4\pi c)\pi^2 - B^2)}{4\pi c} \qquad \text{(where } \vec{B} \text{ is short for } \vec{\nabla}\times A\text{).}$$

$$= \frac{(4\pi c)}{2}\pi^2 + \frac{1}{2}\frac{B^2}{4\pi c} + \pi^i \partial_i A_0 \qquad = \frac{4\pi c}{2}\pi^2 + \frac{1}{2}\frac{(\vec{\nabla}\times\vec{A})^2}{4\pi c} + \pi^i \partial_i A_0$$

The first two terms $\frac{1}{8\pi c}\left[(4\pi c\,\pi^i)^2 + B^2\right] = \frac{1}{8\pi c}(E^2 + B^2)$ are interpreted as energy density/c

Now, in $\quad S = \int d^4x \,(\pi^i \partial_0 A_i - \mathcal{H})$

The variation w.r.t. $A_0$ gives $\partial_i \pi^i = 0$, or $\vec{\nabla}\cdot\vec{E} = 0$ (Gauss's law).

This has no time dependence, it is an equation of constraint.

If we retain source term, $\Delta S = \int d^4x \, \frac{1}{c^2} J^\mu A_\mu$

then $\frac{\delta S}{\delta A_0} = -\partial_i \pi^i + \frac{1}{c^2} J^0 = 0 \qquad \vec{\nabla}\cdot\vec{E} = \frac{4\pi}{c} J^0 = 4\pi\rho$

so Gauss's Law is an equation of constraint (makes sense, no $\frac{\partial}{\partial t}$'s here!)

Energy-Momentum tensor $T^{\lambda}{}_{\mu} = \frac{\partial \mathcal{L}}{\partial(\partial_\lambda \phi)}\partial_\mu \phi - \delta^\lambda_\mu \mathcal{L}$

Use this with $\phi \to A_\nu$

$$(4\pi c)\, T^{\lambda}{}_{\mu} = \frac{\partial \mathcal{L}}{\partial(\partial_\lambda A_\nu)}\partial_\mu A_\nu - \delta^\lambda_\mu \mathcal{L}$$

↑ to avoid repeating in each term

$$= -(\partial^\lambda A^\nu - \partial^\nu A^\lambda)\partial_\mu A_\nu - \delta^\lambda_\mu\left(-\tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\right)$$

$$= -F^{\lambda\nu}\partial_\mu A_\nu + \tfrac{1}{4}\delta^\lambda_\mu F_{\alpha\beta}F^{\alpha\beta}$$

or $(4\pi c)\, T^{\lambda\mu} = -F^{\lambda\nu}\partial^\mu A_\nu + \frac{1}{4}\eta^{\lambda\mu}F_{\alpha\beta}F^{\alpha\beta}$

This tensor is conserved, but neither symmetric nor gauge invariant.

We can try to improve this by noting that we may add to $T^{\lambda\mu}$ another tensor $t^{\lambda\mu}$

(i) $\partial_\lambda t^{\lambda\mu} = 0$

(ii) $\int d^3x\, t^{0\mu} = 0$

so that $\tilde{T}^{\lambda\mu} = T^{\lambda\mu} + t^{\lambda\mu}$ is conserved and defines the same "charge" as $M^{\lambda\mu}$. If we find such tensor then we may try to adjust it so that $\tilde{T}^{\lambda\mu} = \tilde{T}^{\mu\lambda}$ and is gauge invariant.

Consider $(4\pi c)\, t^{\lambda}{}_{\mu} = F^{\lambda\nu}\partial_\nu A_\mu$. It has

$$4\pi c\, \partial_\lambda t^\lambda{}_\mu = \partial_\lambda F^{\lambda\nu}\partial_\nu A_\mu + F^{\lambda\nu}\partial_\lambda\partial_\nu A_\mu$$

Since we are considering the sourceless case, $\partial_\lambda F^{\lambda\nu} = 0$. The second term vanishes by symmetry of $\partial_\lambda\partial_\nu$ times antisymmetry of $F^{\lambda\nu}$. Also

$$\int d^3x\, F^{0\nu}\partial_\nu A_\mu = \int d^3x\, F^{0i}\partial_i A_\mu = -\int d^3x\, \partial_i F^{0i} A_\mu = 0 \quad \big(\underset{\text{(no source)}}{\partial_\lambda F^{0\lambda} = 0}\big).$$

So we adopt a symmetric and gauge invariant expression for the energy-momentum tensor (dropping tilde):

$$T^{\mu\nu} = -\frac{1}{4\pi c}\left[F^{\mu\lambda}F^{\nu}{}_{\lambda} - \tfrac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right]$$

Interpretation

$$4\pi c\, T^0{}_0 = -F^{0i}F_{0i} + \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}$$

$$= \vec{E}^2 + \frac{1}{2}\left(-E^2+B^2\right) = \vec{E}^2 - \frac{1}{2}\left(\vec{E}^2-\vec{B}^2\right) = \frac{1}{2}\left(\vec{E}^2+\vec{B}^2\right)$$

so, energy density $u = \frac{1}{8\pi}\left(\vec{E}^2+\vec{B}^2\right)$ $\left(T^{00}=\frac{u}{c}\right)$

$4\pi c T^{i0}$ = energy flux

$$4\pi c\, T^{i0} = -F^{i\lambda}F^0{}_\lambda = F^{ij}F^{0j} = \left(-\epsilon^{ijk}B^k\right)\left(-E^j\right)$$

$$= \left(\vec{E}\times\vec{B}\right)^i$$

Let $\vec{S} = \frac{c}{4\pi}\left(\vec{E}\times\vec{B}\right)$ "Poynting vector" $\left(T^{i0} = \frac{1}{c^2}S^i\right)$.

Now $\partial_\lambda T^{\lambda 0} = 0 \Rightarrow$ the "current" $\left(\frac{u}{c}, \vec{S}\right)$ is conserved $\boxed{\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S} = 0}$

(Check nagging "c"s $\partial_\lambda T^{\lambda 0} = 0 \Rightarrow$

$\rightarrow \partial_0 T^{00} + \partial_i T^{i0} = 0 \rightarrow \frac{1}{c}\frac{\partial}{\partial t}\frac{u}{c} + \partial_i\left(\frac{1}{c^2}S^i\right) = 0$ ✓✓).

(Dimensions: $[T^{i0}] = \frac{P}{L^3}$ $[S^i] = \left(\frac{L}{T}\right)^2\frac{P}{L^3} = \frac{\frac{L}{T}\cdot P}{L^2\, T} = \frac{E}{L^2\cdot T}$ = energy flux ✓).

$T^{0i}$ : momentum density $= T^{i0}$

$T^{ij}$ : momentum flux

> see below

Note: $\partial_\lambda T^{\lambda\mu} = 0$ holds even in the presence of charges IF we add to $T^{\mu\nu}_{EM}$ the contribution from the rest of the system PROVIDED the whole action is invariant under space-time translations.

Alternatively $\partial_\lambda T^{\lambda\mu}_{EM} = -\partial_\lambda T^{\lambda\mu}_{rest} \neq 0$

Rather than giving the rest of the Lagrangian and computing $T^{\mu\nu}_{rest}$, which depends on the details of "rest", we can compute directly $\partial_\lambda T^{\lambda\mu}_{EM}$ assuming $\partial_\lambda F^{\lambda\nu} = \frac{4\pi}{c}j^\nu$ (ie, with sources). The result (checked in next page) is

$$\partial_\lambda T^{\lambda\mu}_{EM} = -\frac{1}{c^2}F^{\mu\nu}j_\nu$$

For $\mu=0$ $(\times c^2)$ $\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S} = -\vec{E}\cdot\vec{j}$

Interpretation of $\vec{j}\cdot\vec{E}$: Consider $\int d^3x\, \vec{j}\cdot\vec{E}$ and use $\vec{j} = q\vec{v}\,\delta^3(\vec{x}-\vec{x}(t))$

Then $\int d^3x\, \vec{j}\cdot\vec{E}(x) = q\vec{v}\cdot\vec{E}(\vec{x}(t))$

and recall $\dfrac{d\vec{p}}{dt} = q\left(\vec{E} + \dfrac{\vec{v}}{c}\times\vec{B}\right)$ so $\vec{v}\cdot\dfrac{d\vec{p}}{dt} = q\,\vec{v}\cdot\vec{E}$

Moreover $\dfrac{d\vec{p}}{dt} = \dfrac{d}{dt}\dfrac{m\vec{v}}{\sqrt{1-v^2/c^2}} = m\dfrac{d\vec{v}}{dt}\dfrac{1}{\sqrt{1-v^2/c^2}} + m\vec{v}\dfrac{\vec{v}\cdot\frac{d\vec{v}}{dt}}{c^2(1-v^2/c^2)^{3/2}}$

so $\vec{v}\cdot\dfrac{d\vec{p}}{dt} = \dfrac{m\,\frac{1}{2}\frac{d\vec{v}^2}{dt}}{(1-v^2/c^2)^{3/2}} = \dfrac{d}{dt}\dfrac{mc^2}{\sqrt{1-v^2/c^2}} = \dfrac{dE_{kin}}{dt}$

Combining with the above

$$\frac{d}{dt}\Big[\underbrace{\int_V d^3x\, u + E_{kin}}_{\text{total energy in volume } V}\Big] = -\int_V d^3x\, \vec{\nabla}\cdot\vec{S} = \underbrace{-\oint_{\partial V} ds\, \hat{n}\cdot\vec{S}}_{\text{energy escaping } V \text{ through boundary } \partial V}$$

Note: $\vec{S}$ may be non-zero even for static fields! For example, see previous example of static "$E\times B$" field. In this case $\frac{\partial u}{\partial t}=0$ so $\vec{\nabla}\cdot\vec{S}=0$, and while not vanishing it does not add energy to any region of space (and $\vec{S} = \vec{\nabla}\varphi$ for some $\varphi$ ?).

Calculation for previous page:

$$(-4\pi c)\,\partial_\lambda T^{\lambda\mu} = \partial_\lambda\left(F^{\lambda\alpha}F^{\mu}{}_{\alpha} - \tfrac{1}{4}\eta^{\lambda\mu}F^{\alpha\beta}F_{\alpha\beta}\right)$$

$$= \frac{4\pi}{c} j_\alpha F^{\mu\alpha} + X^\mu$$

where

$$X^\mu = F_{\lambda\alpha}\partial^\lambda F^{\mu\alpha} - \tfrac{1}{2}F_{\alpha\beta}\partial^\mu F^{\alpha\beta}$$

$$= F_{\lambda\alpha}\partial^\lambda F^{\mu\alpha} - \tfrac{1}{2}F_{\alpha\beta}\left(-\partial^\alpha F^{\beta\mu} - \partial^\beta F^{\mu\alpha}\right) \quad \text{by homogeneous equation}$$

$$= F_{\lambda\alpha}\partial^\lambda F^{\mu\alpha} + F_{\alpha\beta}\partial^\beta F^{\mu\alpha}$$

$$= 0$$

Hence

$$\partial_\lambda T^{\lambda\mu} = -\frac{1}{c^2}F^{\mu\alpha}j_\alpha$$

Often defined: $\vec{g} = T^{0i} = \frac{1}{c^2}\vec{S}$ = density of momentum

and let $\sigma^{ij} = c\,T^{ij}$, the "Maxwell stress tensor"

($T^{\mu\nu}$ is often called the "stress-energy tensor", another name for "energy-momentum tensor").

Then $\partial_\lambda T^{\lambda i} = \frac{1}{c}\frac{\partial}{\partial t}T^{0i} + \partial_j T^{ji} = \frac{1}{c}\left(\frac{\partial g^i}{\partial t} + \partial_j \sigma^{ji}\right) = 0$

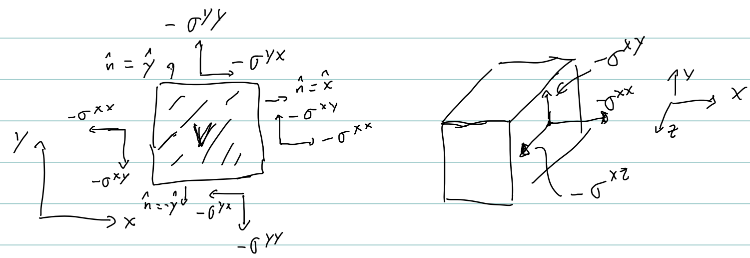
Dimensions: $[T^{ij}] = \frac{p}{L^3}$ $\quad [cT^{ij}] = \frac{p}{L^2 T}$ = momentum flux = force/area.

Since $\frac{d}{dt}\int_V d^3x\, g^i = \frac{d}{dt}(\text{momentum})^i = \text{force}^i$ (force on V)

$$= -\oint_S \sigma^{ji}\,\hat{n}^j\, dS$$

$\Rightarrow$ $\sigma^{ji}$ is the $i$-th component of force <u>into</u> the volume on the surface element $dS$, with normal $n^j$

Forces:




In terms of $\vec{E}$ & $\vec{B}$:

$$\sigma^{ij} = -\frac{1}{4\pi}\left[F^{i\lambda}F^{j}{}_{\lambda} - \frac{1}{4}\eta^{ij}F^{\alpha\beta}F_{\alpha\beta}\right]$$

$$= -\frac{1}{4\pi}\left[F^{i0}F^{j0} - F^{ik}F^{jk} + \frac{1}{2}\delta^{ij}\left(-E^2 + B^2\right)\right]$$

$$= -\frac{1}{4\pi}\left[E^iE^j - \underbrace{\epsilon^{ikm}\epsilon^{jkn}B^mB^n}_{\delta^{ij}B^2 - B^iB^j} + \frac{1}{2}\delta^{ij}\left(-E^2+B^2\right)\right]$$

$$= -\frac{1}{4\pi}\left[E^iE^j + B^iB^j - \frac{1}{2}\delta^{ij}\left(E^2+B^2\right)\right]$$

so, for example: $\sigma^{xx} = -\frac{1}{8\pi}\left(E_x^2 - E_y^2 - E_z^2 + B_x^2 - B_y^2 - B_z^2\right)$

$\sigma^{xy} = -\frac{1}{4\pi}\left(E_xE_y + B_xB_y\right)$

Note: This is the same convention for $\sigma^{ij}$ as Landau-Lifshitz, opposite sign for Jackson.

With sources: we've seen above

$$\partial_\lambda T^{\lambda}_{EM\ \mu} = -\frac{1}{c^2} F_{\mu\nu} J^\nu$$

We will integrate this over space and compare with $\partial_\lambda T^{\lambda}_{rest\ \mu}$ $\quad T^{\lambda\mu}_{rest}$ = charged particles contribution

$$-\frac{1}{c^2}\int d^3x\, F_{\mu\nu} J^\nu \to -\frac{1}{c^2}\int d^3x\, F_{\mu\nu}\, q \frac{dx^\nu}{dt}\, \delta^3(\vec{x}-\vec{x}(t))$$

$$= -\frac{1}{c^2}\, q F_{\mu\nu}(\vec{x}(t)) \frac{dx^\nu}{dt}$$

Recall $\frac{dp_\mu}{dt} = \frac{q}{c}\frac{dx^\nu}{dt} F_{\nu\mu}$

so $\quad -\frac{1}{c^2}\int d^3x\, F_{\mu\nu} J^\nu = -\frac{1}{c}\frac{dp_\mu}{dt}$

$$\Rightarrow \int d^3x\, \partial_\lambda T^{\lambda}{}_{\mu} + \frac{1}{c}\frac{dp_\mu}{dt} = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{c}\int_V d^3x\, T^{0\mu} + \frac{1}{c}p^\mu\right) = -\oint_{\partial V} T^{i\mu}\, \hat{n}^i\, ds$$

The $\mu=0$ component is as before: $T^{00} = \frac{u}{c}$, $T^{i0} = \frac{1}{c^2}S^i$

so, as before, we have $\quad \frac{d}{dt}\left(\int d^3x\, u + \underbrace{cp^0}_{=E_{rest}}\right) = -\oint \vec{S}\cdot\hat{n}\, ds$ ← area element (lowercase)

↑ Poynting vector (uppercase) → sorry!

That is $\quad \frac{d}{dt}\left(\text{Total energy in volume } V\right) = $ (energy flow into volume V through $\partial V$ per unit time).

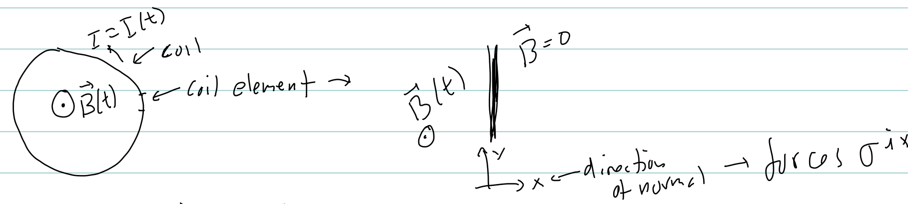
For $\mu = i$

$$\frac{d}{dt}\left(\frac{1}{c}\int d^3x\, g^i + \frac{1}{c}p^i\right) = -\oint \frac{1}{c}\sigma^{ij} n^j ds$$

or $\quad \frac{d}{dt}\left(\int d^3x\, g^i + p^i_{rest}\right) = -\oint \sigma^{ij} n^j ds$

or $\quad \frac{d}{dt}\left(\begin{matrix}\text{Total momentum}\\ \text{in volume } V\end{matrix}\right) = $ (momentum flow into volume V through $\partial V$ / time)

Example:




$$\sigma^{xx} = \frac{1}{8\pi}\left(B_z^2 + 0 \ldots\right) = \frac{1}{8\pi} B_z^2 = \frac{1}{8\pi} B^2$$

$$\sigma^{yx} = \frac{1}{8\pi}\left(B_x B_y + \ldots\right) = 0$$

$$\sigma^{zx} = 0$$

So the $\frac{\text{force}}{\text{area}}$ on the coil has magnitude $\frac{B^2}{8\pi}$, pointing radially outward.

(The force on the field, $-\int \sigma \cdot n \, ds$ points in, then Newton's 3rd gives force on coil).

Of course we can do $|d\vec{F}| = |dq \frac{\vec{v}}{c} \times \vec{B}| = \frac{1}{c} I B \, dL$ over length $dL$ of wire.

Now Ampere's law: $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J}$ $\quad \oint \vec{B} \cdot d\vec{l} = \frac{4\pi}{c} I$ or $B = \frac{4\pi}{c} I / 2\pi R = \frac{2I}{Rc}$

or $dF = \frac{I}{c} B \, dL = \frac{1}{c}\left(\frac{cRB}{2}\right) B \cdot dL = (2\pi R \, dL)\frac{B^2}{4\pi}$ or $\frac{F}{\text{area}} = \frac{B^2}{4\pi}$ ouch.

Refs: Lechner, Chap 5
Jackson, 7.1-7
L&L, Chap 6, 46-48
Chaichian: Chap 4

# EM waves

We consider time dependent (propagating) solutions of Maxwell's equations in free space (no sources).

Consider 1st the 4-vector potential. We have established that in free space

$$\partial^2 A_\mu + \partial_\mu (\partial\cdot A) = 0$$

We choose to impose the covariant gauge condition ("Lorentz gauge") $\partial\cdot A = 0$ so that

$$\partial^2 A_\mu = 0 \qquad \text{(wave equation)}$$

## Plane-waves

A simple solution is

$$A_\mu(x) = a_\mu e^{-ik\cdot x} = a_\mu e^{i\vec{k}\cdot\vec{x} - ik^0x^0} = a_\mu e^{i\vec{k}\cdot\vec{x} - i\omega t} \qquad k^\mu = (k^0, \vec{k}) = (\tfrac{\omega}{c}, \vec{k})$$

where $a_\mu$ and $k_\mu$ are constants. Note that $\partial^2 A_\mu = 0$ requires $k^2 = 0$ or $(k^0)^2 = \vec{k}^2 \Leftrightarrow \omega = \pm c|\vec{k}|$

$A_\mu$ as written is complex. We always implicitly assume we take the real part:

$$A_\mu(x) = Re(a_\mu e^{-ik\cdot x}) = a_\mu e^{-ik\cdot x} + a_\mu^* e^{ik\cdot x}$$

Now, we insist that $\partial_\mu A^\mu = 0 \Rightarrow \partial_\mu (a^\mu e^{-ik\cdot x}) = -ik_\mu a^\mu e^{-ik\cdot x} = 0$

$$\Rightarrow k_\mu a^\mu = 0$$

For example, for wave propagation in $x^3 = z$ direction $k^\mu = (\frac{\omega}{c}, 0, 0, k) = k(1,0,0,1)$ and $k_\mu a^\mu = 0 \Rightarrow a^0 = a^3$

The $\vec{E}$ field is $F^{i0}$. Compute $F_{i0} = \partial_i A_0 - \partial_0 A_i = -i(k_i a_0 - k_0 a_i)\, e^{-ik\cdot x}$

or $E^i = F^{i0} = -i(k^i a^0 - k^0 a^i)\, e^{-ik\cdot x}$

For this example $k^\mu = k(1,0,0,1)$ and $a^3 = a^0$ so $E^3 = -i(k^3 a^0 - k^0 a^3) = 0$

and $E^{1,2} = -i(k^{1,2} a^0 - k^0 a^{1,2})\, e^{-ik\cdot x} = ik a^{1,2}\, e^{-ik\cdot x}$

Also $B^3 = -F^{12} = i(k^1 a^2 - a^2 k^1)\, e^{-ik\cdot x} = 0 \quad B^1 = -F^{23} = i(k^2 a^3 - k^3 a^2)\, e^{-ik\cdot x} = -ika^2 e^{-ik\cdot x}$

and $B^2 = -F^{31} = i(k^3 a^1 - k^1 a^3) e^{-ik\cdot x} = ika^1 e^{-ik\cdot x}$

Note that $B^2 = E^1$, $B^1 = -E^2$. So $\vec{E} \perp \vec{B}$, $|\vec{E}| = |\vec{B}|$ and both $\perp$ $\vec{k}$: $\vec{B} = \hat{k} \times \vec{E}$

with $\hat{k} = \frac{\vec{k}}{|\vec{k}|}$

Clearly only $a_{1,2}$ have physical meaning. Why do we still have $a^0 = a^3$ in $A_\mu$?
Recall if $A_\mu$ satisfies the covariant gauge condition $\partial \cdot A = 0$
we can still make a gauge transformation

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda$$

with $\partial \cdot A' = 0$ provided $\partial^2 \Lambda = 0$. So we can add to our solution

$$\Lambda(x) = i\lambda e^{-ik\cdot x} \quad \to \quad \partial_\mu \Lambda = k_\mu \lambda e^{-ik\cdot x}$$

so that $\quad a'_\mu = a_\mu + k_\mu \lambda$

or $\quad a'^\mu = (a^0 + k\lambda, a^1, a^2, a^3 + k\lambda)$

and choosing $\lambda = -a^0/k$ we obtain $a'^\mu = (0, a^1, a^2, 0)$

So we may as well drop $a^0 = a^3$, and drop the prime.

It is convenient to define a unit vector $\vec{\epsilon} = \dfrac{\vec{a}}{|\vec{a}|}$. It satisfies $\vec{k}\cdot\vec{\epsilon} = 0$. Also
$\epsilon^\mu = (0, \vec{\epsilon})$ satisfies $k \cdot \epsilon = 0$, $\epsilon^2 = -1$

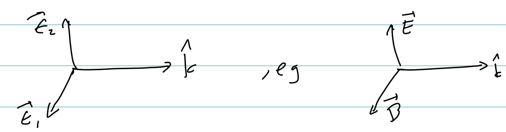
$\vec{\epsilon}$ is the "polarization" vector. We can write generally the equations found for the z-propagation example:

$$\vec{E} = E_0 \vec{\epsilon} e^{-ik\cdot x} \qquad \vec{B} = \hat{k} \times \vec{E} \qquad \vec{k}\cdot\vec{\epsilon} = 0 \qquad \vec{\epsilon}^2 = 1$$

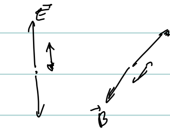
but now $\vec{k}$ is arbitrary.

Plane polarized wave: there are two independent solutions $\vec{\epsilon}_{1,2}$ to $\vec{k}\cdot\vec{\epsilon} = 0$, $\vec{\epsilon}^2 = 1$.
that is two unit vectors in the plane $\perp$ to $\hat{k}$.
E.g. for $\hat{k} = (0,0,1)$ $\quad \vec{\epsilon}_1 = (1,0,0)$ & $\vec{\epsilon}_2 = (0,0,1)$

These, and any linear combination with relatively real coefficients are plane polarized waves




At a fixed point in space, $\vec{E} = \underbrace{\left(E e^{i\vec{k}\cdot\vec{x}}\right)}_{\text{fixed}} \vec{\epsilon} e^{-i\omega t}$ oscillates along $\vec{\epsilon}$ :




For general $\hat{k}$ direction, choose a vector $\vec{\epsilon}_1$ + $\epsilon_1^2 = 1$ & $\hat{k}\cdot\vec{\epsilon}_1 = 0$. Then take $\vec{\epsilon}_2 = \hat{k} \times \vec{\epsilon}_1$

L.L. 48
Jackson 7.2

Circular Polarization: In terms of $\vec{\epsilon}_{1,2}$ above, let

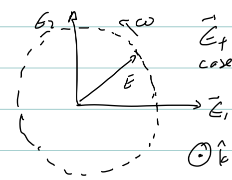
$$\vec{\epsilon}_{\pm} = \frac{\vec{\epsilon}_1 \pm i\,\vec{\epsilon}_2}{\sqrt{2}}$$

Then $\vec{\epsilon}_+ \cdot \vec{\epsilon}_+^* = \vec{\epsilon}_- \cdot \vec{\epsilon}_-^* = 1$ and $\vec{\epsilon}_+ \cdot \vec{\epsilon}_-^* = 0$, and $\vec{\epsilon}_-^* = \vec{\epsilon}_+$.

Now $\vec{E}(\vec{x},t) = E\,\vec{\epsilon}_+\, e^{+i\vec{k}\cdot\vec{x} - i\omega t}$

as seen at fixed point is a vector $\vec{E}$ of fixed magnitude rotating with angular speed $\omega$ in the $\vec{\epsilon}_1$-$\vec{\epsilon}_2$ plane

with $E = |E|e^{i\varphi}$ $\quad \mathrm{Re}\,\vec{E} = \frac{|E|}{\sqrt{2}}\left(\vec{\epsilon}_1 \cos(\omega t - i\vec{k}\cdot\vec{x} - \varphi) + \vec{\epsilon}_2 \sin(\omega t - i\vec{k}\cdot\vec{x} - i\varphi)\right]$




$\vec{\epsilon}_+$ case "right circular polarization"

for $\vec{\epsilon}_-$ case reverse direction (clockwise) "left circular polarization"

Elliptical polarization is a simple extension: $\vec{E} = (E_1\vec{\epsilon}_1 + i E_2\vec{\epsilon}_2)\, e^{i\vec{k}\cdot\vec{x} - i\omega t}$ with $E_1/E_2$ = real. The $E_1 = E_2$ case is circular.

Exercise: for $\hat{k} = \hat{z}$, $\vec{\epsilon}_1 = \hat{x}$ verify that $\frac{(E^1(\vec{x},t))^2}{E_1^2} + \frac{(E^2(\vec{x},t))^2}{E_2^2} = 1$ which describes an ellipse.

Lechner: Chap 5
L&L 48
Jackson: 7.1 (partly)

## Energy

Since $\vec{B} = \hat{k} \times \vec{E}$ we can get immediately

$$u = \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2) = \frac{1}{4\pi} E^2 \qquad \text{energy density}$$

$$\vec{S} = \frac{c}{4\pi}(\vec{E} \times \vec{B}) = \frac{c}{4\pi} E^2 \hat{k} = cu\hat{k} \qquad \text{Poynting vector (energy flux)}$$

The momentum density $\vec{g}$ is related to the energy density (for plane waves!)

$$\vec{g} = \frac{1}{c^2}\vec{S} = \frac{u}{c}\hat{k}$$

The relation $U = gc$ parallels $E_{kin} = \sqrt{(\vec{p}c)^2 + (mc^2)^2} \longrightarrow |p|c$ for the massless particle case. Such particle can move only at the speed of light, since

$$E_{kin} = \frac{mc^2}{\sqrt{1 - v^2/c^2}}, \quad \vec{p} = \frac{m\vec{v}}{\sqrt{1 - v^2/c^2}}$$

can stay non-zero as $m \to 0$ if we take $|\vec{v}| \to c$ simultaneously

So $\vec{g} = \frac{u}{c}\hat{k}$ can be interpreted as the momentum and energy make up of many non-interacting massless particles, all propagating at the common speed $c$. This interpretation is fleshed out in QM-version, the particles = photons.

Finally, $\sigma_{ij} = -\frac{1}{4\pi}\left( E^i E^j + B^i B^j - \frac{1}{2}\delta^{ij}(\vec{E}^2 + \vec{B}^2) \right)$

For $\hat{k} = \hat{z}$ clearly vanishes unless $i = j = 3 = $ "z" for which $\sigma_{33} = \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2) = u$

Recall this is force on surface $\perp$ to "3" (ie to $\hat{k}$) along "3": if we put a perfectly absorbing plane parallel to $\hat{\varepsilon}_1 - \hat{\varepsilon}_2$ plane, it will experience force/area $= u$ = radiation pressure

### Time-averages

For most cases of interest $1/\omega$ is such a short time that measurements of energy density and other such quantities automatically average over many cycles. Then we really care about the average quantities:

$$\bar{f} = \frac{1}{T}\int_0^T dt\, f(t) \quad \text{where } T \text{ is either exactly one period } \frac{2\pi}{\omega} \text{, or it's large } T \gg \frac{2\pi}{\omega}$$

Then if $\vec{E} = \frac{1}{2}(\vec{E}_0(\vec{x})e^{-i\omega t} + c.c.)$, we have from $\bar{u} = \frac{1}{4\pi}\overline{\vec{E}^2}$ and $\vec{E}^2 = \frac{1}{4}\vec{E}_0^2 e^{-2i\omega t} + \frac{1}{4}\vec{E}_0^{*2} e^{2i\omega t} + \frac{1}{2}\vec{E}_0 \cdot \vec{E}_0^*$ so that $\overline{\vec{E}^2} = \frac{1}{2}\vec{E}_0 \cdot \vec{E}_0^*$ and $u = \frac{1}{8\pi}\vec{E}_0 \cdot \vec{E}_0^*$ (The normalization is arbitrary: we could have chosen $\vec{E} = \frac{1}{\sqrt{2}}\vec{E}_0(x)e^{-i\omega t} + c.c.$).

Notes:

1. The waves above $\propto e^{-ik\cdot x}$ are monochromatic. Linearity of wave-equation means linear combinations are still solutions. Most general

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3}\left(a_\mu(k)\, e^{-ik\cdot x} + c.c.\right)$$

("c.c." = complex conjugate)

with $k^0 = |\vec{k}|$ and $k\cdot a = 0$. One may shift $a_\mu \to a_\mu + \lambda k_\mu$ freely, in particular to choose to make $a_0 = 0$, just as above. It is customary to expand in the two independent polarization vectors $\epsilon_\mu^{(i)}$, thus

$$A_\mu(x) = \sum_{i=1}^{2} \int \frac{d^3k}{(2\pi)^3}\left( a^{(i)}(k)\, \epsilon_\mu^{(i)}(k)\, e^{-ik\cdot x} + c.c.\right)$$

(In fact, for second quantization this is the starting point, except that there usually

$$A_\mu(x) = \sum_{i=1}^{2} \int \frac{d^3k}{2E\,(2\pi)^3}\left( \tilde{a}^{(i)}(k)\, \epsilon^{(i)}(k)\, e^{-ik\cdot x} + c.c.\right)$$

The factor of $2E$ is for Lorentz invariance of $\frac{d^3k}{E}$. To see this, recall $k^0 = |\vec{k}|$) so $\int d^4k\, \delta(k^2)\,\theta(k^0) = \int \frac{d^3k}{2E}$ ).

2. One can work directly from Maxwell's equations and get $\vec{E}$ & $\vec{B}$ waves: probably done in your UG course, but see next page.

3. For complex vectors, the magnitude is $\vec{a}\cdot\vec{a}^*$ or $a_\mu a^{*\mu}$. Beware of metric: $\epsilon_\mu \epsilon^{*\mu} = -1$ !

Jackson 7.3
Scheck 4.4
Chaichian 4.4

Waves in media; reflection and refraction at interfaces; polarization at interfaces.

We will study E&M of/in media next quarter. For now we draw from previous knowledge. While we could still use a vector potential — since the homogeneous Maxwell equations are unchanged — the presence of a medium in which $\vec{E}$ and $\vec{B}$ propagate breaks Lorentz invariance – it defines a preferred frame.

We introduce dimensionless permittivity $\epsilon$ ($\vec{D}=\epsilon\vec{E}$) and permeability $\mu$ ($\vec{B}=\mu\vec{H}$) so that the source free Maxwell equations read

$$\epsilon\,\vec{\nabla}\cdot\vec{E}=0 \quad (\leftarrow \text{was } \rho) \qquad (1) \qquad\qquad \nabla\times\vec{E}+\frac{1}{c}\frac{\partial\vec{B}}{\partial t}=0 \qquad (3)$$

and

$$\frac{1}{\mu}\vec{\nabla}\times\vec{B}-\frac{1}{c}\epsilon\frac{\partial\vec{E}}{\partial t}=0 \quad (\leftarrow \text{was } \vec{J}) \qquad (2) \qquad\qquad \vec{\nabla}\cdot\vec{B}=0 \qquad (4)$$

Taking $\vec{\nabla}\times$ of (3) and using $\left[\vec{\nabla}\times(\vec{\nabla}\times\vec{a})\right]^i=\epsilon^{ijk}\partial_j\epsilon^{kmn}\partial_m a^n$

$$=\left(\delta^{im}\delta^{jn}-\delta^{in}\delta^{jm}\right)\partial_j\partial_m a^n$$
$$=\left[\vec{\nabla}(\vec{\nabla}\cdot a)-\nabla^2\vec{a}\right]^i$$

we get

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{E})-\nabla^2\vec{E}+\frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})=0$$

and using (2) in this, $\vec{\nabla}\times\vec{B}=\frac{\mu\epsilon}{c}\frac{\partial\vec{E}}{\partial t}$, and (1) ($\vec{\nabla}\cdot\vec{E}=0$), obtain

$$\frac{\mu\epsilon}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}-\nabla^2\vec{E}=0$$

This is the wave equation with velocity $v^2=\frac{c^2}{\mu\epsilon}$. With $n^2=\mu\epsilon$, this is $v=\frac{c}{n}$. $n$, we will see, is the index of refraction of the medium.

Similarly, taking $\vec{\nabla}\times$ (eq (2))

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{B})-\nabla^2\vec{B}-\frac{\mu\epsilon}{c}\frac{\partial}{\partial t}\vec{\nabla}\times\vec{E}=0$$

and using (4) and (3)

$$\Rightarrow\quad \frac{1}{v^2}\frac{\partial^2\vec{B}}{\partial t^2}-\nabla^2\vec{B}=0$$

again the wave equation with velocity $v=\frac{c}{n}$.

Plane wave solutions are as before,

$$\vec{E}(\vec{x},t)=\vec{E}_0e^{i(\vec{k}\cdot\vec{x}-\omega t)}+\text{c.c.}\qquad \vec{B}(\vec{x},t)=\vec{B}_0e^{i(\vec{k}'\cdot\vec{x}-\omega' t)}+\text{c.c.}$$

where $\vec{k}^2-\frac{\omega^2}{v^2}=0$ and $\vec{k}'^2-\frac{\omega'^2}{v^2}=0$

These must satisfy (1)–(4):

$$(1)\Rightarrow\ \vec{\nabla}\cdot\vec{E}=0\ \Rightarrow\ \vec{k}\cdot\vec{E}_0=0 \qquad (4)\ \vec{\nabla}\cdot\vec{B}=0\ \Rightarrow\ \vec{k}'\cdot\vec{B}_0=0$$

(3) $\vec\nabla\times\vec E+\frac{1}{c}\frac{\partial\vec B}{\partial t}=0 \Rightarrow \vec k\times\vec E_0\, e^{i(\vec k\cdot\vec x-\omega t)} - \frac{\omega'}{c}\vec B_0 e^{i(\vec k'\cdot\vec x-\omega' t)} = 0$

which requires $\vec k'=\vec k$, $\omega'=\omega$ and $\vec k\times\vec E_0=\frac{\omega}{c}\vec B_0 = \frac{v}{c}|\vec k|\vec B_0 = \frac{1}{n}|\vec k|\vec B_0$ or

$$\hat k\times\vec E_0=\frac{1}{n}\vec B_0$$

(4) $\vec\nabla\times B-\frac{\mu\epsilon}{c}\frac{\partial\vec E}{\partial t}=0 \Rightarrow \vec k\times\vec B_0+\frac{\mu\epsilon}{c}\omega\vec E_0=0 \Rightarrow \vec k\times\vec B_0=-n|\vec k|\vec E_0$

$$\hat k\times\frac{\vec B_0}{n}=-\vec E_0$$

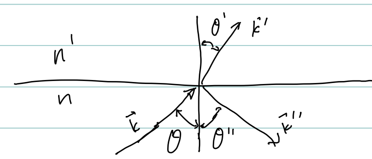
Summarizing, $\vec E_0$, $\vec B_0$ and $\hat k$ are mutually perpendicular, with $|\vec B_0|=n|\vec E_0|$ and $\frac{\vec B_0}{n}=\hat k\times\vec E_0$

In terms of polarization vectors

$$\vec E=E_0\vec\epsilon\, e^{i(\vec k\cdot x-\omega t)} \qquad \vec B=nE_0(\hat k\times\vec\epsilon)\,e^{i(\vec k\cdot\vec x-\omega t)}$$

with $\vec\epsilon\cdot\vec\epsilon^{*}=1$ and $\vec k\cdot\vec\epsilon=0$.

Two media and interface




Incident wave

$$\vec E=\vec E_0 e^{i(\vec k\cdot\vec x-\omega t)} \qquad \vec B=n(\hat k\times\vec E_0)e^{i(\vec k\cdot\vec x-\omega t)}$$

Refracted wave

$$\vec E'=\vec E_0' e^{i(\vec k'\cdot\vec x-\omega' t)} \qquad \vec B'=n'(\hat k'\times\vec E_0')\,e^{i(\vec k'\cdot x-\omega' t)}$$

Reflected wave

$$\vec E''=\vec E_0'' e^{i(\vec k''\cdot x-\omega'' t)} \qquad \vec B''=n(\hat k''\times\vec E_0'')\,e^{i(\vec k''\cdot\vec x-\omega'' t)}$$

Let the boundary be the plane $z=0$. The fields $\vec E$ & $\vec B$ are given by any of these at the boundary so we must have $\omega=\omega'=\omega''$ for common time dependence. Note that this means

$$|\vec k''|=|\vec k|=\frac{\omega}{v}=\frac{\omega}{c}n \qquad \text{and} \qquad |\vec k'|=\frac{\omega}{v'}=\frac{\omega}{c}n'=\frac{n'}{n}|\vec k|$$

Moreover, at $z=0$

$$\vec k\cdot\vec x=\vec k'\cdot\vec x=\vec k''\cdot\vec x$$

That is, the projection on the xy plane is equal:

$$|k|\sin\theta=|k'|\sin\theta'=|k''|\sin\theta''$$

It follows that $\theta''=\theta$ and $\frac{\sin\theta'}{\sin\theta}=\frac{|k|}{|k'|}=\frac{n}{n'}$ or $\boxed{n'\sin\theta'=n\sin\theta}$

Snell's Law of refraction.

Polarization: There is additional information coded in the boundary conditions. For this we need

(i) Normal components of $\vec{D} = \epsilon\vec{E}$ and $\vec{B}$ are continuous (from $\vec{\nabla}\cdot\vec{D}=0$ & $\vec{\nabla}\cdot\vec{B}=0$)

(ii) Tangential components of $\vec{E}$ and $\vec{H} = \frac{1}{\mu}\vec{B}$ are continuous (from $\vec{\nabla}\times\vec{E}$ & $\vec{\nabla}\times\vec{H}$ eqs).

These will be derived when we study continuous media. For now, use them: let $\hat{n}$ be normal to the boundary. Then

$$\hat{n}\cdot\left[\epsilon(\vec{E}_0+\vec{E}_0'') - \epsilon'\vec{E}_0'\right]=0 \qquad (1)$$

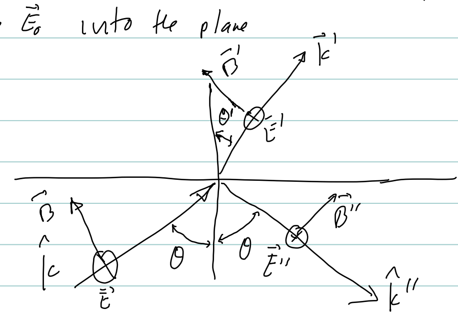
$$\hat{n}\cdot\left[n\hat{k}\times\vec{E}_0 + n\hat{k}''\times\vec{E}_0'' - n'\hat{k}'\times\vec{E}_0'\right]=0 \qquad (2)$$

$$\hat{n}\times\left[\vec{E}_0+\vec{E}_0''-\vec{E}_0'\right]=0 \qquad (3)$$

$$\hat{n}\times\left[\frac{n}{\mu}\hat{k}\times\vec{E}_0 + \frac{n}{\mu}\hat{k}''\times\vec{E}_0'' - \frac{n'}{\mu'}\hat{k}'\times\vec{E}_0'\right]=0 \qquad (4)$$

Two cases (the general one is a linear combination of those):

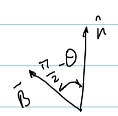
(i) Polarization vector $\vec{E}_0$ of incident wave (linearly polarized) perpendicular to plane of incidence that is plane of $\hat{k}$ and $\hat{n}$: take $\vec{E}_0$ into the plane




We have 1+1+3+3 equations for 3+3 unknowns ($\vec{E}_0'$ & $\vec{E}_0''$): the system is overdetermined. We look for solutions with the polarizations all into the plane (we could derive this). Then

(3) $\Rightarrow$ $E_0 + E_0'' - E_0' = 0$

(4) $\Rightarrow$




$$\frac{n}{\mu}(E_0 - E_0'')\cos\theta - \frac{n'}{\mu'}E_0'\cos\theta' = 0$$

Solving for $E_0'$ & $E_0''$

from (3) $E_0' = E_0 + E_0''$, from (4) $\frac{n}{\mu}(E_0 - E_0'')\cos\theta - \frac{n'}{\mu'}(E_0+E_0'')\cos\theta' = 0$

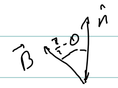
$$\Rightarrow \frac{E_0''}{E_0}\left(\frac{n}{\mu}\cos\theta + \frac{n'}{\mu'}\cos\theta'\right) = \frac{n}{\mu}\cos\theta - \frac{n'}{\mu'}\cos\theta' \quad\Rightarrow\quad \boxed{\frac{E_0''}{E_0} = \frac{\frac{n}{\mu}\cos\theta - \frac{n'}{\mu'}\cos\theta'}{\frac{n}{\mu}\cos\theta + \frac{n'}{\mu'}\cos\theta'}}$$

$\vec{E}_0 \perp$ to $\hat{n},\hat{k}$ plane

$$\text{Then } \frac{E_0'}{E_0} = 1 + \frac{E_0''}{E_0} = \frac{2\frac{n}{\mu}\cos\theta}{\frac{n}{\mu}\cos\theta + \frac{n'}{\mu'}\cos\theta'} \quad\Rightarrow\quad \boxed{\frac{E_0'}{E_0} = \frac{2n\cos\theta}{\frac{n}{\mu}\cos\theta + \frac{n'}{\mu'}\cos\theta'}}$$

One often writes $\cos r = \sqrt{1-\sin^2\theta'} = \sqrt{1-\frac{n^2}{n'^2}\sin^2\theta} = \frac{1}{n'}\sqrt{n'^2 - n^2\sin^2\theta}$ so that it is all expressed in terms of incident data.

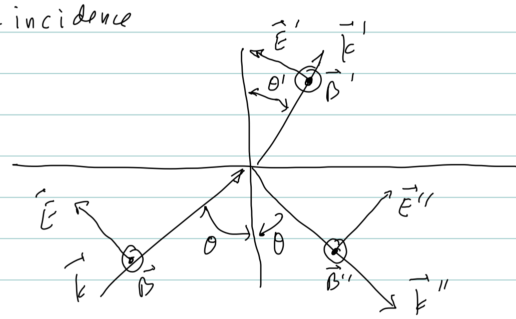
What about (2)?




so $\hat n\cdot\vec B = \sin i\, B$ $\Rightarrow$ $n(\underbrace{E_0+E_0''}_{E_0'})\sin\theta - n' E_0'\sin\theta' = 0$

$\Rightarrow$ $n\sin i = n'\sin r$   Snell's Law again

(ii) $\vec E_0$ parallel to plane of incidence




Now (3) and (4) give

$$(E_0-E_0'')\cos\theta - E_0'\cos\theta' = 0$$

$$\frac{n}{\mu}(E_0+E_0'') - \frac{n'}{\mu'}E_0' = 0$$

Algebra: (4): $E_0' = \frac{\mu'}{\mu}\frac{n}{n'}(E_0+E_0'')$ in (3) gives

$$(E_0-E_0'')\cos\theta - \frac{\mu'}{\mu}\frac{n}{n'}(E_0+E_0'')\cos\theta' = 0 \quad\Rightarrow\quad \boxed{\frac{E_0''}{E_0} = \frac{\cos\theta - \frac{\mu'}{\mu}\frac{n}{n'}\cos\theta'}{\cos\theta + \frac{\mu'}{\mu}\frac{n}{n'}\cos\theta'}}$$

$\vec E_0 \parallel$ to $\hat k$ & $\hat n$ plane

Then $\frac{E_0'}{E_0} = \frac{\mu'}{\mu}\frac{n}{n'}\left(1+\frac{E_0''}{E_0}\right)$

$$\boxed{\frac{E_0'}{E_0} = \frac{2\frac{\mu'}{\mu}\frac{n}{n'}\cos\theta}{\cos\theta + \frac{\mu'}{\mu}\frac{n}{n'}\cos\theta'}}$$

[Jackson: 7.4]

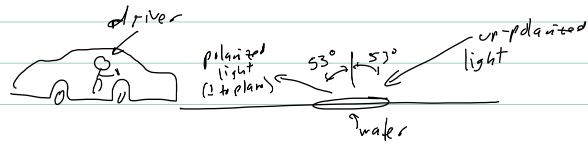
Brewster angle: in this case the reflected wave vanishes for $\cos\theta_B = \frac{\mu'}{\mu}\frac{n}{n'}\cos\theta_B' = \frac{\mu'}{\mu}\frac{n}{n'}\sqrt{1-\frac{n^2}{n'^2}\sin^2\theta_B}$

Consider simpler (but often found) case $\mu'/\mu = 1$. Then

$$\left(\frac{n'}{n}\cos\theta_B\right)^2 = 1-\frac{n^2}{n'^2}\sin^2\theta_B = 1-\frac{n^2}{n'^2}+\frac{n^2}{n'^2}\cos^2\theta_B \Rightarrow \cos^2\theta_B = \frac{1-n^2/n'^2}{\frac{n'^2}{n^2}-n^2/n'^2} = \frac{1-\frac{n^2}{n'^2}+n'^2/n^2-n'^2/n'^2}{n'^2/n^2-n^2/n'^2} = 1-\frac{n'^2/n^2-1}{n'^2/n^2-n^2/n'^2} = 1-\sin^2\theta_B$$

$$\tan^2\theta_B = \frac{n'^2/n^2-1}{1-n^2/n'^2} = \left(\frac{n'}{n}\right)^2 \qquad \boxed{\theta_B = \tan^{-1}(n'/n)}$$

For water $n=1.33$ and air $n=1.00$, $\theta_B = 0.93$ or $53°$.



Jackson 7.4

# Total internal reflection.

You know from elementary courses that for (using Snell's Law)

$$\sin\theta' = \frac{n}{n'}\sin\theta \geq 1 \qquad \text{critical at incident angle } \sin\theta_T = \frac{n'}{n}$$

there is no refracted wave. This is the phenomenon of "total internal reflection". We can understand some aspects of this in more detail with the tools developped.

The refracted wave is $\vec{E}' = \vec{E}_0' \, e^{i(\vec{k}'\cdot\vec{x} - \omega t)}$

Now $\vec{k}'\cdot\vec{x} = |\vec{k}'|\left(z\cos\theta' + x_{\parallel}\sin\theta'\right)$

with $\cos\theta' = \sqrt{1-\sin^2\theta'} = i\sqrt{\sin^2\theta' - 1} = i\sqrt{\left(\frac{\sin\theta}{\sin\theta_T}\right)^2 - 1}$

so $e^{i\vec{k}'\cdot\vec{x}} = e^{-|\vec{k}'|\sqrt{\left(\frac{\sin\theta}{\sin\theta_T}\right)^2 - 1}\, z} \; e^{i|\vec{k}'|x\frac{\sin\theta}{\sin\theta_T}}$

$\Rightarrow$ the wave is exponentially damped into the interior of the medium $n'$.

There is no energy flux into $n'$: $\vec{S} \propto \vec{E}\times\vec{B}$ (actually $\vec{E}\times\vec{H} = \vec{E}\times\frac{1}{\mu}\vec{B}$)

So

$$\hat{n}\cdot\vec{S} \propto \hat{n}\cdot\left(\vec{E}_0'\times\vec{B}_0'^*\right) + \text{c.c.} \;\propto\; 2\,\mathrm{Re}\left[\hat{n}\cdot\vec{E}_0'\times\left(\vec{k}'\times\vec{E}_0'\right)^*\right] = 2\,\mathrm{Re}\left[\hat{n}\cdot\left(\vec{k}'|\vec{E}_0'|^2 - \vec{E}_0'^{*}\,\underbrace{(\vec{k}'\cdot\vec{E}_0')}_{0}\right)\right]$$

$$= 2\,\mathrm{Re}(\hat{n}\cdot\hat{k}')\;|\vec{E}_0'|^2$$

But $\hat{n}\cdot\hat{k}' = \cos\theta'$ is purely imaginary for $\theta > \theta_T \Rightarrow \underline{\hat{n}\cdot\vec{S} = 0}$

Note also that $|E_0''/E_0| = 1$ for each polarization, eg, for $\vec{E}_0 \perp$ to plane of incidence

$$\frac{E_0''}{E_0} = \frac{\frac{n}{\mu}\cos\theta - \frac{n'}{\mu'}i|\cos\theta'|}{\frac{n}{\mu}\cos\theta + \frac{n'}{\mu'}i|\cos\theta'|}$$

The reflected wave has same intensity, but experiences a phase shift. The phase shift is different for each of the two plane polarizations, so this can be used to polarize waves.

L&L: 50
Jackson: 7.2

## Degree of Polarization

(See time averages, p. 4 of these notes)

For $\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)} + \text{c.c.}$

the polarization tensor $\rho_{ij}$ is defined by

$$\rho_{ij} \equiv \frac{E_{0i} E_{0j}^*}{|\vec{E}_0|^2}$$

The rank of this matrix is 2, since $\vec{E}_0$ has $\hat{k}\cdot\vec{E}_0 = 0$. Let's specify $\hat{k} = \hat{z}$ so that we can write $\rho$ as a $2\times 2$ matrix. Note that

$$\mathrm{Tr}\,\rho = 1 \,, \qquad \rho^\dagger = \rho \quad (\text{hermitian}). \qquad (A)$$

$$\text{and} \quad \det\rho = 0 \qquad (B)$$

For linear polarization, eg $E_{0y} = 0$ or $E_{0x} = 0$ we have $\rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

For $\vec{E}_0 = E_0 \vec{\epsilon}_\pm = \frac{1}{\sqrt{2}} E_0 (\hat{x} \pm i\hat{y})$

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & \mp i \\ \pm i & 1 \end{pmatrix}$$

Realistic light is not exactly monochromatic: consists of a superposition of frequencies $\omega \pm \Delta\omega$ with $\Delta\omega \ll \omega$. For example,

$$\int_{-\infty}^{\infty} d\omega' \, e^{-i\omega' t} \, e^{-(\omega'-\omega)^2/2(\Delta\omega)^2} = e^{-i\omega t} \int_{-\infty}^{\infty} d\omega' \, e^{-i\omega' t} \, e^{-\omega'^2/2(\Delta\omega)^2}$$

completing the squares $\frac{\omega'^2}{2(\Delta\omega)^2} + i\omega' t = \frac{1}{2(\Delta\omega)^2}\left(\omega' + i(\Delta\omega)^2 t\right)^2 + \frac{(\Delta\omega)^2 t^2}{2} \Rightarrow \propto \underbrace{e^{-i\omega t}}_{\text{fast}} \underbrace{e^{-\frac{(\Delta\omega)^2 t^2}{2}}}_{\text{slow}}$

At one point in space this is a wave packet that moves through the point.
One can generally write (at one point in space)

$$\vec{E}(t) = \vec{E}_0(t)\, e^{-i\omega t} \qquad \text{with } \left|\frac{d\vec{E}_0}{dt}\right| \sim \Delta\omega |\vec{E}_0| \text{, ie, } \vec{E}_0 \text{ varies slowly}$$

$\rho$ is more generally defined as

$$\rho_{ij} = \frac{\overline{E_{0i} E_{0j}^*}}{\overline{|\vec{E}_0|^2}} \qquad \text{where } \overline{f(t)} = \frac{1}{T}\int_0^T dt\, f(t) \quad \text{the time average}$$

For example, above, with $T = \frac{2\pi}{\omega}$

$$\frac{1}{T}\int_0^T e^{-i\omega t - \frac{(\Delta\omega)^2 t^2}{2}} dt = \frac{1}{T}\int_0^T e^{-i\omega t}\left(1 - \frac{(\Delta\omega)^2 t^2}{2} + \cdots\right) dt = 0 - \frac{(\Delta\omega)^2}{\omega^2} + i\pi\frac{(\Delta\omega)^2}{\omega^2} + O\left(\frac{(\Delta\omega)^4}{\omega^4}\right)$$

where we used $\int dt\, e^{-i\omega t} t^2 = \left(\frac{1}{-i}\frac{\partial}{\partial\omega}\right)^2 \int dt\, e^{-i\omega t} = -\frac{\partial^2}{\partial\omega^2}\left(i\frac{e^{-i\omega t}}{\omega}\right) = -i\frac{\partial}{\partial\omega}\left[e^{-i\omega t}\left(-\frac{1}{\omega^2} - \frac{it}{\omega}\right)\right] = i e^{-i\omega t}\left[\frac{2}{\omega^3} - \frac{2it}{\omega^2} - \frac{t^2}{\omega}\right]$

Now the conditions (A) still hold, but (B) does not hold generally.
det $\rho = 0$ is the condition for complete polarization. It is necessary and sufficient.

$$(\overline{E_0^2})^2 \det\rho = \overline{E_{0x}E_{0x}^*}\;\overline{E_{0y}E_{0y}^*} - \overline{E_{0x}E_{0y}^*}\;\overline{E_{0y}E_{0x}^*}$$

Schwarz inequality: a & b functions of time, $\lambda$ a parameter,

$$\overline{|a+\lambda b|^2} \geq 0 \Rightarrow \overline{|a|^2} + |\lambda|^2\overline{|b|^2} + \lambda^*\overline{ab^*} + \lambda\overline{a^*b} \geq 0 \qquad \text{any } \lambda$$

with $\lambda = -\overline{ab^*}/\overline{|b|^2}$ $\Rightarrow$ $\overline{|a|^2}\,\overline{|b|^2} \geq \overline{ab^*}\;\overline{a^*b}$

and the equality is only if $a + \lambda b = 0$ or $a(t) = -\lambda b(t)$ ($\lambda$ is time independent).

So $\det\rho \geq 0$ with $\det\rho = 0$ only for $E_{0x} + \lambda E_{0y} = 0$, ie $E_{0y}$ proportional to $E_{0y}$ with constant (in time) proportionality constant: for $\lambda$ real this is plane polarization for $\lambda = \pm i$, circular polarization, others $\Rightarrow$ ellyptical.

## Stoke Parameters

(There are many definitions, I choose one).

with $\vec{\sigma}$ the Pauli matrices $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

any 2x2 hermitian matrix

$$\rho = \tfrac{1}{2}a_0\mathbb{1} + \tfrac{1}{2}\vec{a}\cdot\vec{\sigma} \qquad \mathrm{Tr}\,\rho = 1 \Rightarrow a_0 = 1$$

$a_1, a_2, a_3$ are "Stoke parameters".

$$\det\rho = \det\tfrac{1}{2}\begin{pmatrix} 1+a_3 & a_1 - ia_2 \\ a_1 + ia_2 & 1-a_3 \end{pmatrix} = \tfrac{1}{4}(1-\vec{a}^2) \geq 0 \Rightarrow \vec{a}^2 \leq 1$$

The degree of Polarization $P \equiv a_1^2 + a_2^2 + a_3^2$ has $P \leq 1$, with $P = 1$ for complete polarization.

Additionally

(i) For $E_{0y} = 0$ ($\vec{E}_0 \propto \hat{x}$) $a_3 = 1$; more generally $\rho$ is normalized to $\overline{|E|^2} \propto \bar{u} \propto \overline{|\vec{S}|}$, ie, beam intensity, so $\frac{1}{2}(1+a_3) = \rho_{xx}$ is the fraction of intensity that passes through an $\hat{x}$-polarization filter, or $\frac{1}{2}(1-a_3) = \rho_{yy}$ through a $\hat{y}$-polarizer

(ii) Similarly $\frac{1}{2}(1+a_1)$ = fraction of intensity through $\hat{x}+\hat{y}$ filter

(iii) Setting $\vec{E}_0 = E_0\vec{\epsilon}_+$ gives $a_2 = 1$: $\frac{1}{2}(1+a_2)$ fraction of intensity that passes through right circular polarizer.

Also useful

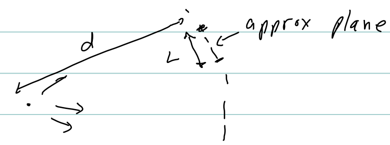
$$a_L = \sqrt{a_1^2 + a_3^2} = \text{degree of linear polarization}$$

$$a_C = a_2 \qquad = \text{" " circular "}$$

L.L: Chap 7: 53
Scheck: 4.5.2
See also Chaichian et al ; 4.8.4

# Eikonal Approximation

For non-plane waves (maybe spherical waves) we expect that at a distance $d$ from sources, and over a region of size $L$, if $d \gg L$ (and $L \gg \lambda$) the wave is approximately a plane wave

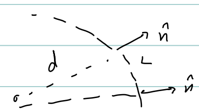



The Eikonal approximation formalizes this, and is the basis for geometrical optics. Take

$$\vec{E} = \vec{E}_0(\vec{x})\, e^{ik\psi(\vec{x}) - ikx^0}$$

$$\vec{B} = \vec{B}_0(\vec{x})\, e^{ik\psi(\vec{x}) - ikx^0}$$

We are assuming still monochromatic waves, with $k = 2\pi/\lambda$. But now what used to be $\exp(ikz) = \exp(ik\,\hat{n}\cdot\vec{x})$ with $\hat{n}$ giving the direction of propagation, became a function $\psi(\vec{x})$. Now we expect $\psi$ to be like $\hat{n}\cdot\vec{x}$ with $\hat{n}$ varying as in




so $\Delta\psi \sim \frac{L}{d}\psi$ or $\frac{\Delta\psi}{\Delta L} \sim \frac{1}{d}\psi \sim \frac{d}{d} \sim 1$ (like $\hat{n}$).

That $\vec{E}_0$ and $\vec{B}_0$ have $\vec{x}$ dependence is also clear, since the polarization must change. But let's just follow the math. Let's use the ansatz in Maxwell's equations for $A_\mu$:

$$A_\mu = a_\mu e^{ik\psi - ikx^0} \qquad a_\mu = a_\mu(\vec{x}) \qquad \psi = \psi(\vec{x})$$

$$\partial_\mu A^\mu = 0 \qquad \partial_\mu a^\mu + ik\left(\partial_\mu\psi - \delta^0_\mu\right) a^\mu = 0 \quad \Rightarrow \quad \vec{\nabla}\cdot\vec{a} + ik\left(\vec{\nabla}\psi\cdot\vec{a} - a^0\right) = 0$$

and

$$\partial^2 A_\mu = 0 \quad \Rightarrow \quad -\nabla^2 a_\mu - 2ik\,\vec{\nabla}a_\mu\cdot\vec{\nabla}\psi + a_\mu\left(k^2(\vec{\nabla}\psi)^2 - ik\nabla^2\psi\right) - k^2 a_\mu = 0$$

Now $k \sim \frac{2\pi}{\lambda} \gg \vec{\nabla}$ for any of these. So we must have

$$(\partial_\mu A^\mu = 0) \rightarrow \quad a^0 = \vec{\nabla}\psi\cdot\vec{a} \qquad \text{and} \qquad (\partial^2 A = 0) \quad (\vec{\nabla}\psi)^2 = 1$$

We still can redefine $A_\mu \rightarrow A_\mu + \partial_\mu\Phi$ with $\partial^2\Phi = 0$. Taking $\Phi = \varphi(\vec{x})\, e^{ik(\psi - x^0)}$

$A_\mu \rightarrow A_\mu + \left(\partial_\mu\varphi + ik(\partial_\mu\psi - \delta^0_\mu)\varphi\right) e^{ik(\psi - x^0)}$. In particular $A_0 \rightarrow A_0 - ik\varphi\, e^{ik(\psi - x^0)}$

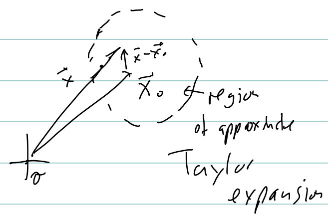
or $a_0 \rightarrow a_0 - ik\varphi \Rightarrow$ set $a_0 = 0$

Summary: $(\vec\nabla\psi)^2 = 1 \qquad \bar a\cdot\vec\nabla\psi = 0 \qquad a^0 = 0$

Now $\vec\nabla\psi$ is a unit vector $\vec\nabla\psi = \hat n$ which depends on $\vec x$, $\hat n = \hat n(\vec x)$;

Expanding $\psi(\vec x) = \psi(\vec x_0) + (\vec x - \vec x_0)\cdot\vec\nabla\psi(\vec x_0) + \cdots = \psi(\vec x_0) + \hat n\cdot(\vec x - \vec x_0) + \cdots = \alpha + \hat n\cdot(\vec x - \vec x_0) + \cdots$

we see that the wave in the region around $\vec x_0$ is




$$\vec A \simeq \vec a(\vec x)\, e^{ik(\hat n\cdot\vec x - x^0) + i\alpha}$$

$$A^0 = 0 \qquad \vec a(\vec x)\cdot\hat n(\vec x) = 0$$

$\vec E$ & $\vec B$:

We have $\vec E = -\dfrac{\partial\vec A}{\partial x^0} = ik\vec A = \vec E_0(\vec x)\, e^{ik(\psi - x^0)}$

and $\vec B = \vec\nabla\times\vec A = ik\,\hat n\times\vec A = \hat n\times\vec E$

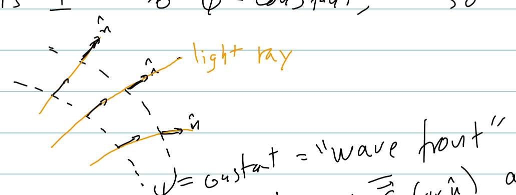
$$\bar u = \frac{1}{8\pi}\overline{(E^2 + B^2)} = \frac{1}{16\pi}(\vec E_0\cdot\vec E_0^* + \vec B_0\cdot\vec B_0^*) = \frac{1}{8\pi}|\vec E_0|^2$$

$$\overline{\vec S} = \frac{c}{8\pi}(\vec E_0\times\vec B_0^*) = \frac{c}{8\pi}\hat n|\vec E_0|^2 \quad \Rightarrow \quad \overline{\vec S} = c\,\bar u\,\hat n$$

or $\bar u = \frac{1}{c}\hat n\cdot\overline{\vec S}$

The interpretation of $\hat n = \vec\nabla\psi$ is clear: it gives the direction of energy flow.

Note that $\vec\nabla\psi$ is $\perp$ to $\psi = $ constant, so




Let light ray be $\vec x(\lambda)$ with $\lambda$ a parameter of the curve (much like $x^\mu(\lambda)$ for world-line).

Then $\dfrac{d\vec x}{d\lambda} \propto \hat n$ where $\hat n = \hat n(\vec x(\lambda))$. The proportionality is such that

$\hat n^2 = 1$: $\dfrac{d\vec x}{d\lambda} = \left|\dfrac{d\vec x}{d\lambda}\right|\hat n$. If $\lambda = s$, the lenght along the trajectory ($ds^2 = d\vec x\cdot d\vec x$)

then $\dfrac{d\vec x}{ds} = \hat n = \vec\nabla\psi$

But $\dfrac{d^2\vec x}{ds^2} = \dfrac{d}{ds}\vec\nabla\psi = \dfrac{dx^i}{ds}\partial_i(\vec\nabla\psi) = \hat n\cdot\vec\nabla\hat n$, but $\hat n\cdot\vec\nabla n^i = n^j\partial_j n^i = n^j\partial_j\partial_i\psi = n^j\partial_i n^j = \frac12\vec\nabla_i(\hat n^2) = \frac12\vec\nabla(1) = 0$

Acceleration $= 0 \Rightarrow$ constant velocity: $\dfrac{d\vec r}{ds} = $ constant $\Rightarrow$ $\vec r(s) = $ straight line

$\Longrightarrow$ geometrical optics.

In a medium replace $k^2(\vec\nabla\psi)^2 = k^2 = \frac{\omega^2}{c^2}$ by $k^2(\nabla\psi)^2 = \frac{\omega^2}{v^2} = n^2 k^2$ or $(\nabla\psi)^2 = n^2$. If medium is not homogeneous $n = n(\vec x)$ and now rays satisfy

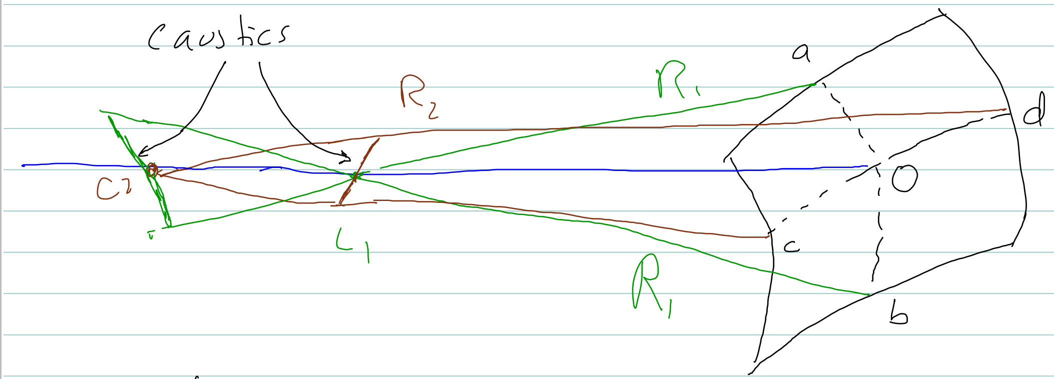
$$\frac{d\vec r}{ds} = \vec\nabla\psi \quad \text{with} \quad \left|\frac{d\vec r}{ds}\right|^2 = n^2 \qquad \text{(this "}n\text{" is index of refraction).}$$

Ref: L.L 54

Note that for $n(\vec x) =$ constant light rays are still straight. But they can curve for non-homogeneous ($n \neq$ constant) media.

Additional information is in $\vec a$ (or $\vec E_0$ & $\vec B_0$): light is also characterized by intensity. One can derive equations for $\vec a$ from next order in Eikonal expansion (so-called transport equations). We will settle for simple energy considerations (as done in most texts).

Consider wave-front $\psi =$ const. Take a sufficiently small portion that we can characterize it by two principal sections of curvature:




Centers of curvature $C_1$ & $C_2$
Radii: $R_1$ & $R_2$

With fixed angles from $C_1$ & $C_2$ we get as we vary $\psi$ a collection of surface segments with area $\propto R_1 R_2$, all having the same light rays crossing which is equivalent to having constant $\int \vec S \cdot \hat n \, ds$. So the collection of surfaces has light intensity $I R_1 R_2 =$ constant ($I =$ energy flux) or

$$I = \frac{\text{Constant}}{R_1 R_2}$$

As the surfaces approach $C_1$ ($C_2$) the radius $R_1$ ($R_2$) shrinks to zero and the surface degenerates into a burning line or "caustic". If $R_1 = R_2$ it's a single point, the "focal point".

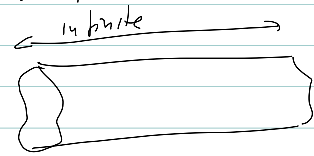
We'll come back to optics when we study diffraction.

Jackson: chap 8, secs 2-4
Schwartz: Chap 9

# Wave guides and cavities

There are many practical uses for waves in regions of space bounded by conductors. The prime examples are

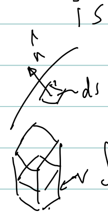
cylindrical wave guides:




symmetric under translations in $\longleftrightarrow$ direction (ie: constant cross section shape)

cylindrical cavities: as above with end caps).
("resonant" cavities)

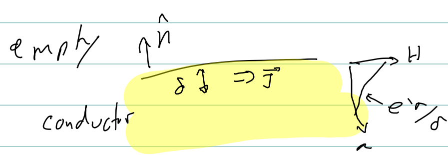
We will assume perfect conductors. For these, $\vec{E}=0$ in the interior. Any excess charge resides on the surface. If the surface charge density is $\sigma$ then



$\vec{\nabla}\cdot\vec{D}=4\pi\rho \implies \hat{n}\cdot\vec{D}=\sigma$

$\int_V \vec{\nabla}\cdot\vec{D} = \int_{\partial V} \hat{n}\cdot D\, ds = 4\pi Q$

In addition, one can show that harmonic ($\sim e^{i\omega t}$) magnetic fields $\vec{H}$ vanish inside the perfect conductor and for realistic conductors there is a "skin depth" over which the field exponentially dies




There is a current density $\vec{J}$ to the depth $\delta$ of the skin.

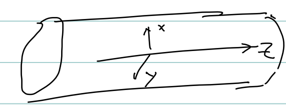
For perfect conductor $\delta\to 0$ and $\vec{J}$ becomes a surface current $\vec{K}$.

The b.c. is $\quad \hat{n}\times\vec{H}=\frac{4\pi}{c}\vec{K}$

The homogeneous eqs give, $\quad n\cdot(\vec{B}_{in}-\vec{B}_{out})=0 \quad \hat{n}\times(\vec{E}_{in}-\vec{E}_{out})=0$
always. We take $\vec{E}_{in}=0=\vec{B}_{in}$, and drop the label: $\vec{n}\cdot\vec{B}=0$ and $\hat{n}\times\vec{E}=0$.

Put $z$-axis along cylindrical axis. Look for solutions to source free Maxwell equations of the form

$$\vec{E}(\vec{x},t) = \vec{\mathcal{E}}(x,y)\,e^{i(kz-\omega t)}$$




$$\vec{B}(\vec{x},t) = \vec{\mathcal{B}}(x,y)\,e^{i(kz-\omega t)}$$

As before
$$\left(\nabla^2 - \frac{\mu\epsilon}{c^2}\frac{\partial^2}{\partial t^2}\right)\begin{pmatrix}\vec{E}\\ \vec{B}\end{pmatrix} = 0$$

which now implies
$$\left[\nabla_\perp^2 + \left(\frac{\mu\epsilon}{c^2}\omega^2 - k^2\right)\right]\begin{pmatrix}\vec{\mathcal{E}}\\ \vec{\mathcal{B}}\end{pmatrix} = 0 \qquad \text{where } \nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

One would guess that the solutions are plane waves with $\vec{\mathcal{E}}$ & $\vec{\mathcal{B}}$ perpendicular to $\hat{z}$ and $\vec{\mathcal{B}} \perp \vec{\mathcal{E}}$. But we will find solutions with non-vanishing $z$-component:

- Transverse electric modes (TE): $\mathcal{E}_z = 0$ but $\mathcal{B}_z \neq 0$
- Transverse magnetic modes (TM): $\mathcal{B}_z = 0$ but $\mathcal{E}_z \neq 0$
- Transverse electromagnetic mode (TEM): $\mathcal{E}_z = 0$ and $\mathcal{B}_z = 0$.

In preparation for this we separate $\vec{\mathcal{E}}_z$ from $\vec{\mathcal{E}}_\perp$
$$\vec{\mathcal{E}} = \vec{\mathcal{E}}_\parallel + \vec{\mathcal{E}}_\perp$$

where $\hat{z}\cdot\vec{\mathcal{E}}_\perp = 0$ $\quad \vec{\mathcal{E}}_\parallel = \hat{z}\,\hat{z}\cdot\vec{\mathcal{E}}_\parallel = \hat{z}\,\mathcal{E}_z$, $\quad \vec{\mathcal{E}}_\perp = \vec{\mathcal{E}} - \hat{z}\,\hat{z}\cdot\vec{\mathcal{E}} = (\hat{z}\times\vec{\mathcal{E}})\times\hat{z}$

and idem for $\vec{\mathcal{B}}$. Simplify discussion by assuming the cavity or wave guide are empty, so that $\mu\epsilon = 1$. The general $\mu\epsilon$ case is left as homework.

To separate $\perp$ from $\parallel$ modes in Faraday $\left( \vec{\nabla}\times\vec{E} - i\frac{\omega}{c}\vec{B} = 0 \right)$ take $\hat{z}\times(\text{eq})$ using

$$\hat{z}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}_\perp \mathcal{E}_z - ik\vec{\mathcal{E}}_\perp \qquad (\text{factor of } e^{i(kz-\omega t)} \text{ understood})$$

(Algebra, not for class: $\epsilon^{n3i}\left(\epsilon^{ijk}\partial_j E_k\right) = \left(\delta^{nj}\delta^{k3} - \delta^{nk}\delta^{j3}\right)\partial_j E_k = \vec{\nabla}(\hat{z}\cdot\vec{E}) - \frac{\partial}{\partial z}\vec{E}$

and $\vec{\nabla}E_z - \frac{\partial}{\partial z}\vec{E} = \left(\partial_x E_z - \partial_z E_x, \partial_y E_z - \partial_z E_y, 0\right) = \vec{\nabla}_\perp E_z - \partial_z\vec{E}_\perp$ ).

$$\Rightarrow \boxed{ik\vec{\mathcal{E}}_\perp + i\frac{\omega}{c}\hat{z}\times\vec{\mathcal{B}}_\perp = \vec{\nabla}_\perp\mathcal{E}_z}$$

Similarly from Ampere's law $\left( \vec{\nabla}\times\vec{B} + \underbrace{\mu\epsilon}_{=1} i\frac{\omega}{c}\vec{E} = 0 \right)$

using $(\vec{\nabla}\times\vec{B})_\perp = ik\,\hat{z}\times\vec{\mathcal{B}}_\perp - \hat{z}\times\vec{\nabla}_\perp\mathcal{B}_z$

[Algebra, not for class:

$$\epsilon^{ijn}\partial_j(\mathcal{B}_n e^{ikz}) = e^{ikz}\left[\epsilon^{ijn}\partial_j\mathcal{B}_n + ik\epsilon^{i3n}\mathcal{B}_n\right]$$

with $i=1,2$ and noting that $\mathcal{B}=\mathcal{B}(x,y)$ only, $\epsilon^{ijn}\partial_j\mathcal{B}_n = \epsilon^{ij3}\partial_j\mathcal{B}_z = \left[(\vec{\nabla}\mathcal{B}_z)\times\hat{z}\right]^i$

so $(\vec{\nabla}\times\vec{B})_\perp = -\hat{z}\times\vec{\nabla}\mathcal{B}_z + ik\,\hat{z}\times\vec{\mathcal{B}}_\perp$ ).

$$\Rightarrow \boxed{i\frac{\omega}{c}\vec{\mathcal{E}}_\perp + ik\,\hat{z}\times\vec{\mathcal{B}}_\perp = \hat{z}\times\vec{\nabla}_\perp\mathcal{B}_z}$$

These 2 equations give $\vec{\mathcal{E}}_\perp$ & $\hat{z}\times\vec{\mathcal{B}}_\perp$ in terms of $\vec{\nabla}_\perp\mathcal{E}_z$ & $\vec{\nabla}_\perp\mathcal{B}_z$.
Solve the simultaneous equations

$$M\begin{pmatrix}\vec{\mathcal{E}}_\perp \\ \hat{z}\times\vec{\mathcal{B}}_\perp\end{pmatrix} = \begin{pmatrix}\vec{\nabla}_\perp\mathcal{E}_z \\ \hat{z}\times\vec{\nabla}_\perp\mathcal{B}_z\end{pmatrix} \qquad \text{with } M = \begin{pmatrix} ik & i\frac{\omega}{c} \\ i\frac{\omega}{c} & ik \end{pmatrix}$$

$$\text{so} \quad \begin{pmatrix}\vec{\mathcal{E}}_\perp \\ \hat{z}\times\vec{\mathcal{B}}_\perp\end{pmatrix} = M^{-1}\begin{pmatrix}\vec{\nabla}_\perp\mathcal{E}_z \\ \hat{z}\times\vec{\nabla}_\perp\mathcal{B}_z\end{pmatrix} \qquad \text{where } M^{-1} = \frac{1}{\left(\frac{\omega}{c}\right)^2 - k^2}\begin{pmatrix} ik & -i\omega/c \\ -i\omega/c & ik\end{pmatrix}$$

(Of course $\vec{\mathcal{B}}_\perp$ is determined, $\vec{\mathcal{B}}_\perp = (\hat{z}\times\vec{\mathcal{B}}_\perp)\times\hat{z}$ ).

So we solve the wave equation for the z-component:

$$\left(\vec{\nabla}_\perp^2 + \gamma^2\right)\psi = 0 \qquad \text{where } \psi = \mathcal{E}_z \text{ or } \mathcal{B}_z \text{ and } \gamma^2 = \left(\frac{\omega}{c}\right)^2 - k^2$$

Boundary conditions: we want them in terms of $\mathcal{E}_z$ & $\mathcal{B}_z$ only.

$\hat{n}\times\vec{E}=0 \Rightarrow \mathcal{E}_z=0$

$\hat{n}\cdot\vec{B}=0$ : Take (Ampere's law) $\cdot$ $(\hat{n}\times\hat{z}) \Rightarrow \hat{n}\cdot\nabla_\perp \mathcal{B}_z = \frac{\partial \mathcal{B}_z}{\partial n}=0$

(Algebra: not in class

$(\hat{n}\times\hat{z})\cdot\left(i\frac{\omega}{c}\vec{\mathcal{E}}_\perp + ik\hat{z}\times\vec{\mathcal{B}}_\perp - \hat{z}\times\vec{\nabla}_\perp \mathcal{B}_z\right)=0$

$\vec{\mathcal{E}}_\perp\cdot\hat{n}\times\hat{z} = (\vec{\mathcal{E}}_\perp\times\hat{n})\cdot\hat{z}=0$ on boundary

$(\hat{n}\times\hat{z})\cdot(\hat{z}\times\vec{\mathcal{B}}_\perp) = -\epsilon^{ij3}\epsilon^{ik3}\hat{n}^j\mathcal{B}_\perp^k = \hat{n}\cdot\vec{\mathcal{B}}_\perp=0$ on boundary ).

So

<u>TE</u>: $\mathcal{E}_z=0$. Then solve $(\nabla_\perp^2+\gamma^2)\mathcal{B}_z=0$
and from this compute

$$\vec{\mathcal{E}}_\perp = -i\frac{\omega}{c\gamma^2}\hat{z}\times\vec{\nabla}_\perp\mathcal{B}_z \quad \text{and} \quad \vec{\mathcal{B}}_\perp = \frac{ik}{\gamma^2}\vec{\nabla}_\perp\mathcal{B}_z$$

<u>TM</u>: $\mathcal{B}_z=0$. Solve $(\vec{\nabla}_\perp^2+\gamma^2)\mathcal{E}_z=0$ and then

$$\vec{\mathcal{E}}_\perp = i\frac{k}{\gamma^2}\vec{\nabla}_\perp\mathcal{E}_z \quad \vec{\mathcal{B}}_\perp = i\frac{\omega}{c\gamma^2}\hat{z}\times\vec{\nabla}_\perp\mathcal{E}_z$$

<u>TEM</u>: $\mathcal{E}_z=\mathcal{B}_z=0$. Then

$$M\begin{pmatrix}\vec{\mathcal{E}}_\perp\\ \hat{z}\times\vec{\mathcal{B}}_\perp\end{pmatrix}=0$$ has solutions only if $\det M=0 \Leftrightarrow \frac{\omega^2}{c^2}=k^2$

Then, from Ampere's law (above) $\vec{\mathcal{B}}_\perp = (\operatorname{sgn} k)\,\hat{z}\times\vec{\mathcal{E}}_\perp$

Faraday $\Rightarrow \vec{\nabla}_\perp\times\vec{\mathcal{E}}_\perp=0$

(Algebra: $\vec{\nabla}\times(\vec{\mathcal{E}}_\perp e^{ikz}) = e^{ikz}\left[\vec{\nabla}_\perp\times\vec{\mathcal{E}}_\perp + ik\hat{z}\times\vec{\mathcal{E}}_\perp\right]$ and $ik\hat{z}\times\vec{\mathcal{E}}_\perp - i\frac{\omega}{c}\vec{\mathcal{B}}_\perp=0$ ).

Gauss: $\vec{\nabla}_\perp\cdot\vec{\mathcal{E}}_\perp=0$

So $\vec{\nabla}_\perp\cdot\vec{\mathcal{E}}_\perp=0$, $\vec{\nabla}_\perp\times\vec{\mathcal{E}}_\perp=0$ and $\vec{\mathcal{E}}_\perp\times\hat{n}\Big|_{\text{boundary}}=0$ is a problem in 2-dimensional electrostatics.

Notes:

(1) For TM & TE modes one must have $\gamma^2 > 0$ to satisfy b.c.'s

Exercise: show this!

Now $(\nabla_\perp^2 + \gamma^2)\psi = 0$ with $\psi\big|_{\partial V} = 0$ or $\frac{\partial\psi}{\partial n}\Big|_{\partial V} = 0$ is an eigenvalue problem: expect discrete solutions $\psi_n(x,y)$ for $\gamma^2 = \gamma_n^2$, $n = 1, 2, \ldots$

This means $\omega$ and $k$ are related by

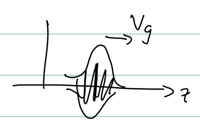
$$\left(\frac{\omega}{c}\right)^2 - k^2 = \gamma_n^2 \qquad \text{or} \qquad \omega = c\sqrt{k^2 + \gamma_n^2}$$

Cut-off frequency: If $\gamma_1 < \gamma_2 < \gamma_3 < \cdots$ (ie, we organize $\gamma_n$'s by their magnitude) then there is a minimum possible frequency for transmission, $\omega_{min} = c\gamma_1$

For $\omega \in [c\gamma_1, c\gamma_2]$ the only propagating mode is $\psi_1$. Similarly, for $c\gamma_2 < \omega < c\gamma_3$ one can have $\psi(x,y) = c_1\psi_1(x,y) + c_2\psi_2(x,y)$ propagating in the wave guide.

Since $E, B \sim e^{ik(x - \frac{\omega}{k}t)}$, the phase velocity is $v_p = \frac{\omega}{k} = c\frac{\sqrt{k^2 + \gamma_n^2}}{k}$ for mode $\psi_n$. So $v_p > c$ and diverges as $k \to 0$ $(\lambda \to \infty)$.

The group velocity is defined [for $\omega = \omega(k)$ a slowly varying function] to give the velocity of the peak of a wave-packet, which is where the energy density is localized:



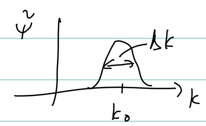

(ignoring $x, y$ dependence) $\psi(z,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\tilde{\psi}(k)\, e^{i(kz - \omega(k)t)}$

$\tilde{\psi}$ can be determined from snapshot at fixed time, say $t = 0$:

$$\psi(z,0) = \int_{-\infty}^{\infty} \frac{dk}{(2\pi)} e^{ikz}\tilde{\psi}(k) \Rightarrow \tilde{\psi}(k) = \int_{-\infty}^{\infty} dz\, e^{-ikz}\psi(z,0)$$

For $\omega =$ constant $\psi(z,t) = \psi(z - \frac{\omega}{k}t, 0)$ so $\frac{\omega}{k} =$ constant has $v_p = v_g$.

If $\tilde{\psi}(k)$ is localized around $k_0$



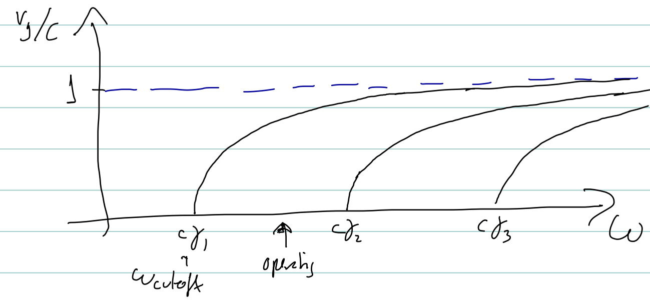

so $\Delta z \sim \frac{1}{\Delta k}$ is the width of the wave-packet, then

$$\omega(k) \simeq \omega(k_0) + (k - k_0)\frac{d\omega}{dk}\Big|_{k_0} + \cdots \quad \text{and} \quad \psi(z,t) = \int \frac{dk}{2\pi}\tilde{\psi}(k)\, e^{i\left[kz - \left(\omega(k_0) + (k-k_0)\frac{d\omega}{dk}\Big|_{k_0}\right)t\right]}$$

$$\text{so } \psi(z,t) \simeq e^{-i\left(\omega_0 - k_0\frac{d\omega}{dk}\Big|_0\right)t}\,\psi\left(z - \frac{d\omega}{dk}\Big|_{k_0}t, 0\right) \Rightarrow v_g = \frac{d\omega}{dk}\Big|_{k_0}$$

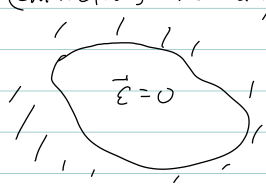
For wave-guides $v_g = \frac{d\omega}{dk} = c\frac{k}{\sqrt{k^2 + \gamma_n^2}} < c$ $\quad \left(\text{or } \frac{v_g}{c} = \frac{\sqrt{\omega^2 - c^2\gamma_n^2}}{\omega}\right)$

Possible group velocities in wave-guide

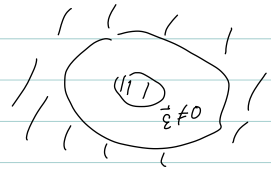



Often design is such that operating frequency $\omega$ is between $\omega_{1,cutoff}$ and $\omega_{2,cutoff}$.

(ii) For TEM there are no solutions other than trivial ($\vec{E}=0$) for a single (connected) boundary

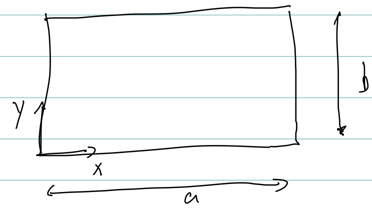



One can have non-trivial solutions with $\geq 2$ disconnected boundaries




This is why co-axial cables or two-wire transmission lines are used in practice.

Example: rectangular cavity




TM: $B_z = 0$, $E_z\big|_{\partial V} = 0$ $\quad \left(\nabla_\perp^2 + \gamma^2\right) E_z(x,y) = 0$

Separation of variables: $E = E_0 X(x) Y(y)$

$$-\gamma^2 = \frac{1}{E}\nabla_\perp^2 E = \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} \qquad \text{with } X(0) = X(a) = 0 \quad Y(0) = Y(b) = 0$$

$$\Rightarrow X(x) = \sin\left(\frac{m\pi}{a}x\right) \quad Y = \sin\left(\frac{n\pi}{b}y\right) \quad \Rightarrow E_z(x,y) = E_{0z}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

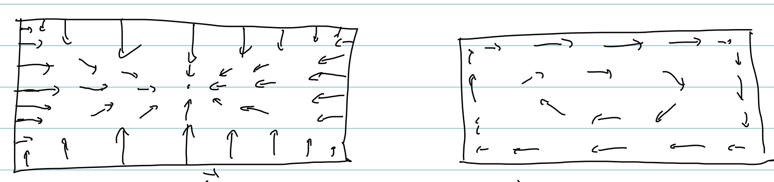
$$\gamma_{mn}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \quad m,n \in \mathbb{Z} \quad mn \neq 0.$$

The transverse fields are

$$\vec{E}_\perp = i\frac{k}{\gamma^2}\vec{\nabla}_\perp E_z = i\frac{k}{\gamma_{mn}^2}E_{0z}\left(\frac{m\pi}{a}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right), \frac{n\pi}{b}\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right), 0\right)$$

$$\vec{B}_\perp = \frac{\omega}{ck}\hat{z}\times\vec{E}_\perp = \frac{i\omega}{c\gamma_{mn}^2}E_{0z}\left(-\frac{n\pi}{b}\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right), \frac{m\pi}{a}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right), 0\right)$$

$TM_{11}$



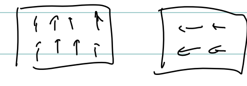
$\vec{E}_\perp$ $\qquad$ $\vec{B}_\perp$

TE: $E_z = 0$ $\quad \frac{\partial B_z}{\partial n}\Big|_{\partial V} = 0$ $\quad \left(\nabla_\perp^2 + \gamma^2\right) B_z(x,y) = 0$

Same but with $\frac{dX}{dx}\Big|_{0,a} = 0$ $\quad \frac{dY}{dy}\Big|_{0,b} = 0$ $\Rightarrow$ cosines

$$B_z(x,y) = B_{z0}\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right) \qquad \gamma_{mn}^2 \text{ as above and } m,n \in \mathbb{Z} \text{ except } m = n = 0.$$

Exercise: compute $\vec{E}_\perp, \vec{B}_\perp$, and plot $TE_{10}$



Circular cross section: need Bessel functions → problem session.

And in notes in "Appendix: Bessel functions".

Resonant cavities.

The discussion is as above, but in addition we have boundary conditions at $z=0,L$ (assuming both ends are capped, and the capping surface is plane).

We now have standing waves so $e^{ikz} \to \cos(kz)$ or $\sin(kz)$

TE: This had $B_z \neq 0$, $E_z = 0$, but now $B_z = 0$ at $z=0,L$ (recall $\vec{B}\cdot\hat{n}=0$).

$$B_z(\vec{x}) = \psi(x,y)\sin\left(\frac{\pi p}{L}z\right) \qquad p \in \mathbb{Z}_+$$

TM: We need $\vec{E}_\perp = 0$ at $z=0,L$, but $\vec{E}_\perp = \frac{1}{\gamma^2}\vec{\nabla}_\perp \frac{\partial E_z}{\partial z} \Rightarrow \left.\frac{\partial E_z}{\partial z}\right|_{0,L} = 0$

$$E_z(\vec{x}) = \psi(x,y)\cos\left(\frac{\pi p}{L}z\right) \qquad p = 0,1,\ldots$$

Note that now $\frac{\omega^2}{c^2} = k^2 + \gamma^2 = \left(\frac{\pi p}{L}\right)^2 + \gamma^2$ completely discretized

For example, box $a \times b \times L$ has

$$\omega_{mnp} = c\sqrt{\left(\frac{\pi p}{L}\right)^2 + \gamma_{mn}^2} = c\sqrt{\left(\frac{\pi p}{L}\right)^2 + \left(\frac{\pi m}{a}\right)^2 + \left(\frac{\pi n}{b}\right)^2}$$

## Appendix: Bessel functions

We may define them through the Laurent series expansion of the "generating function"

$$G(t,z) \equiv e^{\frac{1}{2}z(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} t^n J_n(z) \qquad (A)$$

You may have seen generating functions in the context of Legendre polynomials $P_n(z)$:

$$\frac{1}{\sqrt{1+t^2-2zt}} = \sum_{n=0}^{\infty} t^n P_n(z)$$

(but here the expansion is a Taylor series).

One may recast (A) using $t = ie^{i\varphi}$, so that $t-\frac{1}{t} = i(e^{i\phi}+e^{-i\phi}) = 2i\cos\phi$ and

$$e^{iz\cos\phi} = \sum_{n=-\infty}^{\infty} i^n e^{in\phi} J_n(z) \qquad (B)$$

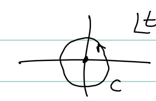
From the generating function we derive properties of $J_n$'s.

- Parity: $e^{iz\cos(\phi+\pi)} = e^{-iz\cos\phi} \Rightarrow \sum_n i^n e^{in\pi} e^{in\phi} J_n(z) = \sum_n i^n e^{in\phi} J_n(-z) \Rightarrow J_n(-z) = (-1)^n J_n(z)$

  $e^{iz\cos(-\varphi)} = e^{iz\cos\varphi} \Rightarrow J_{-n}(z) = (-1)^n J_n(z)$

- Integral representation

$$J_n(z) = \frac{1}{2\pi i}\oint_C e^{\frac{1}{2}z(t-\frac{1}{t})}\frac{dt}{t^{n+1}}$$




Taking the $\phi$ map and a circle

$$J_n(z) = \frac{(-i)^n}{2\pi}\int_0^{2\pi} e^{i(z\cos\phi - n\phi)}\,d\phi$$

which you can stick in a computer to get a numerical value.

- Small $z$ expansion:

$$J_n(z) = \frac{(-i)^n}{2\pi}\int_0^{2\pi} d\phi\, e^{-in\phi}\sum_{k=0}^{\infty}\frac{i^k}{k!}z^k\left(\frac{e^{i\phi}+e^{-i\phi}}{2}\right)^k$$

The 1st nonvanishing term has $|k| = n$, so

$$J_n(z) = \frac{(-i)^n}{2\pi}\cdot\left[2\pi\frac{i^n}{n!}z^n\frac{1}{2^n} + O(z^{n+1})\right. = \frac{1}{2^n n!}z^n + \dots$$

Exercise: get the whole series!

In particular $J_0(0) = 1$, $J_{n\neq 0}(0) = 0$.

· Large $|z|$. Use method of stationary phase.

Conditions $\frac{d}{d\phi}\left(z\cos\phi\right) = z\sin\phi = 0 \quad \Rightarrow \phi = 0$ or $\pi$.

Expansion: $\cos\phi \simeq 1-\frac{1}{2}\phi^2$ and $\cos\phi = -1+\frac{1}{2}(\phi-\pi)^2$

$$J_n(z) \simeq \frac{(-i)^n}{2\pi}\left[\int_{-\infty}^{\infty} d\phi\; e^{iz(1-\frac{1}{2}\phi^2)}\, e^{-in\phi} + \int_{-\infty}^{\infty} d\phi\; e^{-iz(1-\frac{1}{2}\phi^2)}\, e^{-in(\phi+\pi)}\right]$$

Use $\int_{-\infty}^{\infty} dx\; e^{i(ax^2+bx)} = \int_{-\infty}^{\infty} dx\; e^{ia(x+\frac{1}{2}b/a)^2 - ib^2/4a} = \frac{e^{-ib^2/4a}}{\sqrt{-ia}}\int_{-\infty}^{\infty} dx\, e^{-x^2} = \sqrt{\frac{\pi}{-ia}}\, e^{-ib^2/4a}$

So that

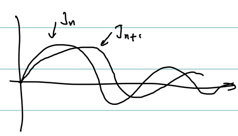
$$J_n(z) \simeq \frac{(-i)^n}{2\pi}\left[\sqrt{\frac{\pi}{iz/2}}\; e^{-in^2/4(-z/2)}\; e^{iz} + (-1)^n\sqrt{\frac{\pi}{-iz/2}}\; e^{-in^2/4(z/2)}\; e^{-iz}\right]$$

$$= \frac{1}{2\pi}\, 2\mathrm{Re}\left[\sqrt{\frac{2\pi}{iz}}\; e^{i(z+\frac{n^2}{2z}-\frac{n\pi}{2})}\right]$$

or

$$J_n(z) \simeq \sqrt{\frac{2}{\pi z}}\cos\left(z-\frac{n\pi}{2}-\frac{\pi}{4}\right)\left(1+O\left(\tfrac{1}{z}\right)\right)$$

So roughly




(and $J_0$ starting from $J_0(0)=1$)

Most importantly, $J_n(z)$ satisfies

$$\left(\frac{d^2}{dz^2}+\frac{1}{z}\frac{d}{dz}+1-\frac{n^2}{z^2}\right)J_n(z)=0 \qquad \text{(C)} \qquad \text{"Bessel equation"}$$

Recall in polar coordinates $ds^2 = d\rho^2+\rho^2 d\theta^2$, $\nabla^2 = \frac{1}{\sqrt{g}}\partial_i\sqrt{g}g^{ij}\partial_j = \frac{1}{\rho}\partial_\rho\rho\partial_\rho + \frac{1}{\rho}\partial_\theta\rho\cdot\frac{1}{\rho^2}\partial_\theta = \partial_\rho^2+\frac{1}{\rho}\partial_\rho+\frac{1}{\rho^2}\partial_\theta^2$

So $J_n$ appears naturally in solutions to $\nabla^2\Psi=\lambda\Psi$ in polar (or cylindrical in 3D) coordinates.

To show (C), take derivatives of $G(t,z)$ in (A)

$$\frac{\partial G}{\partial z} = \frac{1}{2}\left(t-\frac{1}{t}\right)G \qquad \frac{\partial^2 G}{\partial z^2} = \left[\frac{1}{2}\left(t-\frac{1}{t}\right)\right]^2 G$$

$$t\frac{\partial G}{\partial t} = \frac{z}{2}\left(t+\frac{1}{t}\right)G \qquad \left(t\frac{\partial}{\partial t}\right)^2 G = \left\{\left[\frac{z}{2}\left(t+\frac{1}{t}\right)\right]^2 + \frac{z}{2}\left[t-\frac{1}{t}\right]\right\}G$$

So $\left(t\frac{\partial}{\partial t}\right)^2 G - z^2\frac{\partial^2}{\partial z^2}G = z^2 G + z\frac{\partial G}{\partial z}$

Now use the expansion of $G$ in $J_n$'s in (A) with $\left(t\frac{\partial}{\partial t}\right)^2 t^n = n^2 t^n$. This proves (C).

From Gradshteyn & Ryzhik: recursion formulas

$$n J_{n-1}(z) + z J_{n+1}(z) = 2n J_n(z)$$

$$J_{n-1}(z) - J_{n+1}(z) = 2\frac{d}{dz} J_n(z)$$

Exercise: show these.

Also from G&R, as consequence of the above

$$z\frac{d}{dz}J_n + n J_n = z J_{n-1}$$

$$z\frac{d}{dz}J_n - n J_n = -z J_{n+1}$$

$$\left(\frac{1}{z}\frac{d}{dz}\right)^m \left(z^n J_n(z)\right) = z^{n-m} J_{n-m}(z)$$

$$\left(\frac{1}{z}\frac{d}{dz}\right)^m \left(z^{-n} J_n(z)\right) = (-1)^m z^{-n-m} J_{n+m}(z)$$

From the last, with $n=0$, $m=1$ $\quad \frac{1}{z}\frac{d}{dz}J_0(z) = (-1) z^{-1} J_1(z) \Rightarrow J_1(z) = -J_0'(z)$.

- Independent solutions: Bessel equation, being second order and linear, admits two independent solutions. $J_n(z)$ is one. What about the other?

Consider the equation for non-integer $n$. One can find a solution in the form of a power expansion. Let's use $\nu$ for the general case

$$\left(\frac{d^2}{dz^2} + \frac{1}{z}\frac{d}{dz} + 1 - \frac{\nu^2}{z^2}\right) J_\nu = 0 \qquad (C')$$

so $\nu = n$, an integer, is the case studied so far. Assuming $J_\nu$ is regular near $z=0$ we write $J_\nu \sim z^p + \cdots$ close to $z=0$ then

$$p(p-1) + p - \nu^2 = 0 \quad \Rightarrow \quad p = \pm\nu$$

and for a regular solution $p = +\nu$ (assuming $\nu>0$). The regular solution so obtained coincides with $J_n(z)$ when $\nu = n$ an integer.

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\nu+1)} \left(\frac{z}{2}\right)^{2m}$$

Taking $\nu \to -\nu$ when $\nu \neq$ integer gives a second, <u>independent</u> solution.

But for $\nu = n$, $J_{-n}(z) = (-1)^n J_n(z)$ as we have seen, and it is <u>not</u> independent

A $2^{nd}$ solution for $\nu$ = integer: the Neumann function, $N_\nu(z)$, is often used as the $2^{nd}$ independent solution (rather than $J_{-\nu}(z)$).

$$N_\nu(z) \equiv \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}$$

Since this solves (C′) and has a limit as $\nu \to n$ (integer), this can be used as the $2^{nd}$ solution so that the general solution is

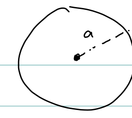
$$\psi(z) = a\, J_n(z) + b\, N_n(z) \qquad a, b = \text{constants}$$

Notice that $N_n(z)$ is not regular at the origin. For $n \geq 1$ clearly $N_n \sim z^{-n}$. For $n=0$, take $\nu = \epsilon$ and expand

$$N_0(z) \sim \frac{z^\epsilon \cos(\epsilon\pi) - z^{-\epsilon}}{\sin(\epsilon\pi)} = \frac{(1+\epsilon \ln z) - (1-\epsilon \ln z)}{\epsilon\pi} = \frac{2}{\pi}\ln z$$

For our purposes (presently) we can ignore $N_n(z)$: we are looking to solve $(\nabla^2 + \gamma^2)\psi = 0$ inside a circle that includes the origin.

Wave guide/cavity with circular cross section




We want to solve

$$(\nabla_\perp^2 + \gamma^2)\psi = 0$$

Use polar coordinates $\rho, \theta$ and try separation of variables:

$$\psi(\rho,\theta) = R(\rho)T(\theta)$$

$$\frac{1}{\psi}\nabla_\perp^2\psi = -\gamma^2 \Rightarrow \rho^2\frac{1}{R}\left(\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}\right)R + \frac{1}{T}\frac{d^2T}{d\theta^2} = -\gamma^2\rho^2$$

Since $T(\theta+2\pi)=T(\theta)$ we take $T(\theta) = e^{in\theta}$ with $n$ an integer $\Rightarrow$

$$\left(\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho} - \frac{n^2}{\rho^2}+\gamma^2\right)R = 0$$

Let $z = \gamma\rho$, so that as function of $z$, $R(z)$ satisfies Bessel's equation : $R(z) = J_n(z) = J_n(\gamma\rho)$.

Boundary conditions: $R(a)=0$ (for TM mode, $E_z| = 0$) or $\left.\frac{dR}{d\rho}\right|_a = 0$ (for TE modes, $\left.\frac{\partial B_z}{\partial n}\right|_{\partial V} = 0$).

(i) $R(a) = 0 \Rightarrow J_n(\gamma a) = 0$. We saw that $J_n$ has an infinite number of positive zeroes. Let $0 \le z_{n1} \le z_{n2} \le z_{n3}$ be the zeroes $J_n(z_{nm}) = 0$. The condition $J_n(\gamma a)$ gives

$$\gamma_{nm} a = z_{nm} \Rightarrow \gamma_{nm} = z_{nm}/a$$

The most general solution is

$$E_z = \sum_{n,m} c_{nm} e^{in\theta} J_n(\gamma_{nm}\rho) \qquad \left(\text{with } \left(\frac{\omega}{c}\right)^2 = k^2 + \gamma_{nm}^2 = k^2 + \frac{z_{nm}^2}{a^2}\right)$$

The zeroes of first few $J_n$'s are tabulated. For example, $z_{01} = 2.4 \quad z_{11} = 3.8 \quad z_{02} = 5.5$

(ii) $R'(a) = 0 = J_n'(\gamma a) = 0$. Since $J_n$ is continuous and has $\infty$ number of zeros, it's derivative must also have an infinite number of zeros. We already know $J_0'(z) = -J_1(z)$ so the $\gamma^2$'s of the $n=0$ TE modes are degenerate with those of the $n=1$ TM modes.

Let $0 \le z'_{n1} \le z'_{n2} \le z'_{n3} \le \cdots$ be $J_n'(z'_{nm}) = 0$, then

$$B_z = \sum_{n,m} c_{nm} e^{in\theta} J_n(\gamma_{nm}\rho) \quad \text{with } \gamma_{nm} = \frac{z'_{nm}}{a}$$

And, eg, $z'_{11} = 1.8 \quad z'_{21} = 3.1 \quad z'_{01} = z_{11} = 3.8$

See table attached.

For future reference

Inverting the series.

In many cases $\psi(\rho,\theta)$ is fixed (boundary value problems)

Then one needs to use this information to determine the coefficients of the expansion.

The idea is this: use completeness of the eigenfunctions,

i.e. $\langle n|m\rangle = \delta_{nm}$ then $\sum_n |n\rangle\langle n| = \mathbb{1}$

or $$\int_0^a \rho\, d\rho \int_0^{2\pi} d\theta \left(J_n(z_{nm}\tfrac{\rho}{a})\, e^{in\theta}\right)^* \left(J_{n'}(z_{n'm'}\tfrac{\rho}{a})\, e^{in'\theta}\right) = N_{nm}\, \delta_{nn'}\, \delta_{mm'}$$

where the normalization factor is $N_{nm} = \int_0^a \rho\, d\rho \int_0^{2\pi} d\theta\, \left|J_n(z_{nm}\tfrac{\rho}{a}) e^{in\theta}\right|^2$

or $$N_{nm} = 2\pi a^2 \int_0^1 dx \left[x J_n^2(z_{nm}x)\right]$$

ASIDE (Note that it matters but this integral can be done: take Bessel $\times$ $J' = \frac{dJ_n}{dz}$ (drop n's for now):

$$J'J'' + \frac{1}{z}(J')^2 + \left(1-\frac{n^2}{z^2}\right) J J' = 0$$

Now $J'J'' = \frac{1}{2}(J'^2)'$ so $\frac{1}{2z^2}\frac{d}{dz}\left(z^2 J'^2\right) = \frac{1}{2}(J'^2)' + \frac{1}{z}(J')^2$, i.e., the first two terms. Multiply by $2z^2$:

$$\frac{d}{dz}\left(z^2 J'^2\right) + (z^2-n^2)(J^2)' = 0$$

Now rewrite $(z^2-n^2)(J^2)' = \frac{d}{dz}\left[(z^2-n^2)J^2\right] - 2zJ^2$ and integrate

$$\int_0^{z_{nm}} dz\, zJ^2 = \frac{1}{2}\left[z^2 J'^2 + (z^2-n^2)J^2\right]_0^{z_{nm}}$$

Changing variables $z = x z_{nm}$, $\int_0^1 dx\, x J_n^2(x z_{nm}) = \frac{1}{2}\left[x^2 J'^2 + \left(x^2 - \frac{n^2}{z_{nm}^2}\right)J^2\right]\Big|_0^1 = \frac{1}{2}\left[J_n'(z_{nm})\right]^2$

And from $z\frac{dJ_n}{dz} - nJ_n = -zJ_{n+1}$ one can rewrite this as $\frac{1}{2}\left[J_{n+1}(z_{nm})\right]^2$ ).

So our orthonormal set is $|nm\rangle = \dfrac{e^{in\theta} J_n(z_{nm}\frac{\rho}{a})}{\sqrt{N_{nm}}}$

If $|\psi\rangle = \sum c_{nm} |nm\rangle \Rightarrow c_{nm} = \langle nm|\psi\rangle$ or

$$c_{nm} = \int_0^a \rho\, d\rho \int_0^{2\pi} d\theta\, \frac{e^{-in\theta} J_n(z_{nm}\rho/a)}{\sqrt{N_{nm}}}\, \psi(\rho,\theta)$$

# Table of First 700 Zeros of Bessel Functions — $J_l(x)$ and $J'_l(x)$

By CURTIS L. BEATTIE

(Manuscript received August 20, 1957)

*The zeros of the Bessel functions and Bessel function deriviatives are identified by standard waveguide notation which also serves as a code for more general mathematical applications.*

The possibilities of low-loss transmission using the $TE_{01}$ (circular electric) mode in circular cylindrical pipe of a diameter large compared to the wavelength has made the study of other modes of such a waveguide important. In order to find phase and attenuation constants of various modes for both solid and ring or helix walls, the zeros of the Bessel functions $J_l(x)$ and $J'_l(x)$ are essential.

In the table given here the first seven hundred roots of Bessel functions $J_l(x) = 0$ and $J'_l(x) = 0$ have been computed and arranged in the order of the magnitude of the arguments corresponding to the roots. In the table $l$ is the order of the Bessel function and $m$ is the serial number of the zero of either $J_l(x)$ or $J'_l(x)$, not counting $x = 0$. In waveguide applications the zeros of $J_l(x)$ correspond to transverse magnetic modes of propagation (TM modes) and those of $J'_l(x)$ to transverse electric modes (TE modes). The designations TM and TE appear in the table for the benefit of those who will use this table in waveguide research and serve as a code designating $J_l(x)$ and $J'_l(x)$ for those who are interested in a more general application of the mathematics.

The roots of the Bessel functions were calculated from the *Tables of the Bessel Functions of the First Kind of Orders, $J_0$ through $J_{51}$*, computed by the Staff of the Computation Laboratory of Harvard University, published by the Harvard University Press, 1946–1948.

This table was first formulated horizontally in the ascending order of the function and vertically in the ascending number of the root. Since the increments in each direction are of a predictable magnitude, the possibility of having neglected a root is virtually eliminated.

All 700 roots were calculated by means of a linear interpolation formula but checked and corrected for six-place accuracy for the first 300 roots. For arguments above 25, where the tabulated values were for each 0.01 only, the following Newton-Bessel formula was used:

$$J_n(x) = J_n(x_0 + h\mu) = J_n(x_0) + \mu B + \frac{\mu(\mu - 1)(C - A)}{4} + \cdots$$

where

$$\begin{aligned} \mu &= (x - x_0)/h \\ A &= J_n(x_0 + h) + J_n(x_0) \\ B &= J_n(x_0 + h) - J_n(x_0) \\ C &= J_n(x_0 + 2h) + J_n(x_0 - h). \end{aligned}$$

Spot checking was done with the first three terms of the Taylor's series expansion:

$$0 \cong J_n(x) + \frac{h}{2}[J_{n-1}(x) - J_{n+1}(x)] + \frac{h^2}{8}[J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)] + \cdots.$$

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2. Gray, A. and Mathews, G. B., *Treatise on Bessel Functions*, Macmillan, London, 1895. Table III. The first 40 roots of $J_0(x) = 0$, with the corresponding values of $J_1(x)$. (Ten decimal places.) Table IV. The first 50 roots of $J_1(x) = 0$, with the corresponding maximum or minimum values of $J_0(x)$. (Sixteen decimal places.)
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TABLE

| | Mode* $l$–$m$ | Value† | | Mode* $l$–$m$ | Value† |
|---|---|---|---|---|---|
| 1 | TE 1–1 | 1.841184 | (48 | TM 1–4 | 13.323692 |
| 2 | TM 0–1 | 2.404826 | (49 | TE 0–4 | 13.323692 |
| 3 | TE 2–1 | 3.054237 | 50 | TM 9–1 | 13.354300 |
| (4 | TM 1–1 | 3.831706 | 51 | TM 6–2 | 13.589290 |
| (5 | TE 0–1 | 3.831706 | 52 | TE 12–1 | 13.878843 |
| 6 | TE 3–1 | 4.201189 | 53 | TE 5–3 | 13.987189 |
| 7 | TM 2–1 | 5.135622 | 54 | TE 8–2 | 14.115519 |
| 8 | TE 4–1 | 5.317553 | 55 | TM 4–3 | 14.372537 |
| 9 | TE 1–2 | 5.331443 | 56 | TM 10–1 | 14.475501 |
| 10 | TM 0–2 | 5.520078 | 57 | TE 3–4 | 14.585848 |
| 11 | TM 3–1 | 6.380162 | 58 | TM 2–4 | 14.795952 |
| 12 | TE 5–1 | 6.415616 | 59 | TM 7–2 | 14.821269 |
| 13 | TE 2–2 | 6.706133 | 60 | TE 1–5 | 14.863589 |
| (14 | TM 1–2 | 7.015587 | 61 | TE 13–1 | 14.928374 |
| (15 | TE 0–2 | 7.015587 | 62 | TM 0–5 | 14.930918 |
| 16 | TE 6–1 | 7.501266 | 63 | TE 6–3 | 15.268181 |
| 17 | TM 4–1 | 7.588342 | 64 | TE 9–2 | 15.286738 |
| 18 | TE 3–2 | 8.015237 | 65 | TM 11–1 | 15.589848 |
| 19 | TM 2–2 | 8.417244 | 66 | TM 5–3 | 15.700174 |
| 20 | TE 1–3 | 8.536316 | 67 | TE 4–4 | 15.964107 |
| 21 | TE 7–1 | 8.577836 | 68 | TE 14–1 | 15.975439 |
| 22 | TM 0–3 | 8.653728 | 69 | TM 8–2 | 16.037774 |
| 23 | TM 5–1 | 8.771484 | 70 | TM 3–4 | 16.223466 |
| 24 | TE 4–2 | 9.282396 | 71 | TE 2–5 | 16.347522 |
| 25 | TE 8–1 | 9.647422 | 72 | TE 10–2 | 16.447853 |
| 26 | TM 3–2 | 9.761023 | (73 | TM 1–5 | 16.470630 |
| 27 | TM 6–1 | 9.936110 | (74 | TE 0–5 | 16.470630 |
| 28 | TE 2–3 | 9.969468 | 75 | TE 7–3 | 16.529366 |
| (29 | TM 1–3 | 10.173468 | 76 | TM 12–1 | 16.698250 |
| (30 | TE 0–3 | 10.173468 | 77 | TM 6–3 | 17.003820 |
| 31 | TE 5–2 | 10.519861 | 78 | TE 15–1 | 17.020323 |
| 32 | TE 9–1 | 10.711434 | 79 | TM 9–2 | 17.241220 |
| 33 | TM 4–2 | 11.064709 | 80 | TE 5–4 | 17.312842 |
| 34 | TM 7–1 | 11.086370 | 81 | TE 11–2 | 17.600267 |
| 35 | TE 3–3 | 11.345924 | 82 | TM 4–4 | 17.615966 |
| 36 | TM 2–3 | 11.619841 | 83 | TE 8–3 | 17.774012 |
| 37 | TE 1–4 | 11.70600$\underline{5}$ | 84 | TE 3–5 | 17.788748 |
| 38 | TE 6–2 | 11.73493$\bar{6}$ | 85 | TM 13–1 | 17.80143$\bar{5}$ |
| 39 | TE 10–1 | 11.770877 | 86 | TM 2–5 | 17.959819 |
| 40 | TM 0–4 | 11.791534 | 87 | TE 1–6 | 18.015528 |
| 41 | TM 8–1 | 12.225092 | 88 | TE 16–1 | 18.063261 |
| 42 | TM 5–2 | 12.338604 | 89 | TM 0–6 | 18.071064 |
| 43 | TE 4–3 | 12.681908 | 90 | TM 7–3 | 18.287583 |
| 44 | TE 11–1 | 12.826491 | 91 | TM 10–2 | 18.433464 |
| 45 | TE 7–2 | 12.932386 | 92 | TE 6–4 | 18.637443 |
| 46 | TM 3–3 | 13.015201 | 93 | TE 12–2 | 18.745091 |
| 47 | TE 2–4 | 13.170371 | 94 | TM 14–1 | 18.899998 |

* TM designates a zero of $J_l(x)$; TE designates a zero of $J'_l(x)$; in each case $l$ corresponds to the order of the Bessel function and $m$ is the number of the root.

† $\bar{5}$ in last place indicates higher value and $\underline{5}$ indicates lower value in rounding off for fewer decimal places.

TABLE — *Continued*

| | Mode* $l$–$m$ | Value† | | Mode* $l$–$m$ | Value† |
|---|---|---|---|---|---|
| 95 | TM 5–4 | 18.980134 | 150 | TM 4–6 | 24.019020 |
| 96 | TE 9–3 | 19.004594 | 151 | TE 3–7 | 24.144897 |
| 97 | TE 17–1 | 19.104458 | 152 | TM 9–4 | 24.23388$\bar{5}$ |
| 98 | TE 4–5 | 19.196029 | 153 | TM 15–2 | 24.269180 |
| 99 | TM 3–5 | 19.40941$\bar{5}$ | 154 | TM 2–7 | 24.270112 |
| 100 | TE 2–6 | 19.512913 | 155 | TE 22–1 | 24.28938$\underline{5}$ |
| 101 | TM 8–3 | 19.554536 | 156 | TE 1–8 | 24.311327 |
| (102 | TM 1–6 | 19.615859 | 157 | TM 19–1 | 24.338250 |
| (103 | TE 0–6 | 19.615859 | 158 | TM 0–8 | 24.352472 |
| 104 | TM 11–2 | 19.615967 | 159 | TE 17–2 | 24.381913 |
| 105 | TE 13–2 | 19.883224 | 160 | TM 12–3 | 24.49488$\bar{5}$ |
| 106 | TE 7–4 | 19.941853 | 161 | TE 8–5 | 24.587197 |
| 107 | TM 15–1 | 19.994431 | 162 | TM 7–5 | 24.934928 |
| 108 | TE 18–1 | 20.144079 | 163 | TE 14–3 | 25.001972 |
| 109 | TE 10–3 | 20.223031 | 164 | TE 11–4 | 25.008519 |
| 110 | TM 6–4 | 20.320789 | 165 | TE 6–6 | 25.18392$\bar{5}$ |
| 111 | TE 5–5 | 20.57551$\underline{5}$ | 166 | TE 23–1 | 25.322921 |
| 112 | TM 12–2 | 20.789906 | 167 | TM 16–2 | 25.417019 |
| 113 | TM 9–3 | 20.807048 | 168 | TM 20–1 | 25.417141 |
| 114 | TM 4–5 | 20.826933 | 169 | TM 5–6 | 25.430341 |
| 115 | TE 3–6 | 20.972477 | 170 | TE 18–2 | 25.495558 |
| 116 | TE 14–2 | 21.015405 | 171 | TM 10–4 | 25.509450 |
| 117 | TM 16–1 | 21.085147 | 172 | TE 4–7 | 25.589760 |
| 118 | TM 2–6 | 21.116997 | 173 | TM 13–3 | 25.705104 |
| 119 | TE 1–7 | 21.164370 | 174 | TM 3–7 | 25.748167 |
| 120 | TE 19–1 | 21.182267 | 175 | TE 2–8 | 25.826037 |
| 121 | TM 0–7 | 21.211637 | 176 | TE 9–5 | 25.891177 |
| 122 | TE 8–4 | 21.229063 | (177 | TM 1–8 | 25.903672 |
| 123 | TE 11–3 | 21.430854 | (178 | TE 0–8 | 25.903672 |
| 124 | TM 7–4 | 21.641541 | 179 | TE 15–3 | 26.177766 |
| 125 | TE 6–5 | 21.93171$\bar{5}$ | 180 | TE 12–4 | 26.246048 |
| 126 | TM 13–2 | 21.956244 | 181 | TM 8–5 | 26.26681$\bar{5}$ |
| 127 | TM 10–3 | 22.04698$\underline{5}$ | 182 | TE 24–1 | 26.355506 |
| 128 | TE 15–2 | 22.142247 | 183 | TM 21–1 | 26.493648 |
| 129 | TM 17–1 | 22.17249$\bar{5}$ | 184 | TE 7–6 | 26.545032 |
| 130 | TM 5–5 | 22.217800 | 185 | TM 17–2 | 26.559784 |
| 131 | TE 20–1 | 22.21914$\bar{5}$ | 186 | TE 19–2 | 26.605533 |
| 132 | TE 4–6 | 22.401032 | 187 | TM 11–4 | 26.773323 |
| 133 | TE 9–4 | 22.501399 | 188 | TM 6–6 | 26.820152 |
| 134 | TM 3–6 | 22.582730 | 189 | TM 14–3 | 26.907369 |
| 135 | TE 12–3 | 22.629300 | 190 | TE 5–7 | 27.010308 |
| 136 | TE 2–7 | 22.671582 | 191 | TE 10–5 | 27.182022 |
| (137 | TM 1–7 | 22.760084 | 192 | TM 4–7 | 27.199088 |
| (138 | TE 0–7 | 22.760084 | 193 | TE 3–8 | 27.310058 |
| 139 | TM 8–4 | 22.945173 | 194 | TE 16–3 | 27.347386 |
| 140 | TM 14–2 | 23.115778 | 195 | TE 25–1 | 27.387204 |
| 141 | TE 21–1 | 23.254816 | 196 | TM 2–8 | 27.420574 |
| 142 | TM 18–1 | 23.256777 | 197 | TE 1–9 | 27.457051 |
| 143 | TE 16–2 | 23.264269 | 198 | TE 13–4 | 27.474340 |
| 144 | TE 7–5 | 23.268053 | 199 | TM 0–9 | 27.493480 |
| 145 | TM 11–3 | 23.275854 | 200 | TM 22–1 | 27.567944 |
| 146 | TM 6–5 | 23.586084 | 201 | TM 9–5 | 27.583749 |
| 147 | TE 10–4 | 23.760716 | 202 | TM 18–2 | 27.697899 |
| 148 | TE 5–6 | 23.803581 | 203 | TE 20–2 | 27.712126 |
| 149 | TE 13–3 | 23.819374 | 204 | TE 8–6 | 27.889270 |

* TM designates a zero of $J_l(x)$; TE designates a zero of $J'_l(x)$; in each case $l$ corresponds to the order of the Bessel function and $m$ is the number of the root.

† $\bar{5}$ in last place indicates higher value and $\underline{5}$ indicates lower value in rounding off for fewer decimal places.

TABLE — *Continued*

| | Mode* $l$–$m$ | Value† | | Mode* $l$–$m$ | Value† |
|---|---|---|---|---|---|
| 205 | TM 12–4 | 28.026710 | 256 | TM 12–5 | 31.459960 |
| 206 | TM 15–3 | 28.102416 | 257 | TE 29–1 | 31.506195 |
| 207 | TM 7–6 | 28.191189 | 258 | TE 6–8 | 31.617876 |
| 208 | TE 6–7 | 28.409776 | 259 | TM 18–3 | 31.650118 |
| 209 | TE 26–1 | 28.418072 | 260 | TM 15–4 | 31.733414 |
| 210 | TE 11–5 | 28.460857 | 261 | TM 5–8 | 31.811717 |
| 211 | TE 17–3 | 28.511361 | 262 | TE 11–6 | 31.838425 |
| 212 | TM 5–7 | 28.626619 | 263 | TM 26–1 | 31.845888 |
| 213 | TM 23–1 | 28.64018$\bar{5}$ | 264 | TE 4–9 | 31.938540 |
| 214 | TE 14–4 | 28.694271 | 265 | TE 20–3 | 31.97371$\bar{5}$ |
| 215 | TE 4–8 | 28.767836 | 266 | TM 3–9 | 32.064853 |
| 216 | TE 21–2 | 28.815590 | 267 | TE 24–2 | 32.109320 |
| 217 | TM 19–2 | 28.831731 | 268 | TE 2–10 | 32.127327 |
| 218 | TM 10–5 | 28.88737$\bar{5}$ | (269 | TM 1–10 | 32.189680 |
| 219 | TM 3–8 | 28.908351 | (270 | TE 0–10 | 32.189680 |
| 220 | TE 2–9 | 28.977673 | 271 | TM 22–2 | 32.210587 |
| (221 | TM 1–9 | 29.046829 | 272 | TM 10–6 | 32.211856 |
| (222 | TE 0–9 | 29.046829 | 273 | TE 14–5 | 32.236970 |
| 223 | TE 9–6 | 29.218564 | 274 | TE 17–4 | 32.310894 |
| 224 | TM 13–4 | 29.270631 | 275 | TE 9–7 | 32.505248 |
| 225 | TM 16–3 | 29.290871 | 276 | TE 30–1 | 32.534220 |
| 226 | TE 27–1 | 29.448163 | 277 | TM 13–5 | 32.731053 |
| 227 | TM 8–6 | 29.545660 | 278 | TM 8–7 | 32.795800 |
| 228 | TE 18–3 | 29.670147 | 279 | TM 19–3 | 32.821803 |
| 229 | TM 24–1 | 29.710509 | 280 | TM 27–1 | 32.911154 |
| 230 | TE 12–5 | 29.728978 | 281 | TM 16–4 | 32.95366$\bar{5}$ |
| 231 | TE 7–7 | 29.790749 | 282 | TE 7–8 | 33.015179 |
| 232 | TE 15–4 | 29.906591 | 283 | TE 21–3 | 33.119162 |
| 233 | TE 22–2 | 29.916147 | 284 | TE 12–6 | 33.131450 |
| 234 | TM 20–2 | 29.961604 | 285 | TE 25–2 | 33.202272 |
| 235 | TM 6–7 | 30.033723 | 286 | TM 6–8 | 33.233042 |
| 236 | TM 11–5 | 30.179061 | 287 | TM 23–2 | 33.330177 |
| 237 | TE 5–8 | 30.202849 | 288 | TE 5–9 | 33.385444 |
| 238 | TM 4–8 | 30.371008 | 289 | TE 15–5 | 33.478449 |
| 239 | TE 3–9 | 30.470269 | 290 | TE 18–4 | 33.503929 |
| 240 | TM 17–3 | 30.473280 | 291 | TM 11–6 | 33.526364 |
| 241 | TE 28–1 | 30.477523 | 292 | TM 4–9 | 33.537138 |
| 242 | TM 14–4 | 30.505951 | 293 | TE 31–1 | 33.561634 |
| 243 | TE 10–6 | 30.53450$\bar{5}$ | 294 | TE 3–10 | 33.626949 |
| 244 | TM 2–9 | 30.56920$\underline{5}$ | 295 | TM 2–10 | 33.716520 |
| 245 | TE 1–10 | 30.601923 | 296 | TE 1–11 | 33.746183 |
| 246 | TM 0–10 | 30.634607 | 297 | TM 0–11 | 33.775821 |
| 247 | TM 25–1 | 30.779039 | 298 | TE 10–7 | 33.841966 |
| 248 | TE 19–3 | 30.824148 | 299 | TM 28–1 | 33.974930 |
| 249 | TM 9–6 | 30.885379 | 300 | TM 20–3 | 33.988703 |
| 250 | TE 13–5 | 30.987394 | 301 | TM 14–5 | 33.99319 |
| 251 | TE 23–2 | 31.013998 | 302 | TM 9–7 | 34.15438 |
| 252 | TM 21–2 | 31.087805 | 303 | TM 17–4 | 34.16727 |
| 253 | TE 16–4 | 31.111945 | 304 | TE 22–3 | 34.26077 |
| 254 | TE 8–7 | 31.155327 | 305 | TE 26–2 | 34.29300 |
| 255 | TM 7–7 | 31.42279$\bar{5}$ | 306 | TE 8–8 | 34.39663 |

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† $\bar{5}$ in last place indicates higher value and 5 indicates lower value in rounding off for fewer decimal places.

TABLE — *Continued*

| | Mode* $l$–$m$ | Value† | | Mode* $l$–$m$ | Value† |
|---|---|---|---|---|---|
| 307 | TE 13–6 | 34.4145$\underline{5}$ | 357 | TE 10–8 | 37.11800 |
| 308 | TM 24–2 | 34.4467$\bar{8}$ | 358 | TM 31–1 | 37.15811 |
| 309 | TE 32–1 | 34.58847 | 359 | TE 18–5 | 37.16040 |
| 310 | TM 7–8 | 34.63709 | 360 | TM 9–8 | 37.40010 |
| 311 | TE 19–4 | 34.69148 | 361 | TM 14–6 | 37.40819 |
| 312 | TE 16–5 | 34.71248 | 362 | TM 23–3 | 37.46381 |
| 313 | TE 6–9 | 34.81339 | 363 | TE 29–2 | 37.55307 |
| 314 | TM 12–6 | 34.82999 | 364 | TE 8–9 | 37.62008 |
| 315 | TM 5–9 | 34.98878 | 365 | TE 25–3 | 37.66491 |
| 316 | TM 29–1 | 35.03730 | 366 | TE 35–1 | 37.66577 |
| 317 | TE 4–10 | 35.10392 | 367 | TM 17–5 | 37.73268 |
| 318 | TM 21–3 | 35.15115 | 368 | TM 20–4 | 37.77286 |
| 319 | TE 11–7 | 35.16671 | 369 | TM 27–2 | 37.78040 |
| 320 | TM 3–10 | 35.21867 | 370 | TE 13–7 | 37.78438 |
| 321 | TM 15–5 | 35.24709 | 371 | TM 7–9 | 37.83872 |
| 322 | TE 2–11 | 35.27554 | 372 | TE 6–10 | 37.99964 |
| (323 | TM 1–11 | 35.33231 | 373 | TM 12–7 | 38.15638 |
| (324 | TE 0–11 | 35.33231 | 374 | TM 5–10 | 38.15987 |
| 325 | TM 18–4 | 35.37472 | 375 | TE 16–6 | 38.21206 |
| 326 | TE 27–2 | 35.38163 | 376 | TM 32–1 | 38.21669 |
| 327 | TE 23–3 | 35.39878 | 377 | TE 22–4 | 38.22490 |
| 328 | TM 10–7 | 35.49991 | 378 | TE 4–11 | 38.26532 |
| 329 | TM 25–2 | 35.56057 | 379 | TM 3–11 | 38.37047 |
| 330 | TE 33–1 | 35.6147$\underline{5}$ | 380 | TE 19–5 | 38.37524 |
| 331 | TE 14–6 | 35.6885$\bar{4}$ | 381 | TE 2–12 | 38.42266 |
| 332 | TE 9–8 | 35.76379 | 382 | TE 11–8 | 38.46039 |
| 333 | TE 20–4 | 35.87394 | (383 | TM 1–12 | 38.47477 |
| 334 | TE 17–5 | 35.93963 | (384 | TE 0–12 | 38.47477 |
| 335 | TM 8–8 | 36.02562 | 385 | TM 24–3 | 38.61452 |
| 336 | TM 30–1 | 36.09834 | 386 | TE 30–2 | 38.63609 |
| 337 | TM 13–6 | 36.12366 | 387 | TM 15–6 | 38.68428 |
| 338 | TE 7–9 | 36.22438 | 388 | TE 36–1 | 38.6905$\bar{5}$ |
| 339 | TM 22–3 | 36.30943 | 389 | TM 10–8 | 38.76181 |
| 340 | TM 6–9 | 36.42202 | 390 | TE 26–3 | 38.79341 |
| 341 | TE 28–2 | 36.46829 | 391 | TM 28–2 | 38.88671 |
| 342 | TE 12–7 | 36.4805$\underline{5}$ | 392 | TM 21–4 | 38.96429 |
| 343 | TM 16–5 | 36.4934$\bar{0}$ | 393 | TM 18–5 | 38.96543 |
| 344 | TE 24–3 | 36.53343 | 394 | TE 9–9 | 39.00190 |
| 345 | TE 5–10 | 36.56078 | 395 | TE 14–7 | 39.07900 |
| 346 | TM 19–4 | 36.5764$\bar{5}$ | 396 | TM 8–9 | 39.2404$\underline{5}$ |
| 347 | TE 34–1 | 36.64051 | 397 | TM 33–1 | 39.2741$\bar{3}$ |
| 348 | TM 26–2 | 36.67173 | 398 | TE 23–4 | 39.39398 |
| 349 | TM 4–10 | 36.69900 | 399 | TE 7–10 | 39.42227 |
| 350 | TE 3–11 | 36.78102 | 400 | TE 17–6 | 39.46277 |
| 351 | TM 11–7 | 36.83357 | 401 | TM 13–7 | 39.46921 |
| 352 | TM 2–11 | 36.86286 | 402 | TE 20–5 | 39.58453 |
| 353 | TE 1–12 | 36.88999 | 403 | TM 6–10 | 39.60324 |
| 354 | TM 0–12 | 36.91710 | 404 | TE 37–1 | 39.71489 |
| 355 | TE 15–6 | 36.95417 | 405 | TE 31–2 | 39.71743 |
| 356 | TE 21–4 | 37.05164 | 406 | TE 5–11 | 39.73064 |

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† $\bar{5}$ in last place indicates higher value and $\underline{5}$ indicates lower value in rounding off for fewer decimal places.

TABLE — *Continued*

| | Mode* $l$–$m$ | Value† | | Mode* $l$–$m$ | Value† |
|---|---|---|---|---|---|
| 407 | TM 25–3 | 39.76179 | 458 | TE 29–3 | 42.16260 |
| 408 | TE 12–8 | 39.79194 | 459 | TM 31–2 | 42.1927$\underline{5}$ |
| 409 | TM 4–11 | 39.85763 | 460 | TE 9–10 | 42.22464 |
| 410 | TE 27–3 | 39.91909 | 461 | TE 14–8 | 42.4258$\bar{5}$ |
| 411 | TE 3–12 | 39.93311 | 462 | TM 36–1 | 42.44014 |
| 412 | TM 16–6 | 39.9525$\bar{5}$ | 463 | TM 8–10 | 42.44389 |
| 413 | TM 29–2 | 39.99080 | 464 | TM 18–6 | 42.46781 |
| 414 | TM 2–12 | 40.00845 | 465 | TM 24–4 | 42.51168 |
| 415 | TE 1–13 | 40.03344 | 466 | TE 7–11 | 42.61152 |
| 416 | TM 0–13 | 40.05843 | 467 | TM 21–5 | 42.62870 |
| 417 | TM 11–8 | 40.11182 | 468 | TM 6–11 | 42.77848 |
| 418 | TM 22–4 | 40.1510$\underline{5}$ | 469 | TM 13–8 | 42.78044 |
| 419 | TM 19–5 | 40.19210 | 470 | TE 40–1 | 42.78537 |
| 420 | TM 34–1 | 40.33048 | 471 | TE 26–4 | 42.8785$\bar{5}$ |
| 421 | TE 15–7 | 40.36510 | 472 | TE 5–12 | 42.89627 |
| 422 | TE 10–9 | 40.37107 | 473 | TE 17–7 | 42.9141$\bar{5}$ |
| 423 | TE 24–4 | 40.55913 | 474 | TE 34–2 | 42.95218 |
| 424 | TM 9–9 | 40.6285$\bar{5}$ | 475 | TM 4–12 | 43.01374 |
| 425 | TE 18–6 | 40.70680 | 476 | TE 12–9 | 43.07549 |
| 426 | TE 38–1 | 40.73879 | 477 | TE 3–13 | 43.0836$\bar{5}$ |
| 427 | TM 14–7 | 40.77283 | 478 | TM 2–13 | 43.1534$\bar{5}$ |
| 428 | TE 21–5 | 40.78864 | 479 | TE 20–6 | 43.17654 |
| 429 | TE 32–2 | 40.79718 | 480 | TE 1–14 | 43.17663 |
| 430 | TE 8–10 | 40.83018 | 481 | TE 23–5 | 43.18255 |
| 431 | TM 26–3 | 40.90580 | 482 | TM 28–3 | 43.18477 |
| 432 | TM 7–10 | 41.03077 | 483 | TM 0–14 | 43.19979 |
| 433 | TE 28–3 | 41.04211 | 484 | TE 30–3 | 43.28071 |
| 434 | TM 30–2 | 41.09278 | 485 | TM 32–2 | 43.29081 |
| 435 | TE 13–8 | 41.11351 | 486 | TM 16–7 | 43.35507 |
| 436 | TE 6–11 | 41.1788$\underline{5}$ | 487 | TM 11–9 | 43.36836 |
| 437 | TM 17–6 | 41.21357 | 488 | TM 37–1 | 43.49352 |
| 438 | TM 5–11 | 41.32638 | 489 | TE 10–10 | 43.60677 |
| 439 | TM 23–4 | 41.33343 | 490 | TM 25–4 | 43.68603 |
| 440 | TM 35–1 | 41.38580 | 491 | TM 19–6 | 43.71571 |
| 441 | TM 20–5 | 41.41307 | 492 | TE 15–8 | 43.72963 |
| 442 | TE 4–12 | 41.42367 | 493 | TE 41–1 | 43.80808 |
| 443 | TM 12–8 | 41.45109 | 494 | TM 22–5 | 43.83932 |
| 444 | TM 3–12 | 41.52072 | 495 | TM 9–10 | 43.84380 |
| 445 | TE 2–13 | 41.56894 | 496 | TE 35–2 | 44.02758 |
| (446 | TM 1–13 | 41.61709 | 497 | TE 8–11 | 44.03001 |
| (447 | TE 0–13 | 41.61709 | 498 | TE 27–4 | 44.03321 |
| 448 | TE 16–7 | 41.64331 | 499 | TM 14–8 | 44.10059 |
| 449 | TE 25–4 | 41.72059 | 500 | TE 18–7 | 44.17813 |
| 450 | TE 11–9 | 41.72863 | 501 | TM 7–11 | 44.21541 |
| 451 | TE 39–1 | 41.76228 | 502 | TM 29–3 | 44.32003 |
| 452 | TE 33–2 | 41.87540 | 503 | TE 6–12 | 44.35258 |
| 453 | TE 19–6 | 41.94459 | 504 | TE 24–5 | 44.37290 |
| 454 | TE 22–5 | 41.98788 | 505 | TM 33–2 | 44.38706 |
| 455 | TM 10–9 | 42.00419 | 506 | TE 31–3 | 44.39653 |
| 456 | TM 27–3 | 42.04674 | 507 | TE 21–6 | 44.40300 |
| 457 | TM 15–7 | 42.06792 | 508 | TE 13–9 | 44.4124$\bar{5}$ |

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TABLE — *Continued*

| | Mode* $l$–$m$ | Value† | | Mode* $l$–$m$ | Value† |
|---|---|---|---|---|---|
| 509 | TM 5–12 | 44.48932 | 561 | TE 20–7 | 46.68717 |
| 510 | TM 38–1 | 44.54601 | 562 | TM 16–8 | 46.71581 |
| 511 | TE 4–13 | 44.57962 | 563 | TE 26–5 | 46.74158 |
| 512 | TM 17–7 | 44.63483 | 564 | TE 10–11 | 48.82896 |
| 513 | TM 3–13 | 44.66974 | 565 | TE 23–6 | 46.8407$\bar{5}$ |
| 514 | TE 2–14 | 44.7145$\bar{5}$ | 566 | TE 44–1 | 46.87409 |
| 515 | TM 12–9 | 44.72194 | 567 | TM 9–11 | 47.04870 |
| (516 | TM 1–14 | 44.75932 | 568 | TE 15–9 | 47.05946 |
| (517 | TE 0–14 | 44.75932 | 569 | TM 19–7 | 47.17400 |
| 518 | TE 42–1 | 44.83043 | 570 | TM 28–4 | 47.1877$\bar{5}$ |
| 519 | TM 26–4 | 44.85670 | 571 | TE 8–12 | 47.22176 |
| 520 | TM 20–6 | 44.95768 | 572 | TE 38–2 | 47.24608 |
| 521 | TE 11–10 | 44.97753 | 573 | TM 7–12 | 47.39417 |
| 522 | TE 16–8 | 45.02543 | 574 | TM 14–9 | 47.4003$\bar{5}$ |
| 523 | TM 23–5 | 45.04521 | 575 | TM 22–6 | 47.42517 |
| 524 | TE 36–2 | 45.10166 | 576 | TM 25–5 | 47.4438$\bar{5}$ |
| 525 | TE 28–4 | 45.18473 | 577 | TE 30–4 | 47.47899 |
| 526 | TM 10–10 | 45.23157 | 578 | TE 6–13 | 47.52196 |
| 527 | TM 15–8 | 45.41219 | 579 | TE 18–8 | 47.59513 |
| 528 | TE 9–11 | 45.43548 | 580 | TM 5–13 | 47.64940 |
| 529 | TE 19–7 | 45.43567 | 581 | TM 36–2 | 47.66568 |
| 530 | TM 30–3 | 45.45267 | 582 | TE 13–10 | 47.6882$\bar{5}$ |
| 531 | TM 34–2 | 45.48156 | 583 | TM 41–1 | 47.69840 |
| 532 | TE 32–3 | 45.51018 | 584 | TM 32–3 | 47.7105$\bar{5}$ |
| 533 | TE 25–5 | 45.55917 | 585 | TE 34–3 | 47.73138 |
| 534 | TM 39–1 | 45.59762 | 586 | TE 4–14 | 47.73367 |
| 535 | TE 22–6 | 45.62431 | 587 | TM 3–14 | 47.81779 |
| 536 | TM 8–11 | 45.63844 | 588 | TE 2–15 | 47.85964 |
| 537 | TE 14–9 | 45.74024 | 589 | TE 45–1 | 47.89542 |
| 538 | TE 7–12 | 45.79400 | (590 | TM 1–15 | 47.90146 |
| 539 | TE 43–1 | 45.85243 | (591 | TE 0–15 | 47.90146 |
| 540 | TM 18–7 | 45.90766 | 592 | TE 27–5 | 47.92033 |
| 541 | TM 6–12 | 45.94902 | 593 | TE 21–7 | 47.93298 |
| 542 | TM 27–4 | 46.02388 | 594 | TM 12–10 | 47.97429 |
| 543 | TE 5–13 | 46.05857 | 595 | TM 17–8 | 48.01196 |
| 544 | TM 13–9 | 46.06571 | 596 | TE 24–6 | 48.05260 |
| 545 | TM 4–13 | 46.1678$\bar{5}$ | 597 | TE 11–11 | 48.21133 |
| 546 | TE 37–2 | 46.17447 | 598 | TE 39–2 | 48.31652 |
| 547 | TM 21–6 | 46.19406 | 599 | TM 29–4 | 48.34846 |
| 548 | TE 3–14 | 46.23297 | 600 | TE 16–9 | 48.37069 |
| 549 | TM 24–5 | 46.24664 | 601 | TM 20–7 | 48.43424 |
| 550 | TM 2–14 | 46.29800 | 602 | TM 10–11 | 48.4471$\bar{5}$ |
| 551 | TE 17–8 | 46.31377 | 603 | TE 31–4 | 48.62201 |
| 552 | TE 1–15 | 46.31960 | 604 | TE 9–12 | 48.63692 |
| 553 | TE 29–4 | 46.33328 | 605 | TM 26–5 | 48.63706 |
| 554 | TE 12–10 | 46.33777 | 606 | TM 23–6 | 48.65132 |
| 555 | TM 0–15 | 46.34119 | 607 | TM 15–9 | 48.72646 |
| 556 | TM 35–2 | 46.57441 | 608 | TM 42–1 | 48.74762 |
| 557 | TM 31–3 | 46.58280 | 609 | TM 37–2 | 48.75542 |
| 558 | TM 11–10 | 46.60813 | 610 | TM 8–12 | 48.82593 |
| 559 | TE 33–3 | 46.62177 | 611 | TM 33–3 | 48.83603 |
| 560 | TM 40–1 | 46.64841 | 612 | TE 35–3 | 48.83910 |

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TABLE—*Concluded*

| | Mode* $l$–$m$ | Value† | | Mode* $l$–$m$ | Value† |
|---|---|---|---|---|---|
| 613 | TE 19–8 | 48.86993 | 657 | TE 6–14 | 50.68782 |
| 614 | TE 46–1 | 48.91645 | 658 | TM 5–14 | 50.80717 |
| 615 | TE 7–13 | 48.97107 | 659 | TM 44–1 | 50.84387 |
| 616 | TE 14–10 | 49.02964 | 660 | TE 4–15 | 50.88616 |
| 617 | TE 28–5 | 49.09560 | 661 | TE 33–4 | 50.90045 |
| 618 | TM 6–13 | 49.11577 | 662 | TM 39–2 | 50.93060 |
| 619 | TE 22–7 | 49.17342 | 663 | TE 48–1 | 50.93760 |
| 620 | TE 5–14 | 49.21817 | 664 | TM 22–7 | 50.93776 |
| 621 | TE 25–6 | 49.26009 | 665 | TE 13–11 | 50.94585 |
| 622 | TM 18–8 | 49.30111 | 666 | TM 3–15 | 50.96503 |
| 623 | TM 4–14 | 49.32036 | 667 | TE 18–9 | 50.97113 |
| 624 | TM 13–10 | 49.33078 | 668 | TE 2–16 | 51.00430 |
| 625 | TE 3–15 | 49.38130 | 669 | TM 28–5 | 51.01228 |
| 626 | TE 40–2 | 49.38586 | (670 | TM 1–16 | 51.04354 |
| 627 | TM 2–15 | 49.44216 | (671 | TE 0–16 | 51.04354 |
| 628 | TE 1–16 | 49.46239 | 672 | TE 37–3 | 51.04919 |
| 629 | TM 0–16 | 49.48261 | 673 | TM 35–3 | 51.0805$\bar{5}$ |
| 630 | TM 30–4 | 49.50618 | 674 | TM 25–6 | 51.0897$\underline{5}$ |
| 631 | TE 12–11 | 49.58340 | 675 | TM 12–11 | 51.2119$\bar{7}$ |
| 632 | TE 17–9 | 49.67443 | 676 | TM 17–9 | 51.35527 |
| 633 | TM 21–7 | 49.68872 | 677 | TE 21–8 | 51.40137 |
| 634 | TE 32–4 | 49.76246 | 678 | TE 11–12 | 51.43311 |
| 635 | TM 43–1 | 49.79610 | 679 | TE 30–5 | 51.43637 |
| 636 | TM 77–5 | 49.82648 | 680 | TE 42–2 | 51.5213$\bar{5}$ |
| 637 | TM 11–11 | 49.83465 | 681 | TE 24–7 | 51.63937 |
| 638 | TM 38–2 | 49.84371 | 682 | TM 10–12 | 51.6532$\bar{5}$ |
| 639 | TM 24–6 | 49.87276 | 683 | TE 27–6 | 51.66288 |
| 640 | TE 47–1 | 49.93717 | 684 | TE 16–10 | 51.68742 |
| 641 | TE 36–3 | 49.94501 | 685 | TM 32–4 | 51.81316 |
| 642 | TM 34–3 | 49.95933 | 686 | TE 9–13 | 51.83078 |
| 643 | TE 10–12 | 50.04043 | 687 | TM 20–8 | 51.86002 |
| 644 | TM 16–9 | 50.04461 | 688 | TM 45–1 | 51.8909$\bar{5}$ |
| 645 | TE 20–8 | 50.13856 | 689 | TE 49–1 | 51.97776 |
| 646 | TM 9–12 | 50.24533 | 690 | TM 8–13 | 52.00769 |
| 647 | TE 29–5 | 50.26756 | 691 | TM 40–2 | 52.0161$\underline{5}$ |
| 648 | TE 15–10 | 50.36251 | 692 | TM 15–10 | 52.0172$\bar{4}$ |
| 649 | TE 8–13 | 50.40702 | 693 | TE 34–4 | 52.03608 |
| 650 | TE 23–7 | 50.40880 | 694 | TE 7–14 | 52.1437$\bar{5}$ |
| 651 | TE 41–2 | 50.45412 | 695 | TE 38–3 | 52.15171 |
| 652 | TE 26–6 | 50.46345 | 696 | TM 23–7 | 52.18166 |
| 653 | TM 7–13 | 50.56818 | 697 | TM 29–5 | 52.1946$\underline{5}$ |
| 654 | TM 19–8 | 50.58367 | 698 | TM 36–3 | 52.1997$\bar{8}$ |
| 655 | TM 31–4 | 50.66103 | 699 | TE 19–9 | 52.26121 |
| 656 | TM 14–10 | 50.67824 | 700 | TM 6–14 | 52.2794$\bar{5}$ |

* TM designates a zero of $J_l(x)$; TE designates a zero of $J'_l(x)$; in each case $l$ corresponds to the order of the Bessel function and $m$ is the number of the root.

† 5 in last place indicates higher value and 5 indicates lower value in rounding off for fewer decimal places.