THE ANALOGY BETWEEN ROTATING AND STRATIFIED FLUIDS

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PART I. GENERAL CONSIDERATIONS

1. Introduction.—The mathematical analogy between stratified fluids and rotating fluids derives from the control of the fluid behavior by the respective constraints of stratification and rotation. In a homogeneous fluid rotating at a constant angular rotation rate, Ω, about a given axis, motions in a plane normal to the axis of rotation are inhibited by the constraint of rotation. On the other hand, if a fluid in a gravitational field is stably stratified, fluid motions parallel to the direction of gravity are inhibited because work must be done against the force of gravity in order to generate such motions.

Under certain restrictive conditions it is possible to draw an exact parallel between the effects of these two constraints on the fluid motion. A wide variety of phenomena in one physical system can be analyzed and understood in terms of the behavior in the analogous system. The increased understanding provided by the different viewpoints is one justification for developing the analogy in some detail. A second reason is that laboratory studies may be more easily pursued in one of the two systems and experimental information pertinent to the other may thereby be obtained.

Yih (28) includes a chapter on rotating fluids in his book on stratified flow and points out the analogy between stratified fluids and rotating fluids. He has also demonstrated the parallel behavior of the two constraining mechanisms in a series of publications referred to in his book. Chandrasekhar (5) presents a detailed analysis of the problems of Bénard convection and Couette flow and makes liberal use of the analogy. He has also indicated a few of the features of hydromagnetic flow which resemble those of rotating fluids. Greenspan (8) devotes an early section in his treatment of rotating fluids to the analogy. In an unpublished manuscript, Robert Dickinson has developed the analogy for a number of particular flows. Perhaps the first reference to the analogy was made by Lord Rayleigh (17) in a discussion of inviscid stability of Couette flow.

In the following pages we shall first show how the constraints in the two physical systems serve to determine the behavior of the flow in the simplest possible situations. The analogy is evident from a straightforward identification of the physical parameters and variables of the two systems. Guided by the knowledge gained from the simple examples, we then develop the analogy between seemingly unrelated flows in fluids which are both rotating and stratified. A series of examples serves to identify specific properties of one system with those of the other. We shall restrict our attention to the case where the axes of rotation and gravity coincide. Also, we shall treat physical problems in which the boundaries are either parallel or perpendicular to the axis of rotation and gravity. Although a number of interesting problems are thereby excluded from consideration, the latter are of interest more in themselves rather than as aids in understanding the analogy.

2. *Two simple models exhibiting the analogy.*—For the first of the simple examples consider a homogeneous, incompressible fluid rotating uniformly about the z-axis. For steady, inviscid flow of small amplitude the Coriolis acceleration is balanced by the pressure gradient, and we have

$$2\Omega \boldsymbol{k} \times \boldsymbol{v} = -\frac{1}{\rho} \nabla p \qquad 1.$$

where Ω is the uniform angular velocity of the frame of reference, $\boldsymbol{v}$ is the relative velocity vector, ρ is the uniform density, $\boldsymbol{k}$ is the unit vector in the z direction, and p includes both the pressure of the fluid and the centrifugal force, $-\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})$. We shall usually refer to p simply as the pressure. The flow described by Equation 1 is called geostrophic flow.

For a stratified fluid in a uniform gravitational field with gravity directed toward negative z, the basic momentum balance for very small motions is hydrostatic balance which is

$$g\rho' = -\frac{\partial p}{\partial z} \qquad 2.$$

Here we have subtracted out the static density and pressure fields so that ρ and p are perturbation quantities.

A second equation for the stratified fluid comes from the incompressibility condition. If the stabilizing vertical density gradient is assumed constant and very large, the equation in the case of steady flow of small amplitude reduces to

$$w \frac{\partial \rho}{\partial z} = 0 \qquad 3.$$

where ρ is the static density field.

We can draw an immediate qualitative analogy between Equation 1 and the pair of Equations 2 and 3 if we assume that the rotating flow is independent of one horizontal direction, y say. Then Equation 1 reduces to the pair of equations

$$-2\Omega v = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad 4.$$

$$2\Omega u = 0 \qquad 5.$$

Apart from physical parameters and dimensional considerations, Equations 4 and 5 are similar in form to Equations 2 and 3 respectively. Hence, we have a mathematical analogy provided that we make the following identifications:

Rotating System (Ω)	v	z	x	w	u
Stratified System (S)	$-\rho'$	x	z	u	w

The relations between the physical parameters will be brought out shortly.

Note that relating v to $-\rho'$ and x to z implies that the vertical, $\partial y/\partial x$, in system Ω is related to the negative density gradient $-\partial\rho'/\partial z$ (or the positive temperature gradient in a Boussinesq (fluid) in system S. This also implies that the gyroscopic constraint, the uniform vertical vorticity, Ω, is to be related to the constraint due to stratification, the negative vertical density gradient, $-\partial\rho'/\partial z$. It is these basic identifications which make the analogy so complete and so useful.

For the second example, consider the simplest perturbation of a geostrophic system. We imagine that the perturbation velocities are independent of all space coordinates so that the momentum equations for the perturbation quantities are

$$\frac{\partial u}{\partial t} - 2\Omega v = 0 \qquad 6.$$

$$\frac{\partial v}{\partial t} + 2\Omega u = 0 \qquad 7.$$

These equations have a nontrivial solution provided that $(u, v) \sim e^{i\omega t}$ where

$$\omega = \pm 2\Omega \qquad 8.$$

The quantity 2Ω is called the inertial or Coriolis frequency and is the fundamental frequency for homogeneous rotating fluids. The corresponding period is called the half-pendulum day since a period of oscillation is executed in half the period of a Foucault pendulum.

The analogous flow in a stratified fluid occurs when a hydrostatic system is perturbed and all perturbation quantities are independent of the space coordinates. The vertical equation of motion and the incompressibility condition become

$$\rho\frac{\partial w}{\partial t} = -g\rho' \qquad 9.$$

$$\frac{\partial \rho'}{\partial t} + w \frac{\partial \rho}{\partial z} = 0 \qquad 10.$$

These have a nontrivial solution provided that $(w, \rho') \sim e^{i\omega t}$ where

$$\omega = \pm \sqrt{\frac{-g}{\rho} \frac{\partial \rho}{\partial z}} \equiv \pm N \qquad 11.$$

The quantity, N, is called the Brunt-Väisälä or buoyancy frequency and is the fundamental frequency for stratified fluids. When the basic density field depends exponentially on z or when the Boussinesq approximation is applicable and the basic stratification is linear in z, the buoyancy frequency is constant.

If the analogy is to be extended to these two oscillatory systems, we must restrict our attention to constant N. We note that in addition to the identifications listed earlier 2Ω and N play similar roles. This latter identification also serves to relate the principal constraints of the two systems quantitatively.

Although the analogy will be extended to fairly complex physical systems, it should be noted at this point that the basic physical identifications are already included in this section. Molecular processes introduce additional parameters in the two systems which must be identified but no basically new parametric identifications emerge when they are included.

For the general development we turn now to the equations of motion for a fluid which is both rotating and stratified and show how the equations can be cast in two forms which are related in a one-to-one fashion mathematically.

3. The equations and the analogy.—The analogy between different physical systems involving fluids which are rotating or stratified, or both, holds rigorously under certain restrictive conditions. We shall derive the conditions for the case where a fluid is both rotating and stratified. In the following it is assumed that the flow can be described adequately when the Boussinesq approximation is made. Then the momentum equations for a stratified fluid in a gravitational field and rotating uniformly about an axis parallel to the direction of gravity take the form

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + 2\boldsymbol{\Omega} x \boldsymbol{v} = -\frac{1}{\rho} \nabla p + g\alpha T \boldsymbol{k} + \nu \nabla^2 \boldsymbol{v} \qquad 12.$$

where $\boldsymbol{v}$ is the velocity vector; $\boldsymbol{\Omega}$ is the angular rotation vector; p incorporates ordinary pressure, the centrifugal force, $-\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})$ and the static part of the gravitational term; $\boldsymbol{r}$ is the coordinate vector; g is the (constant) acceleration due to gravity and acts in the negative z direction; $\boldsymbol{k}$ is the unit vector in the z direction; ρ is the (constant) reference density; and ν is the kinematic viscosity. We shall restrict our attention to the case where $\boldsymbol{\Omega}$ is parallel to $\boldsymbol{k}$. The gravitational term is written in a form appropriate to a Boussinesq fluid. Here α is the coefficient of thermal expansion and T is the perturbation temperature, i.e., the difference between the total temperature and the temperature which exists in the absence of motion.[1]

For a Boussinesq fluid conservation of mass is expressed as the continuity equation

$$\nabla \cdot \boldsymbol{v} = 0 \qquad 13.$$

The first law of thermodynamics can be expressed in terms of the perturbation temperatures as

$$\frac{\partial T}{\partial t} + \boldsymbol{v} \cdot \nabla T + w \frac{\partial \overline{T}}{\partial z} = k \nabla^2 T \qquad 14.$$

where $\partial \overline{T}/\partial z$ is the vertical temperature gradient in the absence of motions and k is the thermometric diffusivity. We shall consider only the case where $\partial \overline{T}/\partial z =$ constant so that the analogy can be made precise. Thus we choose

$$\frac{\partial \overline{T}}{\partial z} = 4 \frac{\Delta T}{L} \qquad 15.$$

where L is the characteristic scale in the vertical direction and $4\Delta T$ is the magnitude of the static temperature difference over the distance, L. The constant, 4, is included in the definition for convenience as we shall see later. Substituting Equation 15 into 14 yields

$$\frac{\partial T}{\partial t} + \boldsymbol{v} \cdot \nabla T + 4 \frac{\Delta T}{L} w = k \nabla^2 T \qquad 16.$$

We shall nondimensionalize Equations 12, 13, and 16 since so much of the analogy can be brought out clearly by so doing. We write the dimensional variables in nondimensional form as

$$\boldsymbol{v} = V \boldsymbol{v}', \qquad \frac{\partial}{\partial t} = \frac{1}{\tau} \frac{\partial}{\partial t'}, \qquad \nabla = \frac{1}{L} \nabla', \qquad p = P p', \qquad T = 2\vartheta T'$$

where the primed variables are dimensionless and V, L, τ, P, and ϑ are dimensional scaling factors. Equations 12, 13, and 16 then take the form

$$\frac{V}{\tau} \frac{\partial \boldsymbol{v}'}{\partial t'} + \frac{V^2}{L} \boldsymbol{v}' \cdot \nabla' \boldsymbol{v}' + 2\Omega V \boldsymbol{k} \times \boldsymbol{v}' = -\frac{P}{L\rho} \nabla' p' + 2g\alpha\vartheta T' \boldsymbol{k} + \frac{\nu V}{L^2} \nabla'^2 \boldsymbol{v}' \qquad 17.$$

[1] A rotating, stratified fluid cannot be in a state of relative rest because of molecular processes. Hence, the motions which are forced by the processes of interest must be sufficiently strong to allow neglect of the slow flow due to molecular processes.

$$\nabla' \cdot v' = 0 \quad 18.$$

$$\frac{2\vartheta}{\tau}\frac{\partial T'}{\partial t'} + \frac{2\vartheta V}{L} v' \cdot \nabla' T' + \frac{4\Delta T V}{L} w' = \frac{2\vartheta k}{L^2} \nabla'^2 T' \quad 19.$$

Now consider two physical systems. In the first, which we denote by system Ω_s, we suppose that the rotational constraint is the basic one and stratification adds a modifying influence. In the second, denoted by S_Ω, stratification is basic and rotation modifies its influence. We shall therefore present two nondimensional forms of Equations 17, 18, and 19, one appropriate to system Ω_s, the other to system S_Ω.

System Ω_s.—In this system we expect that geostrophic balance and inertial waves should determine the nondimensionalization in the sense that the terms which balance each other in these simple flows have no parameters associated with them. Thus we are led to

$$\tau = \Omega^{-1}, \qquad P = L\rho V\Omega \quad 20.$$

The momentum Equation 17 then takes the form

$$\frac{\partial v'}{\partial t'} + \epsilon v' \cdot \nabla' v' + 2_k \times v' = -\nabla' p' + \frac{2g\alpha\vartheta}{\Omega V} T' k + E\nabla'^2 v' \quad 21.$$

where

$$\epsilon = \frac{V}{\Omega L}, \qquad E = \frac{\nu}{\Omega L^2} \quad (\equiv \text{Ekman number}) \quad 22.$$

Using the first of Equations 20 in the heat equation yields

$$\frac{\partial T'}{\partial t'} + \epsilon v' \cdot \nabla' T' + \frac{2\Delta T V}{\Omega \vartheta L} w' = \frac{E}{\sigma} \nabla'^2 T' \quad 23.$$

where

$$\sigma = \frac{\nu}{k} \quad (\equiv \text{Prandtl number}) \quad 24.$$

We expect the rotating flow to be modified by stratification and the latter will be effective primarily through the buoyancy term in Equation 21 and through the vertical stability term in Equation 23. Equating the coefficients of these terms so that they take the same form, we derive

$$\frac{2g\alpha\vartheta}{\Omega V} = \frac{2\Delta T V}{\Omega \vartheta L} \quad \text{or} \quad \frac{V}{\vartheta} = \sqrt{\frac{g\alpha L}{\Delta T}} \quad 25.$$

This last equation shows the relation between the magnitude of the velocity disturbance and the magnitude of the associated thermal (or density) disturbance. Making use of Equation 25, we write Equations 21 and 23 in the following nondimensional form (dropping the primes)

$$\frac{\partial \boldsymbol{v}}{\partial t} + \epsilon \boldsymbol{v} \cdot \nabla \boldsymbol{v} + 2\boldsymbol{k} \times \boldsymbol{v} = -\nabla p + 2FT\boldsymbol{k} + \sigma^{1/2}\tilde{E}\nabla^2 \boldsymbol{v} \qquad 26.$$

$$\frac{\partial T}{\partial t} + \epsilon \boldsymbol{v} \cdot \nabla T + 2Fw = \sigma^{-1/2}\tilde{E}\nabla^2 T \qquad 27.$$

where[2]

$$F = \sqrt{\frac{g\alpha\Delta T}{L\Omega^2}} \equiv \frac{N}{2\Omega} \qquad 28.$$

and

$$\tilde{E} = E\sigma^{-1/2} \qquad 29.$$

The dissipation terms as written in Equations 26 and 27 have a certain symmetry which appears also (and more naturally) in system S_Ω. More commonly used forms for the parametric coefficients of the dissipation terms appear in Equations 21 and 23.

System S_Ω.—We now balance the terms which give hydrostatic balance and buoyancy oscillations. Thus, equating the coefficients of the local time acceleration terms, the pressure term, the buoyancy term, and the vertical stability term yields

$$\tau = \sqrt{\frac{L}{g\alpha\Delta T}}, \qquad V = \sqrt{\frac{g\alpha L}{\Delta T}}\vartheta, \qquad P = g\alpha\vartheta L\rho \qquad 30.$$

Equations 17 and 19 take the form (dropping the primes)

$$\frac{\partial \boldsymbol{v}}{\partial t} + \epsilon \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \frac{2}{F}\boldsymbol{k} \times \boldsymbol{v} = -\nabla p + 2T\boldsymbol{k} + \sigma^{1/2}R\nabla^2 \boldsymbol{v} \qquad 31.$$

$$\frac{\partial T}{\partial t} + \epsilon \boldsymbol{v} \cdot \nabla T + 2w = \sigma^{-1/2}R\nabla^2 T \qquad 32.$$

[2] This parameter is the ratio of the frequencies due to buoyancy and rotational inertia acting separately. The quantity F^2 is sometimes called an internal Froude number. Phillips (13) has proposed calling it the Burger number because Burger (4) recognized its importance in determining the dominant dynamical balances of large scale waves in the atmosphere. However, the parameter is an important one in all problems of rotating stratified flows and it has been used as a basic parameter in such flows for nearly a century.

where

$$\epsilon = \frac{\vartheta}{\Delta T}, \qquad R = \sqrt{\frac{\nu k}{g\alpha \Delta T L^3}} \quad (= R_a^{-1/2}) \qquad 33.$$

and R_a is the Rayleigh number. [No further relations are needed because buoyancy oscillations involve both a thermal and a dynamic balance and these balances suffice to relate the velocity magnitude to the temperature fluctuations. In system Ω_s the basic balance involves only the momentum equations and a separate statement (Equation 25) had to be added for the modification due to stratification.]

We shall work with the sets of Equations 26 and 27 for system Ω_s and 31 and 32 for system S_Ω. The parameter, ϵ, in each set is simply the measure of nonlinearity.

There is an exact analogy between systems Ω_s and S_Ω for two-dimensional flows (e.g., variation with respect to y vanishes) provided that certain restrictions and identifications are made. To see this we write the equations in component form. Thus, for system Ω_s

$$\frac{\partial u}{\partial t} + \epsilon \boldsymbol{v}\cdot\nabla u - 2v = -\frac{\partial p}{\partial x} + \sigma^{1/2}\tilde{E}\nabla^2 u \qquad 34.$$

$$\frac{\partial v}{\partial t} + \epsilon \boldsymbol{v}\cdot\nabla v + 2u = \sigma^{1/2}\tilde{E}\nabla^2 v \qquad 35.$$

$$\frac{\partial w}{\partial t} + \epsilon \boldsymbol{v}\cdot\nabla w - 2FT = -\frac{\partial p}{\partial z} + \sigma^{1/2}\tilde{E}\nabla^2 w \qquad 36.$$

$$\frac{\partial T}{\partial t} + \epsilon \boldsymbol{v}\cdot\nabla T + 2Fw = \sigma^{-1/2}\tilde{E}\nabla^2 T \qquad 37.$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \qquad 38.$$

For system S_Ω we have (re-ordering the equations)

$$\frac{\partial w}{\partial t} + \epsilon \boldsymbol{v}\cdot\nabla w - 2T = -\frac{\partial p}{\partial z} + \sigma^{1/2}R\nabla^2 w \qquad 39.$$

$$\frac{\partial T}{\partial t} + \epsilon \boldsymbol{v}\cdot\nabla T + 2w = \sigma^{-1/2}R\nabla^2 T \qquad 40.$$

$$\frac{\partial u}{\partial t} + \epsilon \boldsymbol{v}\cdot\nabla u - \frac{2v}{F} = -\frac{\partial p}{\partial x} + \sigma^{1/2}R\nabla^2 u \qquad 41.$$

$$\frac{\partial v}{\partial t} + \epsilon \boldsymbol{v} \cdot \nabla v + \frac{2}{F} u = \sigma^{1/2} R \nabla^2 v \tag{42.}$$

$$\frac{\partial w}{\partial z} + \frac{\partial u}{\partial x} = 0 \tag{43.}$$

If $\sigma = 1$, Equations 34 to 38 are identical in form to the set 39 to 43, respectively. For the exact analogy we need only relate the S_Ω variable in the following list with the Ω_s variable which lies above it.

System Ω_s	x	z	u	w	v	T	$\tilde{E}$	F	44.
System S_Ω	z	x	w	u	T	v	R	F^{-1}	

For the special case of steady, linear flow the analogy is valid for all Prandtl numbers because the latter can be incorporated into the scaling of the dependent variables so that σ does not appear explicitly. If local or nonlinear accelerations are present, the restriction, $\sigma = 1$, must be added.

The relations between the variables must extend to the boundary conditions as well. In rotating, stratified flows, the usefulness of the analogy enters via the boundary conditions. E.g., if a flow in Ω_s is driven by vertical vorticity, $\partial v/\partial x$, imposed along a boundary where $z =$ constant, the analogous flow in S_Ω would be driven by a vertical temperature gradient imposed along a boundary where $x =$ constant.

Physically, the significance of the analogy is clear. The constraint of rotation in Ω_s plays the same role as the constraint due to stratification in S_Ω and vice versa. The dissipative processes and the oscillatory motions also fulfill similar functions.

Part II. Types of Flows in Systems Ω and S

1. Systems Ω and S.—When the fluid is rotating but not stratified, F vanishes, the heat equation decouples from the system and system Ω_s becomes system Ω. The pertinent set reduces to

$$\frac{\partial u}{\partial t} + \epsilon \boldsymbol{v} \cdot \nabla u - 2v = -\frac{\partial p}{\partial x} + E \nabla^2 u \tag{1.}$$

$$\frac{\partial v}{\partial t} + \epsilon \boldsymbol{v} \cdot \nabla v + 2u = E \nabla^2 v \tag{2.}$$

$$\frac{\partial w}{\partial t} + \epsilon \boldsymbol{v} \cdot \nabla w = -\frac{\partial p}{\partial z} + E \nabla^2 w \tag{3.}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{4.}$$

When the fluid is stratified but not rotating, the y momentum equation decouples, $F=\infty$, system S_Ω reduces to system S and the pertinent set becomes

$$\frac{\partial w}{\partial t} + \epsilon \boldsymbol{v}\cdot\nabla w - 2T = -\frac{\partial p}{\partial z} + \sigma^{1/2}R\nabla^2 w \qquad 5.$$

$$\frac{\partial T}{\partial t} + \epsilon \boldsymbol{v}\cdot\nabla T + 2w = \sigma^{-1/2}R\nabla^2 T \qquad 6.$$

$$\frac{\partial u}{\partial t} + \epsilon \boldsymbol{v}\cdot\nabla u = -\frac{\partial p}{\partial x} + \sigma^{1/2}R\nabla^2 u \qquad 7.$$

$$\frac{\partial w}{\partial z} + \frac{\partial u}{\partial x} = 0. \qquad 8.$$

2. Linear, inviscid, steady flow.—System Ω reduces to geostrophic flow and we recover the results of section I.2. An important and well-known consequence of geostrophic flow is the Taylor-Proudman theorem (16, 20) which states that under the given restrictions all variables are independent of the coordinate parallel to the axis of rotation. The result is readily deduced from Equations 1 to 4 which reduce to

$$2v = \frac{\partial p}{\partial x}, \qquad u = 0, \qquad \frac{\partial p}{\partial z} = 0, \qquad \frac{\partial w}{\partial z} = 0. \qquad 9.$$

The Taylor-Proudman theorem holds as well for three-dimensional flow in a rotating fluid. Experimentally the result can be observed approximately when a body such as a sphere moves through the fluid. A vertical column of fluid, called a Taylor column, whose cross-section is the maximum horizontal cross-section of the body, moves with the body as if it were a solid. Frictional or transient effects modify this behavior.

For a stratified fluid the flow is hydrostatic and the counterpart of the Taylor-Proudman theorem is that all variables are independent of x. The equations reduce to

$$2T = \frac{\partial p}{\partial z}, \qquad w = 0, \qquad \frac{\partial p}{\partial x} = 0, \qquad \frac{\partial u}{\partial x} = 0. \qquad 10.$$

Corresponding to the Taylor column is the phenomenon called blocking. When a cylinder with axis in the y direction moves in the x direction through the fluid, a horizontal slab of fluid whose thickness is the height of the cylinder moves with the cylinder.

This phenomenon is not observed when the body has a finite length because the fluid can flow horizontally around the body. Hence, the analogy breaks down for three-dimensional flow. The breakdown is considered in detail later.

3. Waves.—Both systems can support wave motions because rotation provides a restoring force in the horizontal direction and stratification provides one in the vertical. Thus for system Ω we have

$$\frac{\partial u}{\partial t} - 2v = -\frac{\partial p}{\partial x} \qquad 11.$$

$$\frac{\partial v}{\partial t} + 2u = 0 \qquad 12.$$

$$\frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} \qquad 13.$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \qquad 14.$$

A wave solution where each of the variables has the form $e^{i(-\omega t+kx+nz)}$ yields the dispersion relation

$$\omega = \pm \frac{2n}{\sqrt{n^2 + k^2}} = \pm 2 \cos \theta \qquad 15.$$

where θ is the angle between the Ω (or z) axis and the direction of propagation. The waves are transverse and in the three-dimensional system they are circularly polarized [Chandraesekhar, (5)]. The frequency is maximum, 2 Ω in dimensional form, for vertically propagating waves. For horizontal "waves" there is no restoring force and the frequency vanishes.

In system S the equations become

$$\frac{\partial w}{\partial t} - 2T = -\frac{\partial p}{\partial z} \qquad 16.$$

$$\frac{\partial T}{\partial t} + 2w = 0 \qquad 17.$$

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} \qquad 18.$$

$$\frac{\partial w}{\partial z} + \frac{\partial u}{\partial x} = 0 \qquad 19.$$

The dispersion relation for a wave form $e^{i(-\omega t+kx+nz)}$ is

$$\omega = \pm \frac{2k}{\sqrt{k^2 + n^2}} = \pm 2 \cos \phi$$

where ϕ is the angle between the horizontal (x) direction and the direction of propagation. The frequency of these transverse waves achieves its maximum value, the buoyancy frequency, N, for horizontally propagating waves, i.e., when the restoring force due to stratification is a maximum.

Because of the anisotropy, when a wave of either system encounters a boundary which is inclined to the direction of the constraint, the reflected wave propagates in a direction which is determined by the constraint rather than by the boundary. This fact can be seen most simply by means of a geometrical construction given by Phillips (14) for a rotating field.

First we rewrite the dispersion relations as

$$n = \pm \alpha k \qquad 21.$$

where $\alpha=\omega/(4-\omega^2)^{1/2}$ in system Ω and $\alpha=(4-\omega^2)^{1/2}/\omega$ in system S. In Figure 1 we have drawn the lines corresponding to Equation 21 for a given value of α (or ω). Also shown is an inclined solid boundary. Now since a reflected wave must have the same frequency as the incoming wave at the boundary, it must lie along one of the two straight lines given by Equation

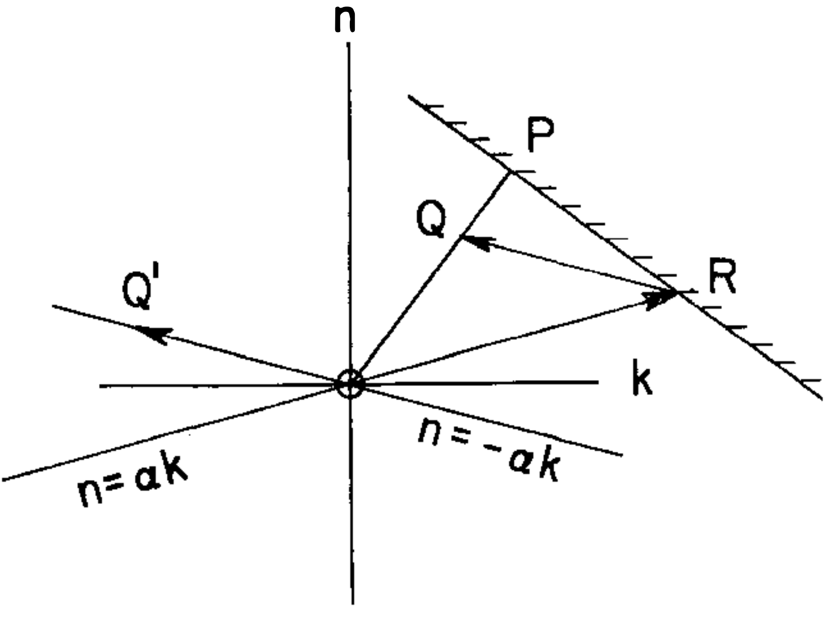

FIG. 1. The reflection of a wave off an inclined boundary for systems Ω and S. For a chosen value of ω, hence α, plane waves must lie on one of the two lines, $n=\pm\alpha k$ which make equal angles with the horizontal axis. The wave component PR parallel to the boundary must have the same magnitude for incident and reflected waves. The incoming wave is OR and the outgoing wave is therefore RQ or OQ' when drawn through the origin.

21. These lines make equal angles with either the vertical or the horizontal direction. Hence, it is evident that the direction of the constraint, i.e., either the horizontal or the vertical for our two systems, determines the line of travel of the reflected wave. To complete the analysis we note that the component of the wave vector parallel to the boundary must have the same length for both the incoming and the outgoing waves. Hence, drawing the perpendicular to the solid boundary through the end point of the incoming wave determines the magnitude of the reflected wave vector. To determine the direction of the reflected wave we need only choose the sign so that the reflected wave propagates back into the fluid rather than through the boundary. Hence, the constraint determines the angle but the boundary determines the magnitude and the sign of the reflected wave vector. In the example shown in the diagram the reflected wave propagates upward to the left. The amplitude of the normal velocity of the reflected wave is the negative of that of the incoming wave to provide cancellation at the boundary. The analogy between the two systems is contained in the identification of α given below Equation 21 or alternatively by keeping α fixed and interchanging n and k.

4. *Linear, steady, dissipative flow.*—Consider system Ω with the fluid occupying the half space $z \leq 0$. A stress or velocity independent of the horizontal coordinates is applied at $z=0$. The steady, linear equations which describe the flow are obtained from Equations 1 and 2 as

$$-2v = E\frac{\partial^2 u}{\partial z^2}, \qquad 2u = E\frac{\partial^2 v}{\partial z^2} \tag{22.}$$

The solution of Equations 22 with u and v vanishingly small as $z \to -\infty$ is

$$u = (u_o \cos \zeta + v_o \sin \zeta)e^{-\zeta} \tag{23.}$$

$$v = (v_o \cos \zeta - u_o \sin \zeta)e^{-\zeta} \tag{24.}$$

where $\zeta = -zE^{-1/2}$, and (u_0, v_0) is the velocity at $z=0$. This solution, first derived by Ekman (6), is an exact solution to the equations of motion. When plotted in the (u, v) hodograph plane, the solution takes the form of a spiral. This so-called Ekman spiral is shown in Figure 2 for the case $v_0=1$, $u_0=0$. The spiral can be observed in a laboratory experiment.

The usefulness of the Ekman solution derives from the fact that it describes the flow in horizontal boundary layers for situations in which horizontal variations are $0(1)$. Vertical variations are $0(E^{-1/2})$ in the Ekman boundary layer and neglect of horizontal derivatives is correct to $0(E^{1/2})$. Eliminating all but one of the dependent variables from Equations 1 to 4 yields (in terms of v)

$$-4\frac{\partial^2 v}{\partial z^2} = E^2\nabla^6 v \tag{25.}$$

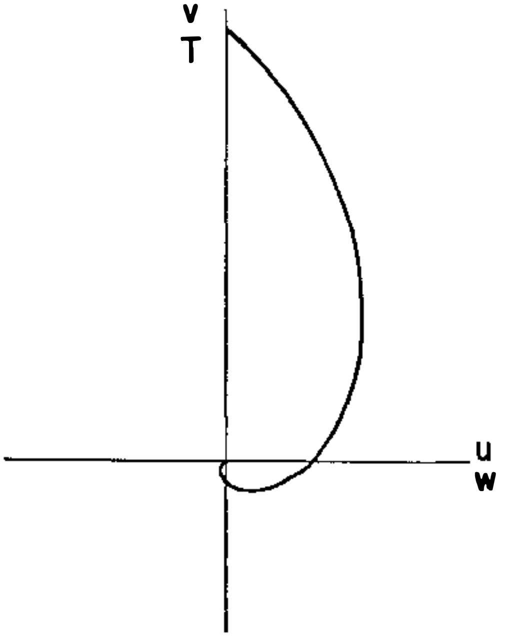

FIG. 2. The Ekman spiral in the (u, v) hodograph plane. At the surface the velocity vector is directed along the v axis. As the depth increases the velocity vector rotates (clockwise for $\Omega > 0$) and decreases in magnitude. For system S the corresponding variables for the buoyancy spiral are (w, T) as shown.

If $E \ll 1$ and all gradients are of $0(1)$, Equation 25 reduces to

$$\frac{\partial^2 v}{\partial z^2} = 0 \qquad 26.$$

to $0(E)$. Equation 26 represents one of the conclusions (the Taylor-Proudman behavior) of geostrophic hydrostatic flow. In the vicinity of horizontal boundaries the variations with respect to z are large and this condition may be expressed by stretching the z coordinate by

$$\frac{\partial}{\partial z} = \pm E^{-1/2} \frac{\partial}{\partial \zeta}, \qquad \frac{\partial}{\partial \zeta} = 0(1) \qquad 27.$$

where the sign is chosen so that ζ increases from the boundary into the fluid. The stretching applied to Equation 25 yields

$$-4 \frac{\partial^2 v}{\partial \zeta^2} = \frac{\partial^6 v}{\partial \zeta^6} \qquad 28.$$

and if boundary solutions with $v \rightarrow 0$ as $\zeta \rightarrow \infty$ are sought, Equation 28 can be integrated twice to yield

$$-4v = \frac{\partial^4 v}{\partial \zeta^4} \tag{29}$$

which is equivalent to Equations 22 when u is eliminated.

In linear dissipative problems the flow can be separated into an interior part (denoted by subscript I) described by the geostrophic, hydrostatic Equation 9, plus a boundary-layer part (denoted by an overbar) given by Equations 22. Boundary conditions are expressed in terms of the total velocity, hence the sum of the interior and boundary-layer parts is given at the boundaries. Since the general solution to the boundary-layer equations is given by Equations 23 and 24, the boundary-layer contribution to the boundary conditions can be expressed in terms of the interior variables. Hence, boundary conditions for the much simpler interior equations can be obtained. The actual procedure follows.

The stretching transformation (Equation 27) used in the continuity Equation 4 yields in terms of the overbar variables

$$\frac{\partial \bar{w}}{\partial \zeta} = \mp E^{1/2} \frac{\partial \bar{u}}{\partial x} \tag{30}$$

Integrating Equation 30 from $\zeta = \infty$ to $\zeta = 0$ and making use of Equations 23 and 24 yield

$$\bar{w}\,|_{\zeta=0} = \pm \frac{E^{1/2}}{2}\left(\frac{\partial \bar{u}_0}{\partial x} + \frac{\partial \bar{v}_0}{\partial x}\right) \tag{31}$$

(For three-dimensional problems $\partial \bar{u}/\partial x$ and $\partial \bar{v}/\partial x$ are replaced by the total horizontal divergence and total vertical vorticity respectively.) But according to the separation procedure described above

$$\bar{v} = v - v_I \tag{32}$$

where v is the total velocity. Hence, Equation 31 becomes

$$[w - w_I]_{\zeta=0} = \pm \frac{E^{1/2}}{2}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} - \frac{\partial v_I}{\partial x}\right]_{\zeta=0} \tag{33}$$

where we have used Equation 9 to set $u_I = 0$.

For example, at a lower horizontal boundary where the positive sign in Equation 33 is valid we have

$$w_{Ib} = w_b - \frac{E^{1/2}}{2}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} - \frac{\partial v_I}{\partial x}\right]_b \tag{34}$$

where the subscript b refers to a bottom boundary. This vertical flow induced in the interior by conditions applied at the boundary is called Ekman suction or pumping.

At an upper horizontal boundary, Equation 33 yields

$$w_{IT} = w_T + \frac{E^{1/2}}{2}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} - \frac{\partial v_I}{\partial x}\right]_T \qquad 35.$$

Since $\partial v_I/\partial z = 0$ by Equation 9, the right hand sides of Equations 34 and 35 are equal and we can write

$$\frac{\partial v_I}{\partial x} = \frac{1}{2}\left[\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right)_T + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right)_b\right] + E^{-1/2}(w_T - w_b) \qquad 36.$$

Thus the interior vertical vorticity is expressed as the average of the horizontal divergences and vertical vorticities applied at the boundaries plus a contribution due to the difference of the applied vertical velocities. Expression 36 is very useful for solutions of rotating problems. An example is described in the next chapter.

A boundary layer can also form in the vicinity of vertical boundaries. Stretching the horizontal coordinate, x, in Equation 25 by means of

$$\frac{\partial}{\partial x} = E^{-1/3}\frac{\partial}{\partial \xi}, \qquad \frac{\partial}{\partial \xi} = 0(1) \qquad 37.$$

where ξ is defined so that it increases from zero with distance from the wall, yields the equation, valid to $0(E^{2/3})$,

$$\frac{\partial^6 v}{\partial \xi^6} = -4\frac{\partial^2 v}{\partial z^2} \qquad 38.$$

The boundary layer within which Equation 38 is applicable is called a Stewartson (19) $E^{1/3}$ layer. The layer is used to satisfy boundary conditions for the vertical and normal velocity components along vertical walls and also to absorb the flow from the Ekman layer when the latter must turn and flow vertically because of the presence of vertical boundaries. Abrupt horizontal changes in the boundary conditions along nonvertical boundaries can also give rise to Stewartson layers.

If the magnitudes of the variables in the $E^{1/3}$ layer are specified so that the layer can absorb the flow from an Ekman layer, the $E^{1/3}$ layer cannot adjust the horizontal velocity tangent to the wall to make it satisfy a general boundary condition. For this purpose it is necessary to introduce another boundary layer, called a Stewartson layer of thickness $E^{1/4}$ and defined by the coordinate stretching

$$\frac{\partial}{\partial x} = E^{-1/4}\frac{\partial}{\partial \eta}, \qquad \frac{\partial}{\partial \eta} = 0(1) \qquad 39.$$

where η increases from zero with distance from the wall. This boundary layer is thicker than the $E^{1/3}$ layer. The existence of the $E^{1/4}$ layer is not formally obvious from an inspection of the equations but can be seen to be necessary when a careful analysis of the magnitudes of the variables is made.

The magnitudes of the variables in the different boundary layers along with the lowest-order dynamical equations are listed in Table I.

The analogous set of equations for system S is evident once one makes the identifications given by I.44. The Ekman layer along a nonvertical boundary goes into the buoyancy layer of thickness $R^{1/2}$ [Prandtl (15), Gill (7)] along a nonhorizontal surface. The buoyancy layer normally adjusts the temperature of the interior fluid to the boundary conditions on a wall (a vertical boundary in the simplest case). The two variables in the hodograph plane for system S are w and T so that the corresponding spiral is not physically observable in the flow. Ekman suction has a counterpoint in buoyancy suction.

The $R^{1/3}$ Stewartson layer in this case adjusts the vertical and horizontal velocities to satisfy boundary conditions on horizontal surfaces. A Stewartson layer of thickness $R^{1/4}$ is necessary to adjust the interior temperature to

TABLE I

	Ekman Layer	$E^{1/3}$ Layer	$E^{1/4}$ Layer
Thickness	$E^{1/2}$	$E^{1/3}$	$E^{1/4}$
Coordinate stretching	$\frac{\partial}{\partial z} = E^{-1/2}\frac{\partial}{\partial \zeta}$	$\frac{\partial}{\partial x} = E^{-1/3}\frac{\partial}{\partial \xi}$	$\frac{\partial}{\partial x} = E^{-1/4}\frac{\partial}{\partial \eta}$
Magnitudes of variables	u, $w = 0(E^{1/2})$ $p = 0(E)$	p, v,	$v = 0(1)$ $w, p =$ $u = 0(E^{1/2})$
Dynamical equations of lowest order	$-2v = \frac{\partial^2 u}{\partial \zeta^2}$ $2u = \frac{\partial^2 v}{\partial \zeta^2}$ $E^{1/2}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial \zeta} = 0$	$2v = E^{-1/3}\frac{\partial p}{\partial \xi}$ $2u = E^{1/3}\frac{\partial^2 v}{\partial \xi^2}$ $\frac{\partial p}{\partial z} = E^{1/3}\frac{\partial^2 w}{\partial \xi^2}$ $\frac{\partial u}{\partial \xi} + E^{1/3}\frac{\partial w}{\partial z} = 0$	$2v = E^{-1/4}\frac{\partial p}{\partial \eta}$ $2u = E^{1/2}\frac{\partial^2 v}{\partial \eta^2}$ $\frac{\partial p}{\partial z} = 0$ $\frac{\partial u}{\partial \eta} + E^{1/4}\frac{\partial w}{\partial z} = 0$

the horizontal mean of the value prescribed along the horizontal boundary.

In system Ω free shear layers, i.e., Stewartson layers of thickness $E^{1/3}$ and $E^{1/4}$, are required to adjust the interior horizontal flow if abrupt changes in the horizontal velocities are prescribed along nonvertical boundaries. Correspondingly, layers of thicknesses $R^{1/3}$ and $R^{1/4}$ are necessary to adjust the interior temperature and vertical velocity if abrupt changes in these quantities are prescribed along nonhorizontal boundaries.

PART III. ANALOGOUS PROBLEMS IN SYSTEMS Ω AND S.

1. Linear, steady, dissipative flows.—A widely studied class of laboratory flows in systems Ω and S is that generated by conditions imposed along boundaries where the Ekman or buoyancy layer is important. The method of solution is that leading to Equation II.36. The present discussion is limited to a physical description of the flow in two simple cases.

Consider a situation where the gap between two vertical walls which are located at $x=\pm 1$ is filled with stratified fluid. For a symmetrical arrangement we suppose that the perturbation temperature at both walls is given by $T_0 \sin \alpha z$, where T_0 and α are positive constants. The boundaries are rigid and all quantities are independent of y. Then by the analogy between systems Ω and S we write the counterpart to Equation II.36 for the vertical perturbation temperature gradient of the interior fluid as[3]

$$\frac{\partial T_I}{\partial z} = \frac{1}{2}\left\{\frac{\partial T}{\partial z}\bigg|_{x=-1} + \frac{\partial T}{\partial z}\bigg|_{x=1}\right\} = \alpha T_0 \cos \alpha z \qquad 1.$$

where the contributions in Equation II.36 due to the values of u and w at the boundaries vanish because the boundaries are rigid. From the analogy to Equations II.34 and II.35 we see that $u_I=0$ everywhere and there is no flow. Hence, the solution corresponds to a simple restratification of the fluid to conform to the applied conditions at the boundaries. There are no boundary layers in this case since they are not needed.

For the anti-symmetric problem we consider the case with $T=T_0 \sin \alpha z$ at $x=-1$ and $T=-T_0 \sin \alpha z$ at $x=1$. Then the analogy to Equation II.36 gives

$$\frac{\partial T_I}{\partial z} = 0 \qquad 2.$$

and the horizontal flow in the interior is obtained from the thermal counterparts of Equations II.34 and II.35 as

[3] For simplicity we have chosen $\sigma=1$ although this is not necessary for these linear steady flows because the Prandtl number can be absorbed into the temperature (25).

$$u_I = -\frac{R^{1/2}}{2}\frac{\partial T}{\partial z}\bigg|_{x=-1} = -\frac{R^{1/2}}{2}\alpha T_o \cos \alpha z. \qquad 3.$$

The vertical velocity and temperature at the boundary layers near the walls are given by the analogues to Equations II.23 and II.24 and are

$$\bar{w} = \mp T_o \sin \alpha z \sin \xi e^{-\xi} \qquad 4.$$

$$\overline{T} = \mp T_o \sin \alpha z \cos \xi e^{-\xi} \qquad 5.$$

where $\xi = R^{-1/2}(1-|x|)$ and the upper (lower) sign is valid near $x = +1(-1)$. In this case the anti-symmetric thermal forcing leaves the temperature of the interior fluid unchanged. However, there are boundary layers in which the fluid in the vicinity of the wall rises where it is heated and sinks where it is cooled. In regions of vertical convergence fluid is pushed out of the boundary layer and an interior flow, given by Equation 3, is generated. The qualitative flow pattern in this case is shown in Figure 3.

For the analogous problems in system Ω the vorticity, $\partial v/\partial x$, is given by $v_* \sin \alpha x$ at $z = \pm 1$ for the symmetric case and by $\mp v_* \sin \alpha x$ at $z = \pm 1$ for the antisymmetric case. In the symmetric problem the relative vorticity of the interior fluid achieves the value $v_* \sin \alpha x$ and there are no boundary layers and no flow in the x or z directions. In the anti-symmetric situation the relative vorticity vanishes in the interior and the (u, v) flow is given by the arrows when Figure 3 is turned on its side.

Although the analogy in more complicated situations exhibits additional interesting behavior, the simplicity of the problem just discussed brings out the parallel between rotating and stratified flows most clearly.

2. Bénard Convection and Couette Flow.—The first mathematical parallel to be drawn between flows in systems Ω and S was presented by Jeffreys (11), though Rayleigh (17) had already pointed out a qualitative similarity. In his article Jeffreys credits G. I. Taylor and R. A. Low for pointing out to him that a mathematical analogy exists between the problems of Bénard convection and Couette flow between rotating cylinders. In these problems the constraint is reversed, i.e., it serves to destabilize the system.

In Bénard convection a fluid layer of infinite horizontal extent is heated uniformly from below and cooled from above. Heat conduction creates a temperature profile which decreases linearly with height from the bottom boundary to the top. Infinitesimal velocity and temperature perturbations are then superimposed on the basic linear temperature field and the stability of the latter is determined by establishing the conditions under which the perturbations can be maintained. The principle of exchange of stabilities was shown by Pellew & Southwell (12) to obtain for the system and the mathematical problem in two-dimensions can be reduced to the single differential equation

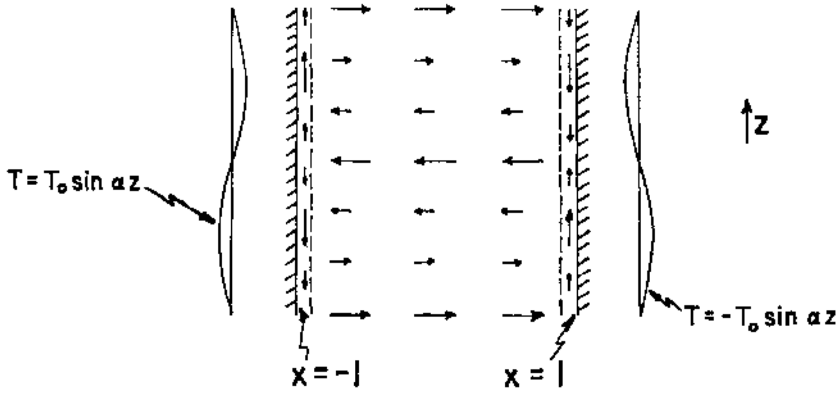

FIG. 3. The circulation in a vertical plane for a slab of stably stratified fluid heated anti-symmetrically at $x = \mp 1$ where $T = \pm T_o \sin \alpha z$. Maximum upflow (downflow) in the buoyancy boundary layers occurs at points of maximum heating (cooling). Maximum buoyancy suction occurs where $\partial T/\partial z$ at the boundaries is maximum.

$$4R_a \frac{\partial^2 w}{\partial x^2} = \nabla^6 w \qquad 6.$$

together with appropriate boundary conditions on w. For perfectly conducting, rigid boundaries the latter are

$$w = 0, \qquad \frac{\partial^4 w}{\partial z^4} + 2 \frac{\partial^4 w}{\partial v^2 \partial z^2} = 0, \qquad \frac{\partial w}{\partial z} = 0, \qquad \text{at } z = \pm 1 \qquad 7.$$

Equation 6 is the thermal analogy of Equation II.25 with the sign of R_a changed because the vertical temperature gradient is negative. As the stability problem is normally posed the coefficient, 4, is incorporated into R_a.

For Couette flow the gap between two concentric right cylinders is filled with fluid and the entire configuration is rotated about the common axis of the cylinders. If the inner cylinder is rotated at a slightly higher rate than the outer, and if the gap thickness is small relative to the mean radius, an azimuthal flow between the cylinders is generated and this mean velocity varies essentially linearly with radius between the values prescribed on the boundaries of the cylinders. The centrifugal force associated with this radially decreasing velocity is destabilizing and serves the same purpose as the destabilizing gravitational force in the thermal problem. The mathematical problem takes the same form in the two problems when the identifications I.44 are made. However, the parameter usually used in the Couette flow problem is a modification of the inverse square of the Ekman number. The parameter is called the Taylor number and includes as a factor the differential rotation of the two cylinders. When the inner cylinder rotates faster than the outer, the multiplying factor is negative thus reflecting the destabilizing effect of the radially decreasing azimuthal velocity.

Jeffreys (11) solved the stability problem for Bénard convection using the assumptions of exchange of stabilities and horizontal periodicity for the perturbation motions. Taylor (21) had already solved the analogous stability problem for Couette flow using the same assumptions and Jeffreys indicated the parallel behavior in his article. The results correspond, as they should, of course, and the critical value of the Rayleigh or Taylor number for the onset of instability is 1708. The analogy breaks down for the general time-dependent problem because in Couette flow the mean azimuthal velocity couples with all three perturbation velocities and instability associated with the shear of the basic velocity profile is possible. This instability mechanism has no counterpart in the thermal problem.

3. *Spin-up versus heat-up.*—Consider an axisymmetric container filled with fluid and rotating uniformly about the axis of symmetry. If the rate of rotation of the container is changed at time $t=0$ to a slightly higher value, the bulk of the fluid will initially lag behind the container but eventually it will also rotate at the new angular velocity. The process of adjustment of the fluid to the new rotation rate has been called "spin-up" by Greenspan & Howard (9) who analyzed the flow for the case where the containers are right cylinders.

The two-dimensional thermal analogy to the spin-up problem is the heat-up problem. Suppose that a contained fluid is stably stratified by a uniform vertical temperature gradient imposed at the side walls. At time $t=0$ the imposed temperature gradient is suddenly increased by a small amount. The response of the fluid can be described physically as follows:

After an interval of the order of a buoyancy period, buoyancy boundary layers form near the side boundaries. At any given level the fluid in the buoyancy layer is warmed and therefore rises. At a slightly higher level the fluid in the buoyancy layer is warmed slightly more and will acquire a larger vertical velocity because of the increased buoyancy force. The boundary layer flow is therefore vertically divergent and fluid is sucked into the buoyancy layer from the interior. For a system heated symmetrically at the sides this buoyancy suction generates a horizontally divergent flow in the interior since fluid goes into both buoyancy boundary layers at the sides. This horizontal divergence generates a flow in the interior which is downward (to compensate for the upward flow in the boundary layers). Fluid at a given level in the interior is replaced by warmer fluid moving downward. (In an enclosed container $R^{1/8}$ and $R^{1/4}$ boundary layers are necessary at the top and bottom boundaries to absorb the horizontal cross flow out of the side boundary layers.) The interior fluid is "heated up" by this indirect circulation.

Since the thickness of the buoyancy layer is $0(R^{1/2})$ the vertical transport in the layer is $0(R^{1/2})$ for unit vertical flow. Hence, the magnitude of the horizontal flow must also be $0(R^{1/2})$. The time for a particle of fluid to flow from the center regions of the system to the lateral boundary is therefore order $R^{-1/2}$ [unit distance/velocity of $0(R^{1/2})$]. Hence the dimensional time of heat-up is $0(\sigma^{1/2}R^{-1/2}N^{-1})$ since unit time corresponds to N^{-1}. With $R \ll 1$, this time is much faster than that due to diffusion alone, $0(\sigma^{1/2}R^{-1}N^{-1})$.

The physical description outlined above parallels that given by Greenspan & Howard (9) for the spin-up process. In the latter the buoyancy layer is replaced by the Ekman layer and the spin-up time is $0(E^{-1/2}\Omega^{-1})$.

PART IV. ANALOGOUS PROBLEMS IN SYSTEMS Ω_S AND S_Ω

The analogy discussed in the examples just presented can be extended by adding stratification to the rotating system and vice-versa. It should be evident that this possibility exists because the parallel between the effects of stratification and rotation has already been exhibited and one might expect that the modifications which occur when these constraints are added to systems which already exhibit analogous behavior will not destroy the analogy.

1. Steady, linear, dissipative flow.—Examples of this type of flow have been presented by Barcilon & Pedlosky (1–3) and by Veronis (25, 26). The simplest example is the modification of the flows discussed in Section III.1.

For the problem in system S_Ω (with $\sigma=1$) we consider once again the configuration with the gap between walls at $x \pm 1$ filled with stratified fluid, but now the entire system is rotating about a vertical axis. The boundary conditions are the same as before and the flow is described by Equations I.39 to I.43 with $\partial/\partial t=0$ and $\epsilon=0$. We suppose that $F=0(R^0)$.

The interior equations, i.e., the parts with uniform derivatives of $0(1)$, are

$$2T = \frac{\partial p}{\partial z}, \qquad w = 0, \qquad \frac{2v}{F} = \frac{\partial p}{\partial x}, \qquad u = 0. \tag{1.}$$

These equations are correct to $0(R)$ and we see that to this order the only velocity possible is a zonal velocity. By eliminating the pressure from Equations 1 we can derive the familiar thermal wind equation

$$\frac{\partial v}{\partial z} = F\frac{\partial T}{\partial x} \tag{2.}$$

Since $u=0$ to $0(R)$, there can be no $0(R^{1/2})$ flux from the buoyancy layers. The solution is thus markedly different from that given in Section III.1 where a horizontal velocity of $0(R^{1/2})$ was generated via the buoyancy layers for the case with anti-symmetric forcing.

In the present case the equations can be ordered by expanding the variables in powers of $R^{1/2}$ and the lowest-order problem for the interior temperature is determined (25) by the equation

$$\nabla^2\left(\frac{\partial^2 T_I}{\partial x^2}+\frac{1}{F^2}\frac{\partial^2 T_I}{\partial z^2}\right)=0 \qquad 3.$$

together with the boundary conditions

$$\frac{\partial T_I}{\partial x}=0, \qquad T_I=T \qquad \text{at } x=\pm 1. \qquad 4.$$

The solution with symmetric boundary conditions, i.e., $T=T_o \sin \alpha z$ on $x=\pm 1$, is

$$T_I=\left(A\cosh \alpha x+B\cosh\frac{\alpha x}{F}\right)\sin \alpha z \qquad 5.$$

where

$$A=\frac{T_o \sinh \alpha/F}{\cosh\alpha \sinh\alpha/F-F\sinh\alpha\cosh\alpha/F}, \qquad B=-\frac{F\sinh\alpha}{\sinh\alpha/F}A \qquad 6.$$

From Equation 2 we obtain

$$v=-\left(A\sinh\alpha x+\frac{B}{F}\sinh\frac{\alpha x}{F}\right)\cos\alpha z. \qquad 7.$$

The remaining (u, w) velocities are $0(R)$ and are thus negligible.

When $\alpha \leq 1$ the solution bears some similarity to the one derived for the pure stratified system in Section III.2. There is effectively no circulation [u and w are $0(R)$] and the temperature is practically constant across the region so that the net effect of the applied boundary conditions is to restratify the fluid. The zonal velocity given by Equation 7 is necessary to balance the horizontal temperature gradient in the rotating system. When $\alpha \gg 1$ the temperature perturbation concentrates near the boundaries and the middle regions of the fluid are more or less unaltered. Hence, the thermal response is quite different from that of system S.

For the anti-symmetric situation the problem is again given by Equations 3 and 4 but with $T=\pm T_o \sin \alpha z$ at $x=\pm 1$. The solution is given by Equations 5, 6, and 7 with sinh and cosh interchanged. The thermal response for the anti-symmetric case differs very much from that of system S. When $\alpha \leq 1$ the interior temperature varies nearly linearly with x between the values imposed at $x=\pm 1$. Recall that in system S the $0(1)$ interior temperature perturbation vanishes. When $\alpha \gg 1$, the thermal response is confined again to boundary regions whose thickness is determined by α. The thermal wind relation, 2, requires a zonal velocity to balance the horizontal temperature gradient.

The corresponding problem for system Ω_S consists of a layer of fluid bounded between plates at $z=\pm 1$ at which a zonal velocity, v, is prescribed by $\pm v_e \sin \alpha x$, with the sign again determining symmetric or anti-symmetric behavior. The solutions parallel those given for system S_Ω and will not be written down here. We note only that the inhibition of buoyancy suction or pumping by the addition of rotation to system S_Ω has as its counterpart the inhibition of Ekman suction or pumping by the addition of stratification to system Ω_e.

The foregoing example is of interest because it provides a picture of the changes in stratified flow brought about by rotation and vice-versa and because it shows how the analogy works (as well as what it means) in a fluid which is both rotating and stratified.

2. *Rotating heat-up versus stratified spin-up.*—Consider the modification due to rotation of the heat-up problem discussed in Section III.3. Assuming that the time scale of the nonrotating heat-up process is appropriate, one can simplify the present problem by stretching the time coordinate by means of the substitution $\partial/\partial t = R^{1/2}\,\partial/\partial_\tau$ where $\partial/\partial_\tau = 0(1)$. An expansion of the variables in powers of $R^{1/2}$ leads to Equations 1 at $0(R^0)$ for the interior variables and to the following set of equations at $0(R^{1/2})$

$$\frac{2v_1}{F} = \frac{\partial p_1}{\partial x}, \qquad \frac{\partial v_0}{\partial \tau} + \frac{2u_1}{F} = 0, \qquad \frac{\partial p_1}{\partial z} = 2T_1,$$
$$\frac{\partial T_0}{\partial \tau} + 2w_1 = 0, \qquad \frac{\partial u_1}{\partial x} + \frac{\partial w_1}{\partial z} = 0 \qquad 8.$$

where subscript 1 denotes $0(R^{1/2})$ variables and subscript 0 denotes the $0(R^0)$ variables governed by Equations 1. In the buoyancy boundary layers the governing equations are

$$-2\overline{T}_e = \sigma^{1/2}\frac{\partial^2 \bar{w}_0}{\partial \xi^2}, \qquad 2\bar{w}_0 = \sigma^{-1/2}\frac{\partial^2 \overline{T}_0}{\partial \xi^2}, \qquad \frac{\partial}{\partial x} = R^{-1/2}\frac{\partial}{\partial \xi} \qquad 9.$$

The velocity boundary conditions are $\mathbf{v}=0$ at the side boundaries. For simplicity we shall choose $T = T_e \sin kz$ at $x = \pm 1$ for $\tau \geq 0$. The task is then to deduce the heat-up time and the spatial form of the response. The analysis is straightforward and follows that of Holton (10) for stratified spin-up. We shall simply state the results here.

We must clarify one point before proceeding further and that is the meaning of "heat-up time." In system S the meaning is clear because the response of the interior of the fluid is independent of the horizontal coordinate and the deduced time response of the interior is valid in the entire domain. In system S_Ω, on the other hand, the time response is a function of the horizontal coordinate. Thus, on the time scale which has been chosen $(R^{-1/2}N^{-1})$, different parts of the fluid will achieve different "final" temperatures as $\tau \to \infty$. By "heat-up time" we mean the e-folding time of the temperature of the fluid locally.

With this point in mind we find that the heat-up time, τ_{hu}, turns out to be

$$\tau_{hu} = \sigma^{1/2}N^{-1}R^{-1/2}\frac{\tanh kF^{-1}}{kF^{-1}} \tag{10.}$$

When $kF^{-1}\rightarrow 0$, τ_{hu} takes on the same value as it does for system S. Hence, either very weak rotation ($F\gg 1$) or very large-scale heating ($k\ll 1$) reproduces the results for the nonrotating heat-up problem. When $kF^{-1}\gg 1$, we find $\tau_{hu}\sim\sigma^{1/2}N^{-1}R^{-1/2}Fk^{-1}$; hence, the heat-up time is reduced. Thus, either very strong rotation ($F\ll 1$) or very small-scale heating at the sides ($k\gg 1$) will reduce the heat-up time drastically.

The stratified spin-up problem can be formulated in an analogous manner and the spin-up time, τ_{su}, is

$$\tau_{su} = \Omega^{-1}E^{-1/2}\frac{\tanh kF}{kF} \tag{11.}$$

with analogous results for the limiting cases.

As we noted above, the interior solution is a function of x. The temperature is given by

$$T_I = T_o\frac{\cosh kx/F}{\cosh k/F}\sin kz(1 - e^{-\tau/\tau_{hu}}). \tag{12.}$$

Again, when $kF^{-1}\rightarrow 0$, we recovei the results for system S, i.e., the temperature adjusts uniformly in x across the fluid. For $kF^{-1}\gg 1$, the adjusted temperature is confined to lateral boundary regions whose thickness is given by Fk^{-1}. Thus either strong rotation or small-scale heating at the sides will tend to concentrate the thermal adjustment to a limited region in the vicinity of the boundaries. This fact is responsible for the reduced heat-up time derived earlier for the case $kF^{-1}\gg 1$ since the thinner region can adjust more quickly. If we redefine the scale of the system in terms of a length given by LFk^{-1}, where L is the dimensional distance between the walls, the heat-up time for this region is the same as for the nonrotating problem, viz., $\sigma^{1/2}N^{-1}R^{-1/2}$.

The limit of strong rotation, $F\ll 1$, in system S_Ω has as its counterpart the limit of strong stratification, $F\gg 1$, in system Ω_S. Hence, the spin-up of a stratified fluid will be confined to a boundary region whose thickness is given by $k^{-1}F^{-1}$. The depth of penetration in the latter case has a direct analogy in the internal radius of deformation, LF, introduced by Rossby (18) to describe the lateral scale of penetration of disturbances near oceanic boundaries or regions of sharp horizontal gradients.

3. Stratified Couette flow versus rotating Bénard convection.—The inhibition of Bénard convection by the rotation of the fluid layer about a vertical axis has been studied in detail by Chandreaskhar (5). In this flow the gravitational force destabilizes the stratified fluid but the imposed rotation acts as a stabilizing force because horizontal pressure gradients tend to be balanced by the Coriolis acceleration; hence, the potential energy available to the system is not released until the constraint is overcome. Analogous behavior is exhibited by stably stratified Couette flow where the differential rotation is destabilizing and the stratification is stabilizing. Thorpe (22) has analyzed the latter problem making liberal use of the analogy and of the results already established by Chandrasekhar (5) and Veronis (23, 24) for the rotating Bénard convection problem.

In the onset of instability in ordinary Bénard convection, the cellular motions which are established carry warm fluid upward and cold fluid downward, thus releasing the potential energy which is generated by the unstable stratification. Associated with these cellular motions are horizontal pressure gradients. When the fluid is rotating, the horizontal pressure gradients are at least partially balanced by Coriolis accelerations which involve motions parallel to the isobars and these motions do not release potential energy. Hence, convection is inhibited. In a similar fashion there is a vertical pressure gradient associated with the cellular motions (azimuthal rolls) in Couette flow which redistribute the angular momentum between the cylinders. When the fluid is stratified, the vertical pressure gradient is at least partially balanced by the buoyancy force, and this balance inhibits the overturning motions necessary for the redistribution of angular momentum. Hence, the onset of instability is inhibited by the stratification.

There are many other characteristics of the two systems which parallel each other. E.g., because a rotating fluid can support oscillatory motions, the onset of instability in rotating Bénard convection may involve oscillatory motions, in contrast to ordinary Bénard convection in which the exchange of stabilities is applicable. For these so-called overstable oscillations to occur the Prandtl number must be smaller than unity. Oscillatory motions can enhance the onset of instability because the local acceleration partially balances the Coriolis acceleration and more of the pressure gradient can thereby be balanced by the direct circulation in time-dependent flow than in steady flow. However, local time variations will generate a phase difference between the temperature and the vertical velocity and this tends to inhibit the release of potential energy. The former, destabilizing, process is more effective than the latter, stabilizing, process when the Prandtl number is less than unity, because a small Prandtl number implies that thermal diffusion is large. The larger the thermal diffusion, the more it tends to balance the vertical convection of heat, hence the more in phase are the thermal fluctuations and the vertical velocities. The stabilizing temperature oscillations are thereby minimized.

In stratified Couette flow vertical motions must carry colder fluid upward and this process requires work. Hence, the more out of phase the temperature fluctuations are with the vertical velocities, the less inhibiting is the stratification to the motions which redistribute the angular momentum. A large value of the Prandtl number implies a small effect of thermal diffusion.

This means that vertical velocity and temperature perturbations are thus out of phase. Hence, the stabilizing effect of the temperature is minimized.

The onset of finite-amplitude instability in rotating Bénard convection (24) has its exact counterpart in the finite-amplitude instability in stratified Couette flow. This instability will also occur optimally for small and large Prandtl number fluids respectively in the two systems. The same physical reasoning which was outlined above for the onset of overstable oscillations is applicable to the finite-amplitude instability.

Part V. The Breakdown of the Analogy in Three-Dimensional Flow

When the boundary temperature in the example discussed in Section III.1 depends on the y direction as well, Equations II.5–II.8 must be augmented by the y equation of motion

$$\frac{\partial v}{\partial t} + \epsilon \mathbf{v}\cdot\nabla v = -\frac{\partial p}{\partial y} + \sigma^{1/2}R\nabla^2 v \qquad 1.$$

and all y variations must be included. The procedure of separating the variables into interior and boundary-layer parts is still appropriate and one finds from the equations for the interior variables that the following relations hold at $0(R^0)$ and $0(R^{1/2})$

$$2T_I = \frac{\partial p_I}{\partial z}, \qquad \frac{\partial p_I}{\partial x} = 0, \qquad \frac{\partial p_I}{\partial y} = 0, \qquad w = 0 \qquad 2.$$

Hence,

$$\frac{\partial T_I}{\partial x} = 0, \qquad \frac{\partial T_I}{\partial y} = 0. \qquad 3.$$

If the applied boundary conditions are harmonic in y as well as z, Equations 3 state that the interior temperature is unaltered at $0(R^0)$ and $0(R^{1/2})$. In other words the fluid is no longer simply restratified when it is heated symmetrically, as it was for the two-dimensional problem. The addition of the third dimension allows the system to respond so that the interior temperature is effectively unaltered by the boundary temperatures.

The lowest-order problem for the interior variables can be formulated by making use of the expansion in powers of $R^{1/2}$ (25). Here we simply state that the lowest-order problem for the u velocity is given by the equations

$$\nabla^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0, \qquad 4.$$

$$u\big|_{x=\pm 1} = \mp \frac{R^{1/2}}{2}\frac{\partial T}{\partial z}\bigg|_{x=\pm 1}, \qquad \frac{\partial u}{\partial x}\bigg|_{x=\pm 1} = 0 \qquad 5.$$

where the temperature in Equation 5 is the specified temperature at the boundaries. The equation which determines the v velocity is the two-dimensional continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \qquad 6.$$

Thus, the lower-order problem yields an unaltered interior temperature and $0(R^{1/2})$ velocities in the interior.

If the boundary temperature is given by

$$T = T_o \sin \alpha y \sin \beta z \qquad \text{at } x = \pm 1 \qquad 7.$$

or

$$T = \pm T_o \sin \alpha y \sin \beta z \qquad \text{at } x = \mp 1 \qquad 8.$$

the problem is straightforward. The forms of u and v are given by

$$u = U(x) \sin \alpha y \cos \beta z, \qquad v = V(x) \cos \alpha y \cos \beta z. \qquad 9.$$

For the case of symmetric forcing, Equation 7 is applicable and the solution is

$$U = \frac{\beta T_o R^{1/2}\{\alpha \cosh \alpha \sinh \sqrt{\alpha^2+\beta^2}\, x - \sqrt{\alpha^2+\beta^2} \cosh \sqrt{\alpha^2+\beta^2} \sinh \alpha x\}}{2\{\alpha \cosh \alpha \sinh \sqrt{\alpha^2+\beta^2} - \sqrt{\alpha^2+\beta^2} \cosh \sqrt{\alpha^2+\beta^2} \sinh \alpha\}} \qquad 10.$$

$$V = \frac{\beta T_o R^{1/2}\{\cosh \alpha \cosh \sqrt{\alpha^2+\beta^2}\, x - \cosh \sqrt{\alpha^2+\beta^2} \cosh \alpha x\}}{2\{\alpha \cosh \alpha \sinh \sqrt{\alpha^2+\beta^2} - \sqrt{\alpha^2+\beta^2} \cosh \sqrt{\alpha^2+\beta^2} \sinh \alpha\}} \qquad 11.$$

For anti-symmetric forcing T is given by Equation 8 at $x = \pm 1$ and the solution is the same as Equations 10 and 11 but with $-\beta$ replacing β and cosh and sinh interchanged throughout.

With $\alpha = \beta = 1$ the solution with anti-symmetric forcing is similar to that discussed in Section III.3. The fluid flows upward (downward) in boundary-layer regions which are in the vicinity of positive (negative) boundary temperatures. In regions where the boundary value of $\partial T/\partial Z$ is positive (negative) fluid is sucked into (pumped out of) the boundary layer. Hence, the flow looks like that of Figure 3. The only modification is a relatively weak v velocity.

For symmetric forcing the two-dimensional problem shows no $0(R^{1/2})$ flow in the interior. (If one were to consider the initial-value problem, there would be an initial period of heat-up in which the restratification would be achieved and after that the flow would cease.) In the steady three-dimensional problem the fluid rises in boundary layers where it is warmed and is pumped into (sucked from) the interior where the boundary value of $\partial T/\partial Z$ is negative (positive). A particle which leaves the boundary layer can enter the interior where it can flow unimpeded to the right or left and return to a portion of the boundary layer where conditions require an inflow. A sketch of two particle trajectories is shown in Figure 4.

If $\alpha \gg 1$ and $\beta \sim 1$, the solutions take the following asymptotic form. For symmetric heating

$$U \approx \pm \tfrac{1}{2}\beta T_o R^{1/2}\{1 + \alpha(1 \mp x)\} e^{-\alpha(1\mp x)}, \qquad x \gtrless 0 \qquad 12.$$

$$V \approx \tfrac{1}{2}\alpha\beta T_o R^{1/2}(1 \mp x) e^{-\alpha(1\mp x)}, \qquad x \gtrless 0. \qquad 13.$$

For anti-symmetric heating the solution is

$$U \approx - \tfrac{1}{2}\beta T_o R^{1/2}\{1 + \alpha(1 \mp x)\} e^{-\alpha(1\mp x)}, \qquad x \gtrless 0 \qquad 14.$$

$$V \approx \mp \tfrac{1}{2}\alpha\beta T_o R^{1/2}(1 \mp x) e^{-\alpha(1\mp x)}, \qquad x \gtrless 0. \qquad 15.$$

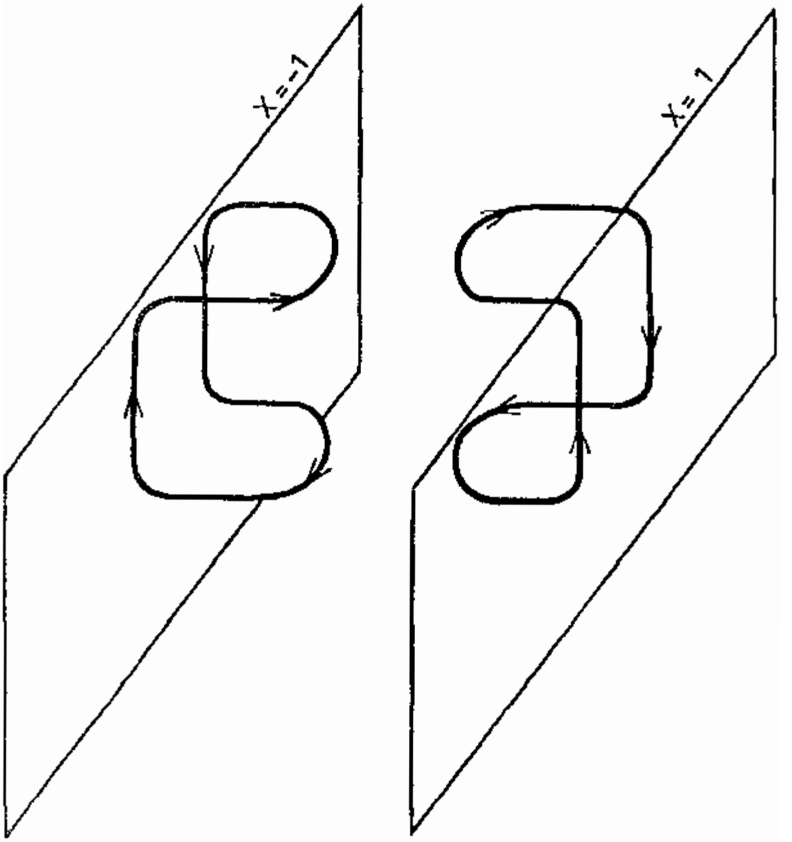

FIG. 4. A schematic picture of particle trajectories in the regions $x \lesseqgtr 0$ for a stably stratified fluid with symmetric boundary temperatures $T = T_0 \sin y \sin z$ at $x = \pm 1$. A particle enters (leaves) the boundary layer in a region where $\partial T/\partial z > 0 (< 0)$ at the boundary. Flow in the interior occurs in a horizontal plane where the temperature is constant. The thermal adjustment of a particle occurs in the boundary layer.

In the symmetric case the flow trajectories look like those of Figure 4 but they are confined to boundary regions whose thickness is $0(\alpha^{-1})$. The y-scale of the flow is also $0(\alpha^{-1})$, of course. The amplitude of the velocities is considerably larger, by $0(\alpha)$, than that of the flow with $\alpha=1$.

For anti-symmetric heating trajectories again look like those of Figure 4 but the direction of one of the trajectories must be reversed. In this case the interior fluid finds it more convenient to flow laterally to an adjacent position on the same boundary rather than to cross to the opposite boundary where the appropriate inflow position is much farther away.

When the corresponding problem in system Ω is formulated, the solution shows that the addition of the third dimension does not alter the qualitative two-dimensional behavior of the flow. The reason is that flow in the x direction is subject to the constraint of rotation and the addition of y variations simply adds another constraining direction. In system S, on the other hand, the additional direction is one along which the fluid can flow unconstrained and it can materially alter the behavior of the system.

A similar argument (26) shows that the analogy breaks down when flows in system S_Ω are extended to three dimensions.

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