Turbulence Theory
— An Introduction
P. Diamond

E.) Basics of Fluid Turbulence (30)

Characteristics of Fluid Turbulence:
- broad range of spatiotemporal scales excited
- decay of large scale energy ⇐ need "turbulence input/stirring to maintain stationarity" - The legacy of A.N. Kolmogorov
- energy input dissipated as heat (to maintain stationarity) ⇐ enstrophy
- irreversible mixing occurs ⇐ i.e. passive tracers
- intermittency manifested
  - i.e. spatially coherent structures (i.e. vortices) occur temporally bursts may/may not probe tracer
- self-similarity/scale-similarity:
  - turbulence looks the same on all scales, except the very largest (stirring) and the very smallest (dissipation)

Caveat: Intermittency = memory of large scales, on small
Navier Stokes Equation - Describes Fluid

\[ \frac{\partial V}{\partial t} + V \cdot \nabla V = -\nabla P + \nu \nabla^2 V \]

Advection / Pressure, Viscous / Diffusion / Nonlinearity / Momentum

\[ \nabla \cdot V = 0 \quad \text{incompressibility} \]

Note: Pressure determined from incompressibility

\[ \nabla \left[ \frac{\partial}{\partial t} \left[ \frac{\partial}{\partial x} + V \cdot \nabla V \right] \right] = -\nabla^2 P + \nu \nabla^2 (V \cdot V) \]

More generally, can eliminate \( P \)

\[ \partial_t V_i + (\partial_i - \delta_{i\epsilon} \nabla^2) \partial_j (V_j V_i) = \nu \nabla^2 V_i \]
Key Parameter: Reynolds #

\[ Re = \frac{\nu m^2}{V} \sim \frac{V(L)}{L} \]

- \( Re \) usually referenced to largest scale

\[ l = L_{\text{max}} \]

\[ V(L) = \text{largest scale velocity} \]

- \( Re \) always referenced to a particular scale

\[ L_{\text{max}}, \lambda = \left( \frac{\left\langle V_i V_j \right\rangle}{\left\langle V_i^2 \right\rangle} \right)^{-1/2} \]

(Taylor Scale)

(\( Re = 1 \))

- \( Re \gg 1 \) in turbulent pipe flow, atmosphere, etc.

\[ Re \sim 10^6 - 10^8 \text{, etc.} \]

i.e., planetary boundary layer: \( h_{bu} \sim 1 \text{ km} \)

\[ h_{ls} \sim 10^5 \text{ cm} \]

\[ \Rightarrow 6 \text{ decades} \]

- \( Re \) : measure of ratio of inertial mixing of momentum to collisional mixing.
 Experimental laws of Fully Developed Turbulence

Much/most of turbulence theory is empirically motivated. Experimental data results preceded sophisticated theoretical analyses...

The experimental facts:

1) 2/3 law (Mundane)

In a turbulent flow with $Re \gg 1$, $\langle dU(l)^2 \rangle$ (mean square velocity increment between two scales $l_s$ separated by distance $l$) scales as $l^{2/3}$.

\[ dU(l) = |U(l+\delta) - U(l)| \]

\[ S_2(l) = \langle dU(l)^2 \rangle \sim l^{2/3} \]

2nd order structure function

\[ Re S_2 \]

slopes $3/2$

dissipation range $d(l)$
By law of Finite Energy Dissipation (Profound)

If in an experiment on turbulent flow all the control parameters are kept the same, except the viscosity, which is lowered as much as possible, the energy dissipation per unit mass $dE/dt$ behaves on a way consistent with a finite limit.

What means 'Energy Dissipation Rate'?

\[ F_d = \frac{1}{2} C_d \rho S U^2 \]

\[ \text{Face surface area} \]

\[ \text{C.I.} \]

\[ \frac{1}{A} \]

\[ \rho = \cos(u \rho) U \rightarrow \text{momentum of air of slug?} \]

\[ M = \rho S U^2 + \text{mass} \]

\[ V = U \]

\[ \text{if assume air momentum completely transferred to can} \]

\[ \frac{dP_{\text{can}}}{dt} = F_d = \rho S U^2 \]
\[ C_0(Re) = \text{drag coefficient (slowly varying function of } Re \text{ depends on shape, etc.)} \]

\[ F_d = C_0 \rho S U^2 \]

\[ \therefore P_d = F_d U \]

\[ \Rightarrow P_d = \frac{C_0 \rho S U^3}{2} \]

**Energy dissipation rate**

\[ E = \frac{P_d}{\text{Mass}} \]

(Re: Volume)

\[ = \frac{C_0 \rho U^3}{2} \]

also N.S \[ \Rightarrow \Delta <V^2> \sim -\langle V \cdot \nabla V \rangle \]

\[ \Rightarrow \text{Why should we care?} \]

Note, energy budget:

\[ \frac{\partial V_i}{\partial t} + V_j \frac{\partial V_i}{\partial x_j} = -\nabla p + \rho \nabla^2 V_i \]
\[ \frac{\partial V_i}{\partial t} + \nabla \cdot \mathbf{V}_i = -\nabla P + \mathbf{F} \]

\[ \langle > \rangle \equiv \text{ensemble (fast space-time avg.)} \]

\[ \frac{\partial}{\partial t} \langle V_i^2 \rangle + \langle \nabla \cdot \mathbf{V}_i \rangle = -\nabla \langle P \rangle \]

\[ \text{surface terms} = \langle \nabla \cdot \mathbf{V}_i \rangle \]

\[ \text{upon IBP} \]

\[ \frac{\partial}{\partial t} \langle V_i^2 \rangle = -\nabla \langle \nabla V_i^2 \rangle \]

but \[ E = -\frac{\partial}{\partial t} \langle V_i^2 \rangle \]

\[ E = \nabla \langle \nabla V_i^2 \rangle \]

\[ \text{experiments suggest that } E \rightarrow \text{finite as } r \rightarrow 0 \]

\[ \text{remarkable} \]

\[ \nabla \text{suggests that extremely large } DV \]

\[ \text{forms as } r \rightarrow 0 \]

\[ \text{singular vortex sheets} \]

\[ \Rightarrow \text{singular velocity gradients, formed in} \]

\[ \text{limit of weak viscosity} \]

\[ \text{Heart of turbulence problem is grappling with} \]

\[ \text{Singularity (especially its degree) of velocity gradients} \]
Dissipation Law

N.B.: Singularity formation is at the heart of why turbulence is a "hard" problem.

Re: Dissipation Law:

\[ \varepsilon \sim \frac{U^3}{L} \sim \frac{U^2}{(L/U)} \]

\[ \sim \frac{K.E.}{\text{circulation time}} \text{ per Mass} \]

i.e. in macro circulation time, a finite fraction of (more) kinetic energy is dissipated by viscosity.

\( \Rightarrow \) dissipation time scale is \( (L/U) \).

K.41: \( K \) \text{ theory of turbulence}

Implications.

In the limit of \( Re \to \infty \) all possible symmetries of the Navier-Stokes equation, usually broken by the mechanisms producing the turbulent flow, are restored in a statistical sense at small scales and away from boundaries.
What means?

"Small scales": \( l \ll L_0 \).

Integral scale \( \rightarrow \) characteristic of production.

- symmetries

First, symmetries of Navier-Stokes Eqn. 7.

a) space translations \( \mathbf{r} \rightarrow \mathbf{r} + \mathbf{b} \) (no explicit \( f \) dep.)

b) time translation \( t \rightarrow t + \tau \) (no \( t \) dep.)

c) Galilean boosts (no frame dep.) \( \left\{ \begin{array}{c}
\mathbf{r} \rightarrow \mathbf{r} + \mathbf{v} t \\
\mathbf{v} \rightarrow \mathbf{v} + \mathbf{u}
\end{array} \right. \)

\( \Rightarrow \quad 2\mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla P + \eta \nabla^2 \mathbf{v} \)

cast \( \Rightarrow \quad -\mathbf{v} \cdot \nabla \mathbf{v} + \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla P + \eta \nabla^2 \mathbf{v} \)

d) Parity (left-right) \( x \rightarrow -x \), \( \mathbf{v} \rightarrow -\mathbf{v} \)

e) Rotation (no preferred direction) \( \left\{ \begin{array}{c}
\mathbf{r} \rightarrow \mathbf{R} \mathbf{r} \\
\mathbf{v} \rightarrow \mathbf{R} \mathbf{v}
\end{array} \right. \)
e.) Scaling (for $x \to 0$) $\Rightarrow$ critical; scale elimination

$$y \in \mathcal{Y}, f \Rightarrow \lambda y, \lambda^h y, \lambda^h f$$

c.e. $\frac{\partial v}{\partial t} + v \cdot \nabla v = -\nabla p$

$$x \Rightarrow \lambda^a x$$

$$f \Rightarrow \lambda^b f$$

$$\frac{\partial v}{\partial t} + \lambda^a v \cdot \nabla \lambda^a v = -\nabla \lambda^b p$$

$\Rightarrow$ From $\nabla \cdot v = 0$

$$\lambda^{2a-1} = \lambda^{a-b}$$

$\Rightarrow$ $a = b$ $\Rightarrow$ $b = 1 - a$

Thus, scalings $r \Rightarrow r^a$, $V \Rightarrow \lambda^a V$, $t \Rightarrow \lambda^h t$

Now, turbulence onset $\Rightarrow$ symmetry breaking!

c.e. 1) $kH$: shear breaks translational invariance.

2) Rigid body boundary flow, etc.

3) Flushing toilet, etc.

etc.
However, fully developed turbulence tends to restore symmetry, except near boundaries on small scale. (b.c. → boundary conditions)

c.i.e. if \( \nu(x, t) = \nu(t+\tau) - \nu(t) \)

\[ \frac{\partial \nu}{\partial \tau} = \frac{\partial \nu}{\partial x} \]

(similarly: isotropy, parity

(facilitates scaling approach)

**H2** For \( Re \to \infty \) turbulence at small scales and away from boundaries, the flow is self-similar at small scales, i.e., possesses a unique scaling exponent \( \nu / f \)

\[ \nu(x, \lambda t) \to \lambda \nu(x, t) \]

(\( \nu \) addresses \( \lambda^3 \) law)

**H3** With assumptions similar to H1, the turbulent flow has a finite, nonvanishing mean rate of dissipation \( \varepsilon \) per unit mass.

\[ Re \to \infty \Rightarrow \nu \to 0 \text{ with } V_0 = V_{rms} \text{ be fixed} \]

\[ \varepsilon = V_0^3 / \lambda_0 \]
Alternative (not necessary): Kolmogorov's Second Universality Assumption: In the limit of infinite Reynolds number all the small-scale statistical properties are uniquely and universally determined by the scale \( \ell \) and the mean energy dissipation rate \( \varepsilon \).

\[
\frac{-4}{5} \varepsilon \ell = \langle u'^3 \rangle
\]

**Implications:**

- \( \langle dV \ell \rangle^2 = S_2 \quad ? \)
  
  \( S_2 \sim L^2 / T^2 \quad \text{dimensionally} \)

Now \( \varepsilon \sim L^2 / T^3 \)

\[ \Rightarrow \langle dV \ell \rangle^2 \sim \varepsilon^{2/3} L^{2/3} \Rightarrow \text{recovery of \( k \) law} \]

Also, implies \( h = 1/3 \Rightarrow \text{scaling exponents etc.} \)

\[ H_1, H_2, H_3 \ (2^{\text{nd}} \text{ Universality Assumption}) \Rightarrow H_4 \text{ phenomenology} \]
K 41 Phenomenology

Picture: (Richardson) Cascade / Eddy Mitosis

\[ l_0 \]

\[ l_1 = \alpha l_0 \] (\( \alpha < 1 \))

\[ l_n = \alpha^n l_0 \]

\[ l_d \]

Key Idea: Flux of energy in \( 'scale space' \)
from \( l_0 \) (integral scale) to \( l_d \)
(dissipation scale)

- \( \alpha \) \( \to \) energy flux is self-similar
- \( \alpha \) \( \to \) symmetry restoration.

\( \rightarrow \) energy dissipation \( \rightarrow \) finite limit as \( \alpha \rightarrow 0 \)
(i.e., endpoint re-adjustment)

\( \rightarrow \) K-Similarity \( \rightarrow \) 2/3 law

\[ S_2 = C \left( \frac{l}{l_0} \right)^{2/3} \]

\[ l_0 \rightarrow l_0 + \lambda \]

\( C \rightarrow C \lambda^{-2} \)
Ingredients on K41 Phenomenology:

- $l$: scale parameter, eddy scale
- $V(l)$: $V(l) \sim (\langle 0^2 \rangle)^{1/2}$
  - eddy velocity
  - $0^2 \sim (V(l+\epsilon) - V(l)) \cdot \frac{1}{\epsilon}$

  $\equiv$ longitudinal velocity increment

- $V_0$: rms velocity fluctuation (large scale dominated)
  - $V(\infty) \sim V_0$

- $T(l)$: eddy lifetime/turn-over rate
  - characteristic rate of transfer thru scale $l$

Self-similarity: energy throughput rate is scale invariant

- $\varepsilon = \frac{V(l)^2}{T(l)}$
  - energy dissipation rate
  - $\rightarrow$ scale $l$, life-time

\[ \text{ow, } T(l)? \]
\[ \sim (l) \rightarrow \text{"lifetime" of structure of scale } l \]
\[ \rightarrow \text{ i.e. time for structure to be distorted out of existence} \]

Scales \( l' > l \): 
- Reduced eddy, apply Galilean boost, but don't affect life-time.
- Irrelevant for symmetry under random Galilean transformations.
- Also violates symmetry restoration.

Scales \( l < l' \): 
- Irrelevant as very little energy/sheer in such eddys/scales.
- Scales which:
\[ \begin{align*}
3 & \rightarrow 4 \\
2 & \rightarrow \frac{3}{l/\sqrt{v_0}} \\
\end{align*} \]

\[ T(l) \sim \frac{l}{\Theta(l)} : \text{ cascade local in scale space.} \]
\[ \varepsilon = \frac{\langle \varepsilon \rangle}{3} \]

\[ \Rightarrow \frac{V(\varepsilon) \sim \langle \varepsilon \rangle^{1/3}}{V(\varepsilon)^3 \sim \varepsilon^{2/3}} \]

- Verifies K.41 scaling relation
- For spectrum:

If \( \text{E}(k) = \frac{1}{2} |\text{V}(k)|^2 \)

s/t \( E = \int dk \text{E}(k) \) \{ c.i.e. absorbs density of states \}

then \( V(\varepsilon) = \int \frac{1}{k} dk \text{E}(k) \)

\[ k^{(1-n-1)} \]

\[ V(\varepsilon)^2 \sim \varepsilon^{2/3} = \varepsilon^{2/3} V_{\varepsilon}^{2/3} \]

\[ \Rightarrow \left[ \text{E}(k) = \varepsilon^{2/3} k^{-5/3} \right] \]

Kolmogorov spectrum

\[ e^{\varepsilon} \]

at \( \varepsilon_0 \): \( V_0 \sim \varepsilon^{1/3} \varepsilon_0 \Rightarrow \frac{V_0^3}{\varepsilon_0} = \varepsilon \).
It seems like the content is written in a non-English language or a mix of symbols and text that is difficult to interpret. There might be an equation or a mathematical expression, but the handwriting is unclear. If you have clearer images or a more context, please provide them, and I can assist better.
Counting Degrees of Freedom

How big is the inertial range?

\[ v \sim l_0 \sim \frac{l_0}{(\nu v / \bar{u}^3)^{1/4}} \]

\[ \# L's \sim \frac{1}{(\nu v / \bar{u}^3)^{1/4}} \sim \frac{(\nu v / \bar{u}^3)^{1/4}}{y^{3/4}} \sim \frac{(\nu v / \bar{u}^3)^{1/4}}{y^{3/4}} \sim \text{Re}^{3/4} \]

Number of degrees of freedom for 3D turbulence is:

\[ N \sim \text{Re}^{7/4} \]

This number of grid points to resolve range of scales in numerical simulation.

Now, i.e. atmospheric boundary layer:

\[ l_0 \sim 1 \text{Km} \]

\[ l_1 \sim 1 \text{mm} \]

\[ \text{Re} \sim 10^6 \Rightarrow \frac{\# \text{grid points}}{\text{Re}} \sim 10^{18} \]

\[ \Rightarrow \text{subgrid scale modelling} \ldots \]

**B**: Sometimes able to exploit reduced degrees of freedom models, i.e. when some class of scales slaved to others.
Exercises:

1) Consider a passive scalar with concentration C:

\[ \frac{\partial C}{\partial t} + \mathbf{v} \cdot \nabla C - \nu \nabla^2 C = \tilde{f}_c \]

Concentration rate in H41 turbulence is:

\[ \alpha = C_0 \frac{\nu}{\ell_0} \]

\[ \Rightarrow \alpha = \frac{C_0}{\ell_0} \]

\[ \Rightarrow a) \text{ Calculate } H41 \text{ spectrum for } C. \]

\[ b) \text{ What if } \nu \ll \nu ? \]

\[ \Rightarrow \text{ Consider incompressible turbulence with } M = \frac{V_0}{C_s} \ll 1. \]

\[ \text{Show: } \frac{\lambda_d}{R_0} \sim M^{-1} R_0^{1/4} \]

\[ \Rightarrow \text{ Validity of continuum hydrodynamics gets better at high } R_0. \]
Particle Separation / Richardson Law

Consider 2 particles (text) in \( k-\varepsilon \) turbulence, rate of separation?

- Larger eddies advect both
- Smaller eddies do nothing

Divergence controlled by eddies of scale \( \lambda \sim |x_1 - x_2| \).

If \( \lambda \equiv |x_1 - x_2| \)

\[
\frac{d\lambda}{dt} = v(\lambda) = \varepsilon^{1/3} \lambda^{1/3}
\]

\[
\lambda^{2/3} = \varepsilon^{1/3} t
\]

\[\lambda \sim \varepsilon^{1/2} t^{3/2}\]

Richardson's 3/2 Law

N.B.: Non-diffusive!

\( x^2 = \varepsilon t^3 \Rightarrow T_{sep.} \sim x^2 / \varepsilon^{1/3} \)

N.B.:

- Process is self-accelerating \( \Rightarrow \) large eddies move faster
- Non-diffusive
Turbulent Pipe Flow

Till now → homogeneous flow in a periodic box
→ cascade in scale space (Kolmogorov)

Now → inhomogeneous flow in a pipe
→ momentum transport in a turbulent boundary layer (Prandtl)

Consider turbulent pipe flow:

\[ z \rightarrow U(z, r) \rightarrow \frac{d}{dr} \rightarrow \] logarithmic profile (inertial sublayer)
  (virtually all high Re experiments \( \sim \) profile consistency)
→ linear profile (viscous sublayer)

Common features of pipe flow:

- linear \( \rightarrow \) logarithmic \( U(x) \) profile
- logarithmic profile persists over a broad range of \( Re \)
  \( (Re = 2Ua/\nu) \)
The logarithmic profile "universal" ("law of the wall") increases with increasing $Re$ discontinuously \[ \lambda = 2a \Delta p/l \]

\[ \frac{1}{2} \rho U^2 \]

\[ \text{mean flow energy} \]

then \[ \ln(100 \lambda) \]

\[ \ln R \]

laminar branch

- turbulent resistance curve universal.

2. What is going on?

no slip boundary condition

\[ U = U(x) \to 0 \quad x \to 0 \]

\[ U = U(x) \quad \text{momentum flux to wall} \]
→ momentum flux to wall = stress on the wall

→ wall stress must balance pressure drop for steady flow

\[ \text{Wall stress: } \rho U_f^2 \]

\[ U_f \equiv \text{friction velocity} \]

\[ \frac{\Delta P}{l} = \frac{\rho U_f^2}{\pi l} \]

\[ \text{Force on fluid} = (\text{Pressure drop})A \text{ flow} \]

\[ \text{Force on wall} = \rho U_f^2 A \text{ wall} \]

\[ \frac{\Delta P}{l} = \rho U_f^2 \frac{2\pi a}{l} \]

\[ U_f = \left[ \frac{\Delta P/2\rho}{\pi \left( \frac{a}{2l} \right)^2} \right]^{1/2} \]

Friction Velocity
2) Dimensional reasoning in pipe flow inertial sublayer have.

3) Key point: Assumption of scale invariance.

\[ \text{Re}_d = \frac{\nu}{\sqrt{\kappa}} \leq \frac{1}{4} \]

\[ \sqrt{\kappa} \leq 4 \]

Dimensionless parameters have.

\( \text{Re}_d \) - typical velocity of turbulence in turbulent pipe.
The simplest form for $\frac{dU}{dx}$ is:

$$\frac{dU}{dx} = \frac{U_*}{x}$$

$$\Rightarrow U = \frac{U_* \ln \left( \frac{x}{x_0} \right)}{K} = \frac{U_* \ln x + \text{const.}}{K}$$

- Logarithmic profile (consequence of scale invariance in pipe flow)
- $K \approx 0.4$ universal constant $\rightarrow$ von-Karman constant
- $x_0$ width of viscous sublayer $\approx \frac{v}{U_*}$

2) Physical Reasoning

Stationary flow $\Rightarrow$

Momentum flux to well $= \text{Pressure drop}$
\[ \langle \tilde{V}_x \tilde{V}_z \rangle = U^2_k \]

Reynolds stress

\[ \langle \tilde{V}_x \tilde{V}_z \rangle = \Gamma_p \]

\( \Gamma_p \) momentum flux

\[ \Gamma_p / \rho = U^2_k \]

Now to calculate \( \langle \tilde{V}_x \tilde{V}_z \rangle \):


**take velocity fluctuation as generated by mixing of \( U(x) \), so**

\[ \tilde{V}_z \sim \frac{1}{L} \frac{\partial U}{\partial x} \]

"mixing length"

analogous to Chapman-Enskog expansion, i.e.

\[ L \Rightarrow l_{\text{mix}} \]

\[ \nu \Rightarrow \nu_{\text{th}} \]
Here, scale invariance \( \lambda \sim x \)

\[
\langle v_x v_b \rangle = \langle v_x \rangle \frac{\partial \langle u \rangle}{\partial x}
\]

\[
\approx u_x \frac{\partial \langle u \rangle}{\partial x}
\]

\[
Y_T = u_x x \ \rightarrow \ \text{"eddy viscosity"} \\quad \text{"turbulent viscosity"} \quad \text{key concept}
\]

rate of turbulent transport of momentum

then momentum balance \( \Rightarrow \)

\[
u_x \frac{\partial \langle u \rangle}{\partial x} = u_x^2
\]

\( \Rightarrow \)

\[
U = \frac{u_x}{H} \ln \left( \frac{x}{x_0} \right) \quad \rightarrow \quad \text{logarithmic profile}
\]

\( \rightarrow \text{law of the wall} \)
Some comments:

"Mixing length theory always works . . . provided you know the mixing length . . . ."

- P. D.

why a single value of velocity, i.e. $u_*$?

Consistent with mixing length hypothesis, velocity fluctuations generated by mixing of mean flow gradient, i.e.

$\bar{v} - l \frac{2u}{\partial x} \propto \frac{x}{\partial x}$

consistent. 2. Assumption consistent with:

- logarithmic profile
- scale invariance


**Viscous Sublayer / Cut-off of Mixed Layer?**

\[ \nu = \nu_f < 1 \]

\[ U_{\text{mix}} \leq x \]

\[ x \leq \frac{\nu}{U_{\text{mix}}} = x_0 \]

**Viscous Sublayer Scale.**

In viscous sublayer flow linear:

\[ \frac{\partial u}{\partial x} = u_x^2 \]

\[ U = \frac{U_x^2 x}{\nu} \]

\[ \Rightarrow \text{note effect of turbulence is to:} \]

- flatten profile
- higher transport at fixed wall stress
- reduce central velocity
- limit \( Q \) (quality factor)
- matching for const:

\[ x_0 = \frac{y}{U_t} \quad \text{so} \quad U_t = \frac{U}{\ln \left( \frac{y + y_0}{y_0} \right)} \]

Note: Flow on viscous sublayer is turbulent, but mixing there affected by dissipation range scales \( \propto \) linear profile

New = turbulent dissipation? 

Consider NSE:

\[
\frac{\partial \hat{u}^2}{\partial t} + \hat{v} \cdot \nabla \hat{u}^2 + \frac{\partial}{\partial z} \hat{u} \frac{\partial \hat{u}}{\partial z} + \frac{\partial}{\partial x} \frac{\partial \hat{u}}{\partial x} - \hat{u} \hat{v} \frac{\partial \hat{v}}{\partial z} = -\hat{\nabla} \rho + \nu \hat{\nabla}^2 \hat{u}^2
\]

\[ \hat{u} \text{ and } \hat{v} \Rightarrow \]

\[
\frac{\partial}{\partial t} \left( \hat{u} \hat{v} \right) + \left( \hat{v} \cdot \nabla \hat{u} \right) \hat{v} = -\left( \frac{\partial \hat{u}}{\partial z} \hat{\nabla}^2 \hat{u}^2 \right) + \left( \hat{u} \cdot \hat{\nabla} \hat{v} \right)
\]

\[ i.e. \hat{u}, \hat{v} \]
obviously: \( \langle \nu \partial_x^2 \rangle = \nu \left( \frac{\partial u}{\partial x} \right)^2 \)

small scale dissipation

and

\[ e = (u_x) (u_x)^2 \]

(ignoring \( \nu \))

\[ e = \frac{u_x^3}{x} \]

\( \rightarrow \) sets dissipation rate.

\[ e = \frac{V_0^3}{l} \]

\( l \leftrightarrow x \)

\( \rightarrow \) finite as \( x \rightarrow \infty \) (i.e. viscous sublayer gradient diverges then)

Additional References:

- S. B. Pope, "Turbulent Flows"

- H. Tennekes and J. Lumley, "A First Course in Turbulence"
For net energy budget:

\[ \frac{\partial E}{\partial t} = -\frac{\partial}{\partial x} \left( \langle u_x u_z \rangle \right) - \nu \langle \overline{(u_z)^2} \rangle \]

- Input to fluctuations by relaxation of mean shear flow (Reynolds work)
- Dissipation of fluctuation energy by viscosity

\[ \varepsilon = \frac{\partial}{\partial x} \left( \langle u_x u_z \rangle \right) \]

Turbulent dissipation rate

And using mixing length theory:

\[ \langle u_x u_z \rangle = u_k x \frac{\partial \bar{u}}{\partial x} \]

\[ \varepsilon = \left( u_k x \right) \left( \frac{\partial \bar{u}}{\partial x} \right)^2 = \gamma_+ \left( \frac{\partial \bar{u}}{\partial x} \right)^2 \]

Rate of "heating" by turbulent relaxation of mean flow.
Now interesting to tabulate comparison between Pipe Flow and K41 Problem.

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<th>K41 (Kolmogorov)</th>
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Practical Issues


have: \[ \frac{V}{U} < x < \frac{a}{b} \]

radius can push to \( x = a \) with logarithmic accuracy

\[ U = \frac{U_r}{R} \ln \left( \frac{U_r a}{V} \right) \]

but

\[ U_r = U_r = \left( \frac{a}{b} \frac{\Delta p}{2 \rho} \right)^{1/2} \]

\[ U = \left( \frac{a}{2 \rho \ell h^2} \right)^{1/2} \ln \left( a \left( \frac{a \Delta p}{2 \rho h} \right)^{1/2} / V \right) \]

Convenient to define:

\[ x = \frac{2 a \Delta p / \rho}{1/2 \cdot U^2} \rightarrow \frac{\text{Friction Factor}}{\text{Resistance Coefficient}} \]

\[ \rightarrow \text{Flow KE} \]
\[ \lambda = \frac{2 \pi \Delta p / \rho}{\frac{1}{2} \rho U^2} \]

taking \( R_e = \frac{2 \pi U}{\nu} \)

Can rewrite friction law as:

\[ \frac{\lambda}{\nu} = 88 \ln \left( R_e \sqrt{\frac{\lambda}{\nu}} \right) - 85 \]

phenomenal

Good fit to pipe flow data.
1a) A very strong explosion, with energy released $DE$, creates a spherical blast wave in an atmosphere of pressure $P$, density $\rho$. Use dimensional analysis to derive the radius of the blast front as a function of time, i.e., $r(t)$? When does this scaling fail?

b) A hot surface produces thermal convection above it. Assuming the convection is turbulent, use scaling arguments to calculate the temperature profile above the plate assuming the hot plate drives a surface heat flux $Q$. (See Chapter 5, Landau).
Wave Kinetics

Why Wave "kinetics"?

"A wave is never found alone but is mingled with as many other waves as there are uneven places in the object where the said wave is produced. At one and the same time there will be moving over the greatest wave of a sea innumerable other waves proceeding in different directions."

- Leonardo da Vinci
  Codice Atlantico, c. 1500
From Asian art...

The great wave at Kanagawa

Hokusai