→ Vorticity cont'd

→ 2D

→ 2D vorticity dynamics of interest in geophysical context

c.f. atmospheric thin layer.

- constrained dynamics

- treat fluid as effectively 2-dimensional

Now,

\[ \partial_t \omega + \nabla \cdot \omega = \omega \cdot \nabla V \]

\[ \omega = \omega \xi \]

\[ V = V_x \xi + V_y \zeta \]

\[ \omega \cdot \nabla V = 0 \]

→ no vortex tube stretching
\[ \begin{align*}
\frac{\partial \omega}{\partial t} + \nabla \cdot \omega &= 0 \\
\Delta \omega + \nabla \cdot \omega &= \nu \Delta \omega \\
\omega &= \omega(x, t) \\
\nabla \times \omega &= 0 \\
\nabla \times \nu &= 0 \\
\omega &= -\frac{\partial \psi}{\partial x} \\
\Delta \psi + \nabla \times \frac{\partial \psi}{\partial x} \cdot \Delta \psi - \nu \Delta^2 \psi &= 0 \\
\Delta \psi + \nabla \times \frac{\partial \psi}{\partial x} \cdot \Delta \psi - \nu \Delta^2 \psi &= 0
\end{align*} \]

- Vorticity conserved along particle trajectories.
- System is Hamiltonian

\[ \begin{align*}
\dot{t} &= \phi (x, y, t) \\
\dot{x} &= \frac{\partial \phi}{\partial y} \\
\dot{y} &= -\frac{\partial \phi}{\partial x}
\end{align*} \]
- \( P \) is Hamiltonian.

E.g., conservation phase space density

**Observe:**

\[ P_0 = 1 \]

\[ E = E_k = \int \frac{v^2}{2} \, d^3x = \frac{\int \text{det}_{\rho} \, d^3x}{2} \]

\[ \partial_t \int \psi^* \psi \, d^3x = -\int \psi^* \psi \, \text{div} \, \text{v} \, d^3x \]

\[ = +\int (\psi^* \psi \, \text{v} \cdot \partial \psi) \, d^3x \]

\[ = 0 \]

energy conserved (to \( v \))

equivalent to:

\[ \frac{\partial \psi}{\partial t} = \psi \times v \]

\[ \Rightarrow \int \frac{v^2}{2} \, d^3x = -\int \text{det} \, v \cdot \text{D} \left( w + \frac{v^2}{2} \right) \]
\[ 2 + \int \frac{v^2}{2} = -\int d^3x \cdot \frac{1}{2} \left[ v \left( u + v^2 \right) \right] \]
\[ = 0, \quad \text{for } u, v = 0 \]

Now, however,
\[ \frac{d}{dt} u = 0 \]

trivial to show:
\[ \frac{d}{dt} \omega = 0 \quad \Rightarrow \quad \delta \int \omega^2 = \delta \int \frac{\partial \phi}{\partial t} \phi^2 d^3x = 0 \]

i.e. \[ 2 \text{ mutual quadratic conserved quantities} \]
\[ E = \int \frac{1}{2} \phi^2 \quad \rightarrow \text{energy} \]
\[ S = \int \frac{1}{2} \frac{\partial \phi}{\partial t} \phi^2 \quad \rightarrow \text{enstrophy} \]
\[ \text{mean square vorticity} \]
Dual conservation laws make for profound difference from 3D.

Key is absence of vortex tube stretching.

Relaxation in 2D - Vorticity
Homogenization

- Prandtl - Bachelar Theorem.
- Non-Ideal Process

2D Vorticity tends toward homogenous distribution \( \Rightarrow \) homogenization.

A little viscosity makes a global difference.
Flux Expulsion and Homogenization — Non-Idetical
cont'd

so far have encountered:

- \( S \) reconnecting weak dissipation
  \((\Lambda \gg L)\) has strong effect of
  singularity — BOUNDARY LAYER
- Taylor hypothesis of small flux tubes
  destroyed by stochasticity leaving
  \( \int d^3 x \ a \cdot b \) as robust invariant
  
  \[ \text{diffusive dissipation most effective} \]
  \[ \text{at breaking freezing-in on small} \]
  \[ \text{scales} \]

Another example:

- singular behavior on
  \( \text{close-streamline flow} \)

- Homogenization Theory
  \[ \text{Arndt, Batchelor,} \]
  \[ \text{Weiss} \]
  \[ \text{recall} w \text{ evolution for} \]
  \[ \text{Rhines, Young} \]

\[ \frac{\partial w}{\partial t} + \nabla \cdot \mathbf{q} = \omega \cdot \mathbf{v} + \nabla \cdot \mathbf{D} \mathbf{w} \]

\[ 2 \mathbf{D} \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = \mathbf{0} \quad \omega = \omega (\mathbf{D}) \]

\[ \mathbf{v} = \partial \mathbf{\Phi} \mathbf{\Phi} \]

C. F. A. Davidson
2004
then
\[ \mathbf{u} + \nabla \times \mathbf{v} = \nabla \times \mathbf{v} \]

more generally scalar \( \mathbf{v} \) - vorticity
\[ \nabla \cdot \mathbf{v} + \nabla \times \mathbf{v} \cdot \mathbf{v} = \nabla \cdot \mathbf{v} \]

New:
\[ t \rightarrow 0^+ \quad 2 + t \rightarrow 0 \]
\[ \frac{\partial}{\partial t} \mathbf{x}^2 \cdot \mathbf{v} = \nabla \cdot \mathbf{v} \]

\[ t \rightarrow 0 \quad \frac{\partial}{\partial t} \mathbf{x}^2 \cdot \mathbf{v} = 0 \]

Re-plot \( t \rightarrow 0^+ \)
\[ \mathbf{v} = \mathbf{v}(\mathbf{x}) \]

bounded domain, closed streamline solution
\[ q = q(x) \quad \text{arbitrary solution} \]

can develop arbitrarily \( \rightarrow \) closed streamline \( \rightarrow \)
finite scale \( \mathbf{v}(\mathbf{x}) \) perfect memory

- \( \), finite scale structures develops, no internal streamline communication \( \Rightarrow \) persists

\( \mathbf{v} \rightarrow \mathbf{v} \quad \text{intragr each stream}

\( \text{any smoothing of sharp gradients} \)
"Not all solutions of the Navier-Stokes equations are realized in nature!"

\[ \text{blob in converging shear flow} \]

\[ \text{non-diffusive stretching} \]

\[ \text{produces arbitrarily fine scale structure!} \]

\[ \text{now point is that for } R \neq 0 \]

\[ \text{Re} \gg 1 \]

\[ \text{instead of arbitrarily fine scale structure} \]

\[ \text{must have: } \frac{\partial U}{\partial y} \rightarrow \text{const} \]

\[ \text{at } y \rightarrow \infty \]

\[ \text{small } R \rightarrow \text{global behavior} \]

\[ \Rightarrow \text{finite } R \rightarrow \text{large } Re \Rightarrow \]

\[ \text{Vorticity homogenization} \]

\[ \Rightarrow w \rightarrow \text{const} \]

\[ \text{within } \Omega \]

\[ \Rightarrow \text{highly singular behavior} \]

\[ R = 0 \rightarrow \text{Euler Eqn. (26)} \]

\[ \frac{\partial w}{\partial x} \rightarrow 0 \]

\[ R \neq 0 \rightarrow \text{large } Re \rightarrow \text{Navier-Stokes} \]

\[ \text{Eqn. } \Rightarrow \frac{w}{\partial x} \rightarrow 0 \]

\[ \text{Note contrast!} \]
Issues:

- How long to homogenization? (what means asymptotic)
- Where is \( \partial U \to 0 \) boundary layer thickness?
- Analysis of MHD? - Flux Expansion

\[
E + \nu \times B = m J
\]
\[
\nu = \frac{\partial \phi \times E}{\partial x^3}
\]
\[
B = \frac{\partial \phi}{\partial x^3}
\]

\[
-\frac{1}{2} \partial \phi \cdot \partial \phi + \left( \frac{\partial \phi}{\partial x^3} \right) \times \left( \frac{\partial \phi}{\partial x^3} \right) = m J
\]

\[
E = \frac{1}{2} \partial \phi \cdot \partial \phi - \partial \phi \cdot \frac{\partial \phi}{\partial x^3} + \frac{1}{2} \left[ \left( \frac{\partial \phi}{\partial x^3} \right)^2 \right] \frac{\partial \phi}{\partial x^3}
\]

\[
= \left( \frac{\partial \phi}{\partial x^3} \right) \cdot \partial \phi = m J
\]

\[
\Delta A + \frac{\partial \phi}{\partial x^3} \cdot \partial \phi = m \Delta A
\]

\[
\partial \partial \text{conver} \left\{ \begin{array}{c}
D \cdot V = 0 \\
\eta \to 0
\end{array} \right.
\]

\[
\eta \to 0 \text{ except boundary.}
\]
Prandtl-Batchelor Theorem

Consider a region of 2D incompressible flow (i.e., vorticity advection) enclosed by closed streamlines. Then, if diffusion occurs,

\[ \nabla \cdot \mathbf{u} + \nabla \times \omega + \mathbf{f} = 0 \quad \text{(vortex)} \]

then the vorticity is uniform (homogenization) or if \( \omega \to 0 \) within \( C_0 \).

Proof

\[ \text{N.B.: finite } \nabla \Rightarrow \text{ radically different final state} \]

no comment on "has long"
\[ \mathbf{F} \mapsto \mathbf{v} \quad \text{where} \quad \mathbf{v} \mapsto 0 \]

\[ \mathbf{V} \cdot D (\omega) = \mathbf{v} \cdot D (\omega) \]

For stationarity, enter \( t \to \infty \) defining \( \mathbf{v} \approx 0 \).

- Choose arbitrary closed \( \mathcal{C}_0 \) within \( \mathcal{C}_0 \).
- Here, \( \mathcal{C}_0 \) is streamline.
- \( \mathcal{C}_0 \) is simply connected region, \( \mathcal{C}_0 \) has no holes.
- Stationarity implies \( \mathcal{C}_0 \) is constant along streamlines.
- \( \mathcal{C}_0 \) is specified on \( \partial \mathcal{C}_0 \).
- \( \omega \neq 0 \) on \( \mathcal{C}_0 \). (Ultimately \( \mathcal{C}_0 \) sets \( \partial \mathcal{C}_0 \).)
- \( \mathbf{v} \approx 0 \) on \( \mathcal{C}_0 \).

If \( \mathcal{A}_0 \) is area enclosed by \( \partial \mathcal{A}_0 \),

\[ \int_{\mathcal{A}_0} \nabla \cdot \mathbf{v} \, dA = \int_{\partial \mathcal{A}_0} \mathbf{v} \cdot \mathbf{n} \, ds \]

but

\[ \int_{\mathcal{A}_0} \nabla \cdot \omega \, dA = \int_{\partial \mathcal{A}_0} \mathbf{n} \cdot \mathbf{v} \, ds \]

\[ = \int_{\partial \mathcal{A}_0} \mathbf{n} \cdot (\mathbf{v} \cdot \mathbf{n}) \, ds \]

\[ \mathbf{n} \times \text{normal} \]
\[ \text{streamline, } \mathbf{c}_n. \]

\[ \int_{\mathbf{n}_c} \frac{\mathbf{r}}{\mathbf{n}_c} \cdot d\mathbf{A} = 0 \]

as \( \mathbf{v} \) is along streamline.

\[ \int_{c_n} \mathbf{D} \cdot d\mathbf{c} = 0 \]

\[ \int_{c_n} \mathbf{D} \cdot d\mathbf{c} = 0 \]

new \( \mathbf{v} \) stationary state must have \( \mathbf{w} \) constant along streamline.

\[ \mathbf{w} = \mathbf{w}(\phi) \]

\[ \mathbf{w}_n = \mathbf{w}(\phi_n) \]

\[ \oint_{c_n} \mathbf{D} \cdot d\mathbf{A} = 0 \]

\[ \oint_{c_n} \mathbf{D} \cdot d\mathbf{A} = 0 \]
\[ F^t = \int_{\Sigma} \mathbf{P} \cdot \mathbf{n} \]
\[ = \int_{\Sigma} \mathbf{df} \cdot (\mathbf{Q} \times \mathbf{E}) \]
\[ = \int (\mathbf{E} \times \mathbf{n}) \cdot (\mathbf{Q} \times \mathbf{E}) \]
\[ = -\int_{\Sigma} \mathbf{df} \cdot (\mathbf{Q} \cdot \mathbf{n}) = -\int_{\Sigma} \mathbf{df} \cdot (\mathbf{Q} \cdot \mathbf{n}) \]

\[ 0 = \mathbf{v} \cdot \frac{\partial \mathbf{P}}{\partial \mathbf{n}} \]

\[ \frac{\partial u}{\partial \phi_n} = 0 \]

but if arbitrary \( \Rightarrow \frac{\partial u}{\partial \phi_n} = \) arbitrary \( \Rightarrow \) no variation from line to line

\[ \Rightarrow \mathbf{w} \text{ homogenized} \]

so, expect \( dw \) large at bounding contour \( C_0 \)
\( dw \) to within \( \Rightarrow \) \( dw \) held of boundary
Some Comments:

Homogenization theory looks (magical) \( \rightarrow \) caveat emptor

\[ f \to \infty \Rightarrow \text{time asymptote} \]
\[ q = 2(\theta) \Rightarrow \text{concentric streamlines} \]

How long to achieve configuration?

2. Simply connected domain \( \Rightarrow \) analysis (2 FI)

a. Single structure \( \Rightarrow \) expulsion from neighbors and possible interaction not addressed

i.e. what happens if \( f \) (interference) of boundary layer?

=> stamina intersection

\[ \Rightarrow \text{steps} \]

etc.
4) Key Assumptions:

- Closed, bounding streamline

(viscous dissipation, i.e., can envision):

\[ \text{exact streamline, molecular viscosity} \]

\[ \text{coarse-grained streamline, eddy viscosity} \]

\[ \text{correspond to homogenization of} \]

\[ \text{total vorticity} \]

\[ \text{mean/coarse-grained vorticity} \]

\[ \text{time scales different} \]

\[ \frac{T_c}{T_d} \ll 1 \Rightarrow \frac{D}{D} \ll 1 \]

or \[ \frac{\nu}{\nu} \gg 1 \]

Translational \[ \ll 1 \Rightarrow \text{Rayleigh-Benard} \]

Diffusion

\[ \text{to establish concentric circulation lines} \]

Diffuse across to homogenize \[ \text{but slow} \]

\[ \frac{T_c}{T_d} \ll 1 \Rightarrow \frac{D}{D} \ll 1 \]

\[ \nu \ll \nu \]

\[ Re \gg 1 \]
or equivalently \( \frac{\nabla}{\nabla} \to 1 \)

i.e. \( \text{Re}_\text{eff} \gg 1 \) or \( \text{Ary} \)

related: essential idea is that \( \frac{\text{constant segregated along streamlines}}{\text{established on fluid (\text{Re}) scale}} \)

- dissipation homogenizes on slower (\( \text{\text{Re}} \)) time scale (but this is slow...)

\[ \text{What are the time scales?} \]

- useful to consider differentially rotating sheared flow within closed pattern

\[ \text{e.g.} \quad V_{\text{circ}} = V_{\text{circ}} \approx V_{0} \]

\[ \text{near} \quad n_{\text{cub}, \text{cub}} \]

\[ \text{What is the mixing time scale?} \]

\[ \text{Shear dispersion} \]

\[ \text{Key: Synergism between Shear} \]

C. E. S. Stiglitz, P. H. O. C. and P. W. Terry

\{ Phys. Fluids (B2), 1, 1990 \}

(first noted by C. E. Taylor)
→ 3D structures
- Recall writhe structures
  - tubes
  - rings
  - sheets
  - lines
- Spot → 2D (phase element)
  - helicity
- In 3D, can identify quadratic
  - co-suicide co-variant

\[ H = \int \mathbf{v} \cdot \mathbf{\omega} \, d^3x \rightarrow \text{pseudoscalar} \]

\[ x \rightarrow -x \]

to show conserved:

1. \( \frac{\partial \mathbf{v}}{\partial t} - \mathbf{v} \times \mathbf{\omega} = -\nabla (w + \frac{v^2}{2}) + \nabla \mathbf{v} \)

2. \( \frac{\partial \mathbf{\omega}}{\partial t} = \nabla \times \mathbf{v} \times \mathbf{\omega} + \nabla \times \mathbf{v} \omega \)
\[ \omega_1 + v_2 = 0 \]

\[ 2 + \int d^3x \quad \nabla \cdot \omega = \int d^3x \quad \nabla \cdot \left( \sum_{\text{surf}} \omega \right) + \int d^3x \quad \nabla \cdot \left( \sum_{\text{surf}} \left( \omega + v^2 \right) \right) \]

\[ \int d^3x \quad \nabla \cdot \left( \sum_{\text{surf}} \left( \omega + v^2 \right) \right) \]

For \( \nabla \cdot \sum_{\text{surf}} = 0 \)

\[ 0 = 0 \]

\[ 0 = 0 \]

\[ \int d^3x \quad \left[ \omega \cdot \left( \sum_{\text{surf}} \nabla \times \left( \nabla \times \omega \right) \right) - \int d^3x \quad \left( \nabla \times \left( \nabla \times \omega \right) \right) \cdot \left( \nabla \times \omega \right) \right] = 0 \]
then \( \text{Rot} \rightarrow \text{Screw} \Rightarrow \)

\[ \begin{array}{c}
\vec{v} \\
\rightarrow \quad \omega
\end{array} \]

Solenoid \( \Rightarrow (\vec{v} \cdot \vec{\omega}) \neq 0 \)

Helicity \( \Rightarrow \) helical symmetry in flow
- breaking of reflection's symmetry

And/or: \( \text{Linkage} \leftarrow \text{topological} \)

\[ \begin{array}{c}
\text{2 linked vortex lines} \\
\rightarrow \quad \vec{\omega}
\end{array} \]

\[ \int_{\Omega} \vec{\alpha} \cdot \vec{v} = \left( \int_{\Omega} \omega \cdot \mathbf{e}_3 \right) \star \left( \int_{\Omega} \vec{u} \cdot \mathbf{e}_3 \right) \]
\[ 2 \int d^3x [(v \cdot \nabla) \psi] = 0 + o(v) \]

\{ Fluid helicity conserved. \\
Enviscid invariant. \}

What does it mean?

- Consider a vortex tube

\[ \text{\ldots} \rightarrow \psi \]

i.e. \[ \begin{array}{c}
\text{small loop} \\
\text{deform} \\
twist \end{array} \]
and can show:

\[ \int \mathbf{v} \cdot \mathbf{u} = 2 \Phi_1 \Phi_2 \]