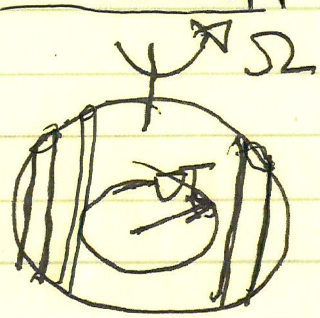
⇒ Further develop:

→ **Rotating Systems!**

- Many astrophysical systems:

  ⇒ convection + rotation



Jupiter

- bands
- $T_{day} \sim 10$ hrs

How do they interact?

- Key point: rotation changes the $\begin{cases} \text{freezing-in law of} \\ \text{vorticity dynamics.} \end{cases}$

Recall:

$$\boxed{\partial_t \underline{\omega} - \nu \nabla^2 \underline{\omega} = \nabla \times \underline{v} \times \underline{\omega}}$$

induction eqn. for vorticity

akin:
$$\underline{E} + \frac{\underline{v} \times \underline{B}}{c} = \eta \underline{J}$$

$$\nabla \times \underline{E} = -\frac{1}{c} \frac{\partial \underline{B}}{\partial t}$$

then

$$\frac{\partial \underline{B}}{\partial t} - \eta \nabla^2 \underline{B} = \underline{\nabla} \times \underline{V} \times \underline{B}$$

Compressible: $\underline{\omega}/\rho$, $\underline{B}/\rho$ frozen in

hereafter incompressible. what of anelastic?

Now, $\partial_t \underline{\omega} + \underline{V}\cdot\underline{\nabla}\,\underline{\omega} - \nu \nabla^2 \underline{\omega} = \underline{\omega}\cdot\underline{\nabla}\,\underline{V}$

$$\frac{d\underline{\omega}}{dt} = \underline{\omega}\cdot\underline{\nabla}\,\underline{V} \qquad \rightarrow \text{ note structure.}$$

Consider 2 test particles at $\underline{r}_1$, $\underline{r}_2$

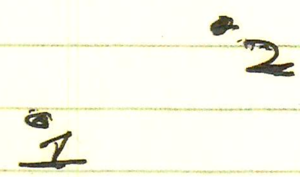
"Frozen into" flow:

$$\frac{d\underline{u}}{dt} = -\gamma\left(\underline{u} - \underline{V}(\underline{r},t)\right) + \tilde{\underline{f}}.$$

so

$$\frac{d\underline{r}_1}{dt} = \underline{V}(\underline{r}_1, t)$$

$$\frac{d\underline{r}_2}{dt} = \underline{V}(\underline{r}_2, t)$$



so let $\delta\underline{l} = \underline{r}_2 - \underline{r}_1$

think of as denoting filament

$$\frac{d}{dt}\delta\underline{\ell} = \underline{v}\left(\underline{r}_+ + \frac{\delta\underline{\ell}}{2}, t\right) - \underline{v}\left(\underline{r}_+ - \frac{\delta\underline{\ell}}{2}, t\right)$$

$$\cong \delta\underline{\ell}\cdot\underline{\nabla}\,\underline{v}$$

so

$$\boxed{\frac{d}{dt}\delta\underline{\ell} = \delta\underline{\ell}\cdot\underline{\nabla}\,\underline{v}}$$

eqn. for frozen in vector filament

→ $d\underline{\ell}$ follows the flow and is stretched by it

→ $d\underline{\ell}$ is vector field "frozen into" the flow.

Since: $\dfrac{d\underline{\omega}}{dt} = \underline{\omega}\cdot\underline{\nabla}\,\underline{v}$

has same form as $\delta\underline{\ell}$ equation, we say vorticity 'frozen into' the flow. Vortex filaments follow, and are stretched by, the flow.

Obviously: $\underline{\text{freezing-in} \leftrightarrow \text{Kelvin's Thm}}$

↓ local, field (vector) condition

↓ integral (oriented) scalar.

Now, in rotating fluid:

$$\Omega = \Omega \hat{z}$$

$$\underset{\substack{\uparrow\\ \text{frame of lab}}}{\underline{v}_L} \rightarrow \overset{\text{fluid}}{\underline{v}} + \underline{\Omega} \times \underline{r}, \quad \underline{\underline{\text{etc.}}}$$

so, for $\underline{\nabla} \cdot \underline{v} = 0$,

centrifugal force (absorb into pressure)

$$\left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla}\, \underline{v}\right) = -\underline{\nabla}\left(\frac{P}{\rho} - \frac{\Omega^2 r^2}{2}\right) + \underline{v} \times 2\underline{\Omega}$$

coriolis force.

$$\underline{v} \cdot \underline{\nabla}\, \underline{v} = \underline{\nabla}\left(v^2/2\right) - \underline{v} \times \underline{\omega}$$

$\Rightarrow$

centrifugal

$$\frac{\partial \underline{v}}{\partial t} = -\underline{\nabla}\left(\frac{P}{\rho} + \frac{v^2}{2} - \Omega^2 r^2\right) + \underline{v} \times \underline{\omega} + \underline{v} \times 2\underline{\Omega}$$

coriolis

relative vorticity; planetary

so

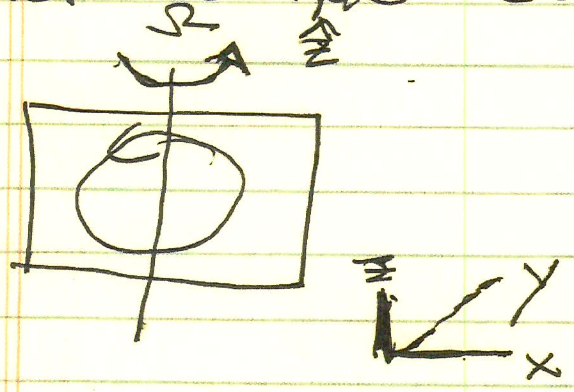
mean vorticity

$$\boxed{\frac{\partial}{\partial t}\left(\underline{\omega} + 2\underline{\Omega}\right) = \underline{\nabla} \times \underline{v} \times \left(\underline{\omega} + 2\underline{\Omega}\right) + \nu \nabla^2 \underline{\omega}}$$

$\therefore$ $-\ \underline{\omega} + 2\underline{\Omega}$ frozen in

$-\int d\underline{a} \cdot \left(\underline{\omega} + 2\underline{\Omega}\right)$ conserved to ν.

So, for simple case:



later consider tilt, etc. → GFD

∴

$$\frac{\partial \underline{\omega}}{\partial t} + \underline{v}\cdot\nabla(\underline{\omega}+2\underline{\Omega}) - \nu\nabla^2\underline{\omega} = (2\underline{\Omega}+\underline{\omega})\cdot\nabla\underline{v}$$

$$\frac{d\underline{\omega}}{dt} - \nu\nabla^2\underline{\omega} = (2\underline{\Omega}+\underline{\omega})\cdot\nabla\underline{v}$$

$$= 2\Omega\frac{\partial}{\partial z}\underline{v} + \underline{\omega}\cdot\nabla\underline{v}$$

For $\Omega \gg |\underline{\omega}|, |\nabla v|$; strong rotation
+ mean vorticity
i.e. system rotation strong compared to relative vorticity.

- motions cannot vary in direction of Ω
- all steady motions in rotating, inviscid slow, are necessarily two dimensional..

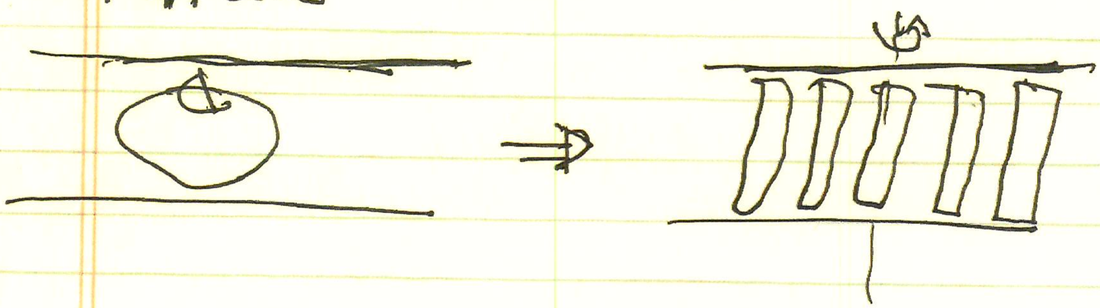
lim $\Omega \gg$ all else $\Rightarrow$

$$\boxed{2\Omega \frac{\partial}{\partial z} \underline{v} \cong 0}$$

$\Rightarrow$ **Taylor-Proudman Theorem (1921, 1916)**

$\Rightarrow$ Rapid rotation "two-dimensionalizes" the flow.

$\Rightarrow$ cells organize into: "Taylor columns," "Proudman Pillars" $\Rightarrow$ i.e. shift in cartoon.



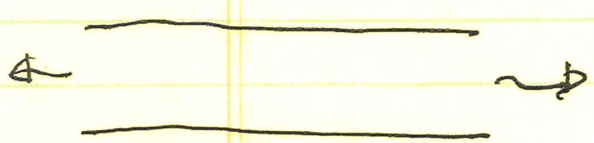
$\Rightarrow$ viscous boundary layer top/bottom
$\nu k^2 \rightarrow$ Ekman Layer
(fluid rotating, tank not)
may enter.

Next: Consider: { – waves in rotating fluids
– Convection in rotating systems.

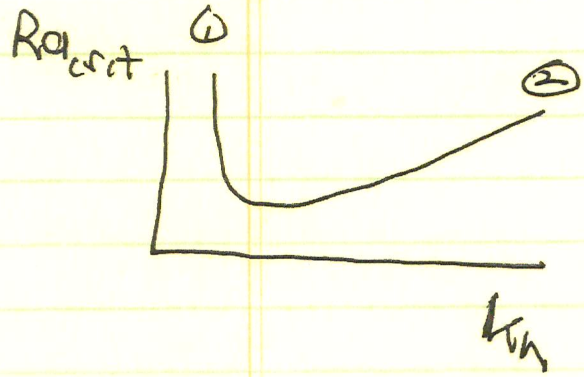
Lecture VII – More Convection   Winter 2018

Aside: Convection in "tall, thin" box
(Question from LC).

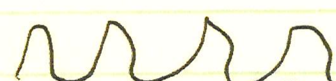
Recall:
- 'lesson' from study of Rayleigh–Benard convection is that boundary conditions matter!
- recall case study was "short, wide" box



- dismissed side
- focus on stress-free no slip top, bottom.

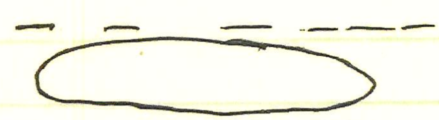


② →



increasing diffusive damping due high k_h

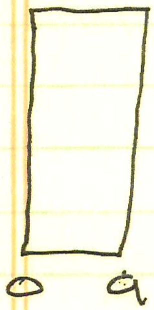
①



increasing diffusive damping due small vertical scales (~ high k_z).

tall, thin box

Now:



→ ignore top/bottm boundary

→ no-slip on side walls is key → and only relevant case

i.e. $V_z \Rightarrow w$; $w(0) = w(a) = 0$

side B.L.

side B.L. due no-slip

- long thin cell must fight no-slip b.c.'s on side wall

- higher k_h will introduce high k_y damping.

∴ - expect high $Ra)_{crit}$ due side-wall no slip, even at $k_h \to k_{h\,min}$.

- curvature of Ra_{crit} vs k_h curve TBD. Speculate rather weak curvature.

- as side surface area >> top surface area, expect top no-slip vs others free comparison not significant.

(a) → Introduction to Rotating Convection
— Intermezzo on Landau Theory.
(b) → Basics of Wakes.

A.) Convection + Rotation

- see Lecture 6 for { Rotation, Freezing-in Law, Taylor-Proudman Thm. }

Key pt: for Ω large enough, flow is two-dimensionalized

→ Inertial Waves
— Rotating Convection

I

→ Inertial Waves → (radial) buoyancy waves in rotating fluid

⇝ Recall Pblm 4, set 1:

Fluid rotates at $\Omega \hat{z}$, $k_\theta = 0$
(convenience), $k_r, k_z \neq 0$

$$\Rightarrow \quad \omega^2 = k_z^2 (4\Omega^2) \Big/ k_z^2 + k_r^2$$

and, with radial b.c., eigenvalue.

→ Quick derivation:

- from vorticity equation (T–P thm.), linearization:

$$\frac{\partial \tilde{\underline{\omega}}}{\partial t} + \cancel{\underline{v}\cdot\underline{\nabla}(\underline{\omega}+2\underline{\Omega})} = 2\Omega\frac{\partial}{\partial z}\tilde{\underline{v}} + \cancel{\tilde{\underline{\omega}}\cdot\underline{\nabla}\,\tilde{\underline{v}}}$$

so

$$\partial_t \tilde{\omega}_z = 2\Omega\,\partial_z \tilde{v}_z$$

and, from EOM

$$\frac{\partial \tilde{\underline{v}}}{\partial t} + \cancel{\tilde{\underline{v}}\cdot\underline{\nabla}\,\tilde{\underline{v}}} = -\frac{\underline{\nabla}}{\rho}p^* + \underline{v}\times 2\Omega\hat{z}$$

$\underline{\nabla}\times\underline{\nabla}\times\underline{v} \Rightarrow$

$$\partial_t \underline{\nabla}(\underline{\nabla}\cdot\tilde{\underline{v}}) - \partial_t \nabla^2\tilde{\underline{v}} = 0 + \underline{\nabla}\times\underline{\nabla}\times\left[\underline{v}\times 2\Omega\hat{z}\right]$$

$$\Rightarrow\ -\partial_t \nabla^2\tilde{\underline{v}} = \underline{\nabla}\times\left(2\Omega\partial_z\tilde{\underline{v}} - \cancel{\underline{v}\cdot\underline{\nabla}\,2\Omega\hat{z}}\right)$$

so, Σ.

$$-\partial_t \nabla^2 \tilde{V}_z = 2\Omega\, \partial_z \tilde{\omega}_z$$

{ axial gradient in $\tilde{\omega}$ $\Rightarrow$ axial acceleration

and!

$$\partial_t \tilde{\omega}_z = 2\Omega\, \partial_z \tilde{V}_z$$

{ stretching vortex $\Rightarrow$ vorticity fluctuation.

$\Rightarrow$ inertial wave dispersion relation.

$$\omega^2 = k_z^2\, 4\Omega^2 / (k_r^2 + k_z^2)$$

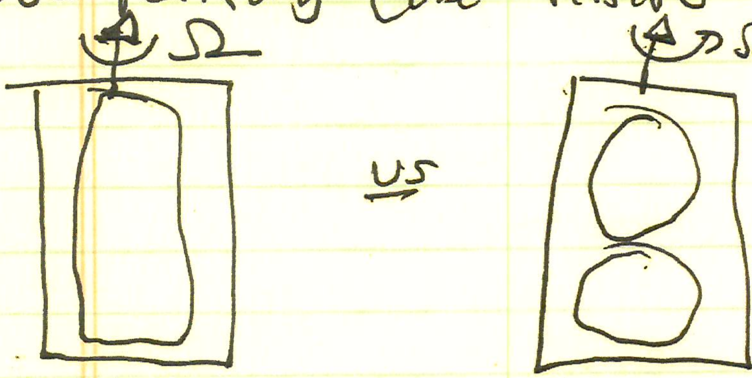
- physics is rotating flow vortex lines don't like being bent ($k_z \neq 0$)
  - $\Rightarrow$ imposes energy penalty for motions with finite k_z
  - $\Rightarrow$ + definite contribution to δW.
- rough picture is one of gyroscopic restoring force (conservation L_z)

- analogous to Alfven waves and field line bending in MHD.

$$\omega^2 = k_\parallel^2 V_A^2$$

- aside: – backward wave: $V_{gr} < 0$
  – $\omega = 0$ finite k_r modes. (shear layers).

⇒ as convection necessitates cellular overturning (i.e. finite k_z),



low k_z (k_z min) motion favored

⇒ tracks T-P thm. conclusion of 2D-ization of the flow.

→ suggests that rotation is (strong) stabilizing effect on convection.

* [Also suggests that another dimensionless number enters comparing rotation to viscous dissipation
⇒ naturally:

$$Ta = 4\Omega^2 d^4/\nu^2 \sim \frac{(2\Omega)^2}{(\nu/d^2)^2} \qquad d \equiv \text{box scale}$$

Taylor Number

- ~ Taylor number captures natural competition between rotation and viscous diffusion.

- ~ Ta joins Ra, Pr as key parameters in convective stability theory

- ~ $Ra_{crit} = Ra_{crit}(Pr, Ta)$ is now stability threshold problem.
  N.B. Ra, Ta both involve ν but are distinct - $\alpha g \Delta T/d$ vs Ω^2.

Can combine stationary convection and inertial wave calculations to obtain basic equations:

$$\partial_t \theta = \Theta W + \kappa \nabla^2 \theta$$

$$\partial_t \nabla^2 W = g\alpha \nabla_h^2 \theta + \nu \nabla^2 \nabla^2 W - 2\Omega \partial_z \omega_z$$

$$\partial_t \omega_z = -2\Omega \partial_z W$$

(derive)

notation as before.

For ideal stability:

$$\boxed{\omega^2 = \left[-g\alpha\beta k_h^2 + (4\Omega^2) k_z^2\right] / k^2}$$

- favors high k_h, low k_z cells

$\Rightarrow$

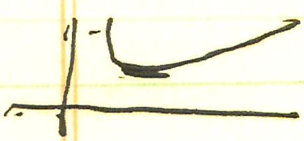
- Taylor columns (thin) and Proudman pillars, as shown in movies.

- stabilizing effect of rotation evident

For viscous, conductive stability with rotation, then:

$$Ra_{crit} = Ra_{crit}(Ta, \underset{\displaystyle k_h h}{\underset{\downarrow}{\alpha}}), \quad \text{for } Pr \sim 1.$$

One can further specify Ra_{α} as Ra_{crit} minimum (vs $\alpha = k_h h$)

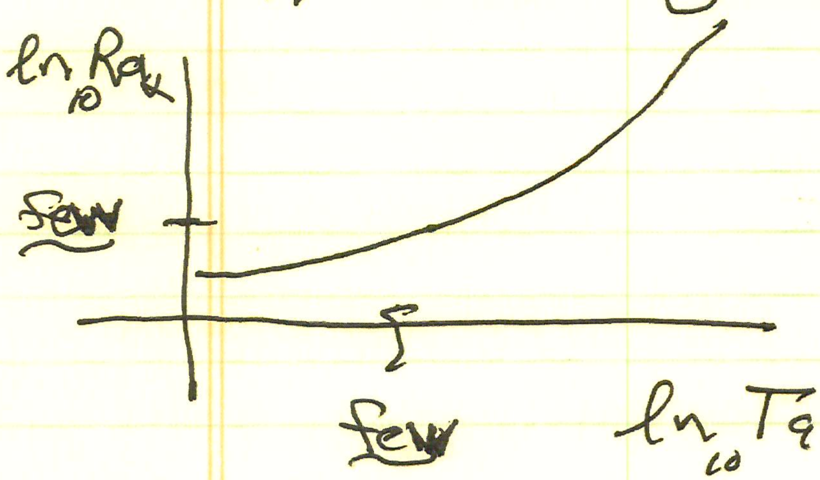


$Ra_\alpha = Ra_\alpha(Ta)$

↓

Taylor # dependence

For all cases: { no-slip / stress free



{ demonstrates stabilizing effect rotation.

→ can develop variational principle for exchange-of-stabilities case. Need also treat over-stable limit.

→ Cultural Aside : Magnetoconvection

Dynamo-generated magnetic fields can feed back on convection.
A particular example is sunspots (i.e. dark – lower T ?! – convection weakened), which are

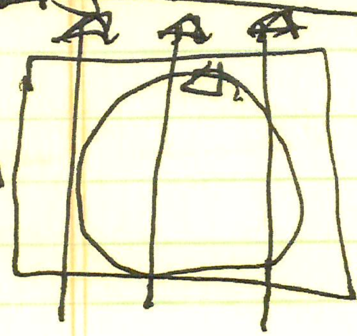
N.B.: Comment on dissipation effects.

associated with strong magnetic fields

⇒ **Magnetoconvection**

$\underline{B_0}$

$\nabla T < 0$ ↑



↓ g

- similar to rotation problem, with 'bending element' due B_0
  ⇒ i.e. energy penalty
- Alfven wave replaces inertial wave.
- expulsion can occur, etc.

Enough linear stability theory!

Intermezzo

Commentary: Landau Equation / Law

← to date, linear stability

→ Nonlinear evolution ?!?

⇒ difficult problem, especially for turbulence...

⇒ seek characterize weakly nonlinear evolution, i.e. for flow shear / KH instability (stabilized by viscosity)