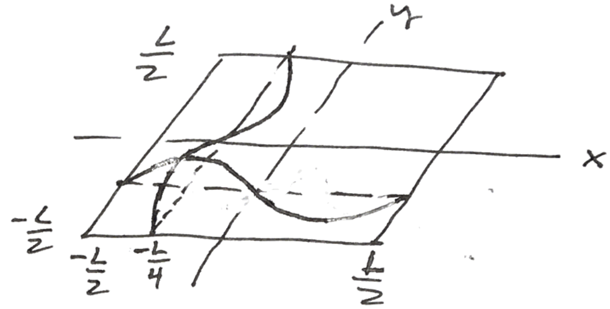
Physics 4E — Quiz 5 Solutions — Spring 2020

Problem 1



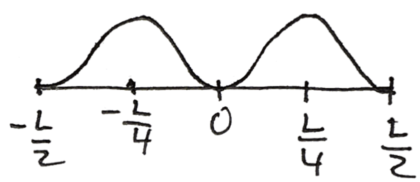
$$\Psi(x,y) = A\sin\left(k_1\left(x+\tfrac{L}{2}\right)\right)\sin\left(k_2\left(y+\tfrac{L}{2}\right)\right)$$

so that $\Psi(x=-\frac{L}{2}, y) = \Psi(x, y=-\frac{L}{2}) = 0$

and we need $\Psi(x=\frac{L}{2}, y) = \Psi(x, y=\frac{L}{2}) = 0$ which requires

$k_1 = \frac{n_1 \pi}{L}$, $k_2 = \frac{n_2\pi}{L}$, n_1, n_2 integers.

The probability is maximal for $x = y = \frac{L}{4}$ $\Rightarrow$

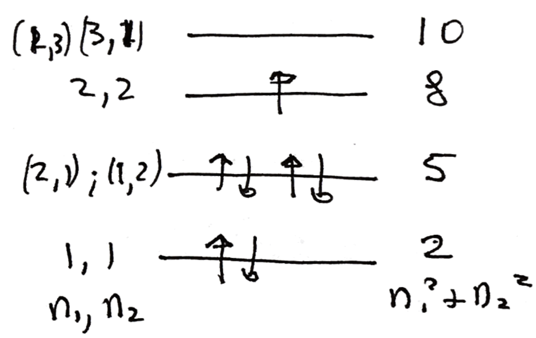


$\Rightarrow$ $\boxed{n_1 = 2,\ n_2 = 2}$

The energy is $E_{n_1,n_2} = \frac{\hbar^2\pi^2}{2m_e L^2}(n_1^2+n_2^2) = \frac{7.62\times\pi^2}{2\times 25}\times 8\,\text{eV} = 12.03\,\text{eV}$

so $\boxed{E_{22} = 12.03\,\text{eV}}$

(b)



the closest state with higher energy is (1,3) or (3,1)

transition to that state will absorb longest wavelength photon.

$$\frac{hc}{\lambda} = E_{3,1} - E_{2,2} = \frac{7.62\times\pi^2}{50}(10-8)\,\text{eV}$$

$$\Rightarrow \boxed{\lambda = \frac{12{,}400}{3.008}\,\text{Å} = 4122\,\text{Å}}$$

(c) shortest wavelength photon emitted: $\frac{hc}{\lambda} = E_{2,2} - E_{1,1} = \frac{7.62\pi^2}{50}(8-2)\,\text{eV}$

$\Rightarrow \boxed{\lambda = 1374\,\text{Å}}$

(d) Occupy all lower states with electrons

$\boxed{\text{need 6 electrons}}$

Problem 2

$$\psi(r,\theta,\phi) = C \cdot r \cdot e^{-3r/a_0} e^{im_\ell\phi} \cos\theta$$

(a) The exponential is always of the form e^{-Zr/na_0}

$\Rightarrow \frac{Z}{n} = 3.$ From the fact that the wavefunction has no nodes and the factor r^1 we deduce that $\boxed{n=2}$ and $\boxed{\ell=1}$ Therefore, $\boxed{Z=6}$

From the $\cos\theta$ factor we deduce that $\boxed{m_\ell = 0}$

(b) Radial probability: $P(r) = r^2 R(r)^2 = C^2 r^4 e^{-6r/a_0}$

Most probable r: $P'(r) = 0 = 4r^3 - \frac{6r^4}{a_0} \Rightarrow \boxed{r = \frac{2}{3}a_0}$

(c) $U(r) = -\frac{ke^2 Z}{r}$

$$\langle \tfrac{1}{r} \rangle = \frac{\int_0^\infty dr \, \frac{1}{r} P(r)}{\int_0^\infty dr \, P(r)} = \frac{\int_0^\infty dr \, r^3 e^{-6r/a_0}}{\int_0^\infty dr \, r^4 e^{-6r/a_0}} = \frac{3! \, 6^5}{6^4 \cdot 4! \, a_0} = \frac{6}{4a_0} = \frac{3}{2a_0}$$

$$\Rightarrow \boxed{\langle U(r) \rangle = -\frac{ke^2 \cdot 6 \cdot 3}{2a_0} = -\frac{9ke^2}{a_0}}$$

(d) For the Bohr atom, with $n=2$:

$\boxed{r_n = \frac{a_0}{Z} n^2 = \frac{a_0}{6} \cdot 2^2 = \frac{2}{3} a_0}$ agrees with (b)

$\boxed{U(r) = -\frac{ke^2 Z}{r_n} = -\frac{ke^2 \cdot 6}{\frac{2}{3}a_0} = -\frac{9ke^2}{a_0}}$ agrees with (c).

Problem 3

In the presence of a magnetic field, the energy of level n is

$$E_n(B) = E_n - \vec{\mu}\cdot\vec{B} = E_n - \mu_z B$$

Ignoring spin, $\mu_z = -\frac{e\hbar}{2m_e} m_\ell = -\mu_B m_\ell \Rightarrow$

$E_n(B) = E_n + \mu_B B m_\ell$. So for the transition 3d to 2p:

$$E_3(B) = E_3 + \mu_B B m_{3\ell} \quad , \quad E_2(B) = E_2 + \mu_B B m_{2\ell}$$

$$\Rightarrow E_3(B) - E_2(B) = E_3 - E_2 + \mu_B B(\underbrace{m_{3\ell} - m_{2\ell}}_{\Delta m_\ell})$$

$m_{3\ell} = \pm 2, \pm 1, 0$ and $m_{2\ell} = \pm 1, 0$. However because of the selection rule $\Delta m_\ell = \pm 1$ or 0 there are only 3 possible photons, with wavelengths: $\lambda_1 = \frac{hc}{E_3 - E_2}$, $\lambda_2 = \frac{hc}{E_3 - E_2 + \mu_B B}$, $\lambda_3 = \frac{hc}{E_3 - E_2 - \mu B}$

$E_3 - E_2 = 1.889\,\text{eV}$, $\mu_B B = 5.79\times 10^{-4}\,\text{eV} \Rightarrow$

(a) $\boxed{\lambda_1 = 6564.3\,\text{Å}, \quad \lambda_2 = 6562.3\,\text{Å}, \quad \lambda_3 = 6566.3\,\text{Å}}$

(b) With spin, $\mu_z = -\mu_B(m_\ell + 2m_s)$, $m_s = \pm\frac{1}{2}$

$$E_3(B) - E_2(B) = E_3 - E_2 + \mu_B B(m_{3\ell} + 2m_{3s} - m_{2\ell} - 2m_{2s})$$

$$= E_3 - E_2 + \mu_B B\Big(\underbrace{(m_{3\ell} + m_{3s}) - (m_{2\ell} + m_{2s})}_{= \Delta(m_\ell + m_s) = 0, 1, \text{ or } -1} + \underbrace{m_{3s} - m_{2s}}_{\substack{\text{max value} = 1 \\ \text{min value} = -1}}\Big)$$

so maximum and minimum difference is: $E_3 - E_2 \pm 2\mu_B B$

$$\Rightarrow \boxed{\lambda_{max} = \frac{hc}{E_3 - E_2 - 2\mu_B B} = 6568.3\,\text{Å} \qquad \lambda_{min} = \frac{hc}{E_3 - E_2 + 2\mu_B B} = 6560.3\,\text{Å}}$$