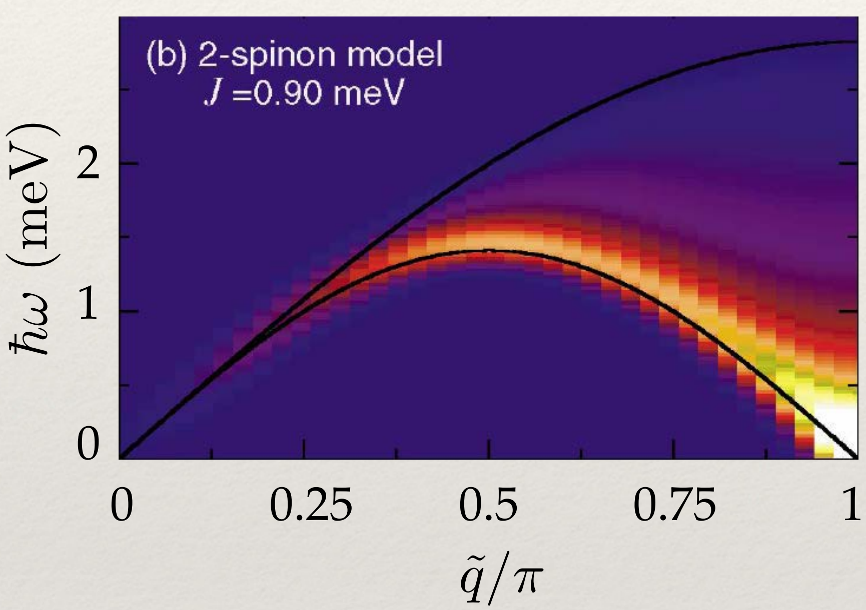
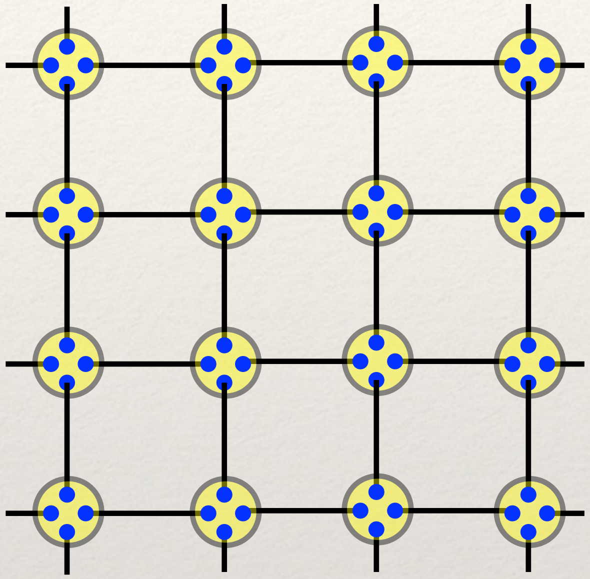
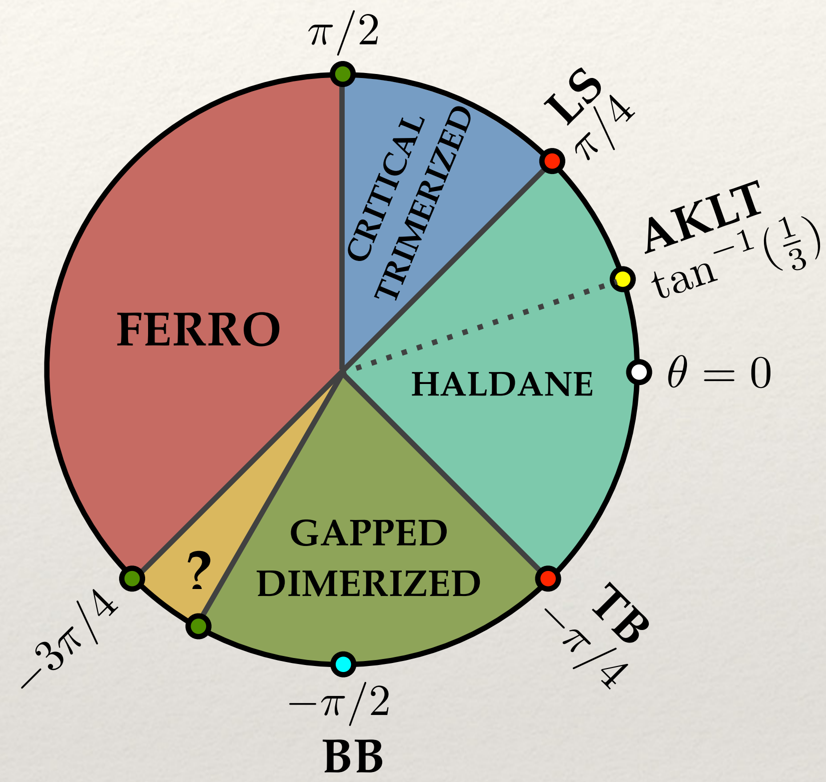
Quantum Magnetism

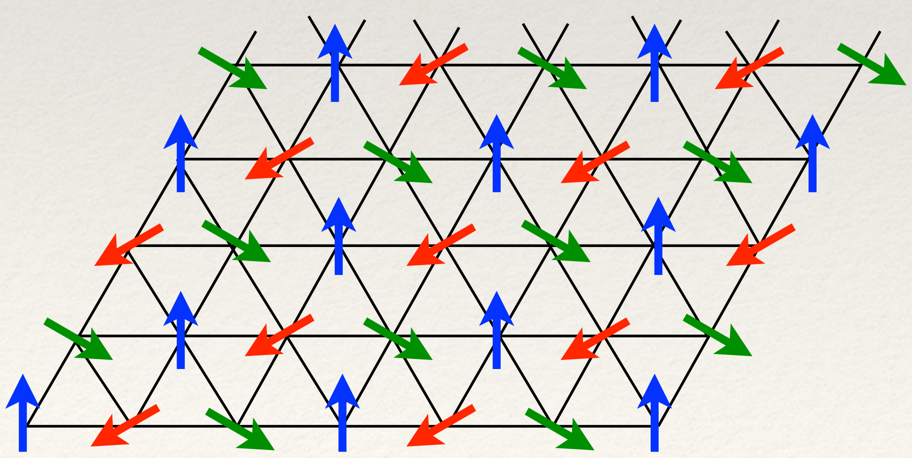
From the Iron Age to the Present

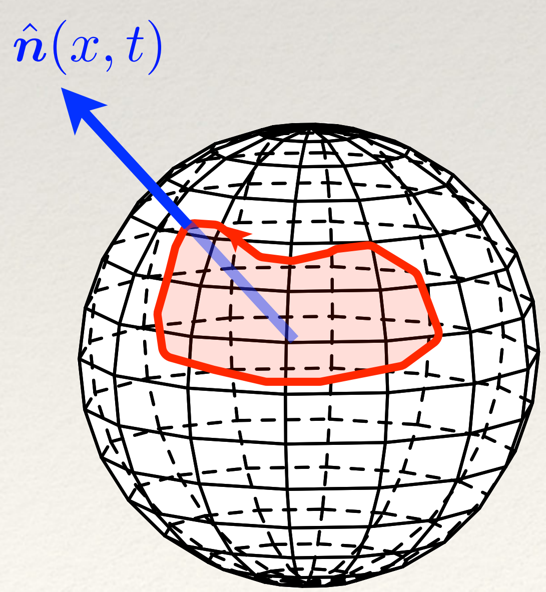
D. P. Arovas, UCSD

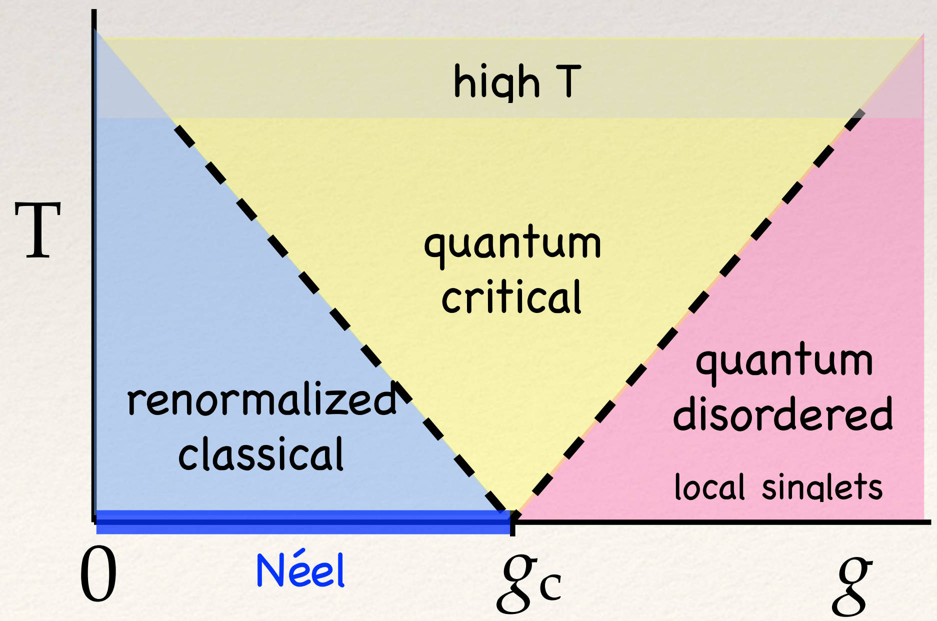










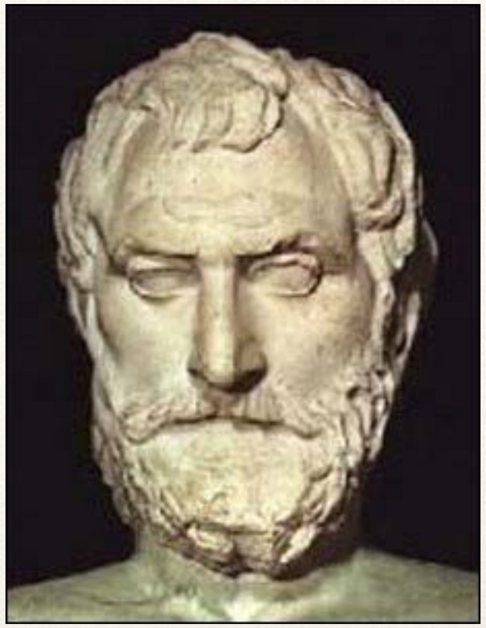


Magnetism in Antiquity

Lodestone : mostly magnetite (Fe_3O_4)

Thales (c. 620 BCE - c. 546 BCE) : first theory of magnetism!

"Thales…says that a [lodestone] has a soul because it causes movement to iron." - Aristotle (4th c. BCE)



Thales of Miletus

lodestone attracting iron

"The lodestone attracts iron."

- Wang Xu (4th c. BCE)

Compass : 2nd c. BCE - 1st c. CE in China

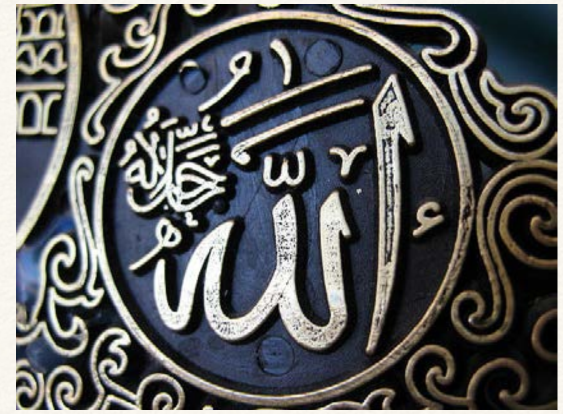
Initially used for divination (Feng Shui), and eventually for navigation.

"south pointing spoon"

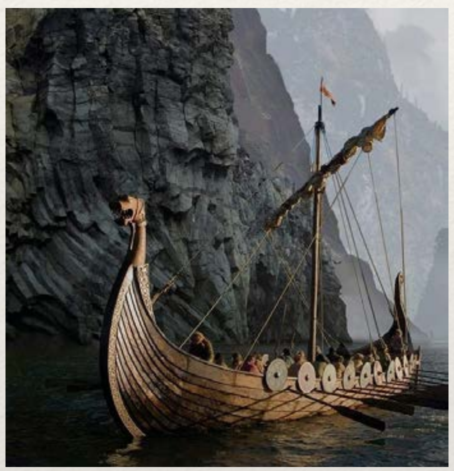

Han dynasty


Jesus

Roman empire

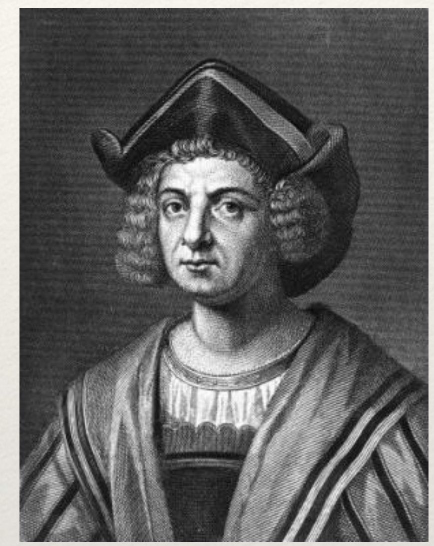

Islam

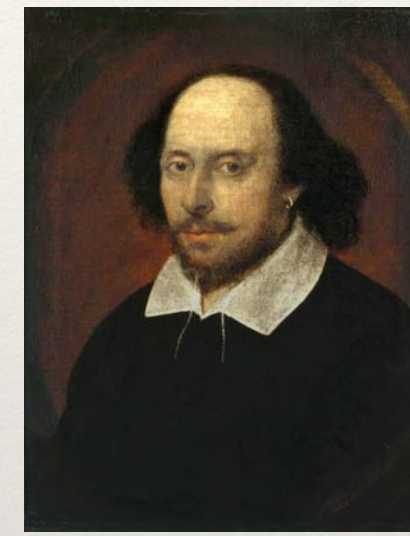

Charlemagne

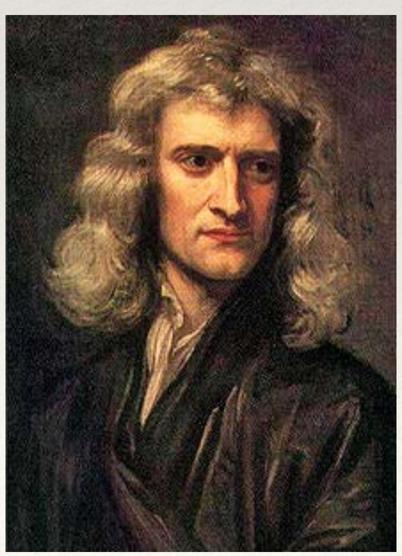

Viking conquests


Crusades


Bubonic plague


Columbus


Shakespeare


Newton


Beethoven


American revolution


Meiji restoration


World War I

The Ising Model (Lenz, 1920)

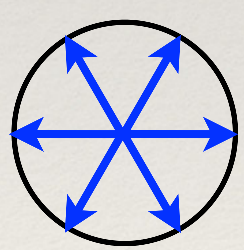
$$H = -\sum_{i<j} J_{ij}\,\sigma_i\,\sigma_j \qquad \text{with} \qquad \sigma_i = \pm 1$$

ferromagnetic : $J_{ij} > 0 \quad \left| \uparrow_i \uparrow_j \right\rangle$; **antiferromagnetic** : $J_{ij} < 0 \quad \left| \uparrow_i \downarrow_j \right\rangle$

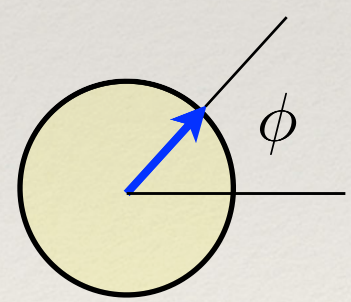
Global symmetry group $\mathbb{Z}_2$: $\sigma_i \to \varepsilon\,\sigma_i$ with $\varepsilon \in \{-1, +1\}$

Other common global symmetries:

p-state clock :

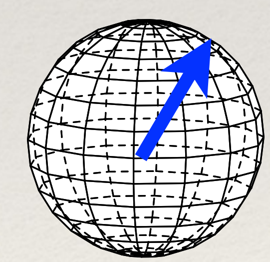


xy :



Heisenberg :

$\hat{\boldsymbol{n}} = (n_1, \ldots, n_N)$



$\hat{\boldsymbol{n}}^2 = 1$

group :	$\mathbb{Z}_p$	$\mathrm{O}(2)$	$\mathrm{O}(N)$
	discrete	continuous	continuous

Spontaneous symmetry breaking

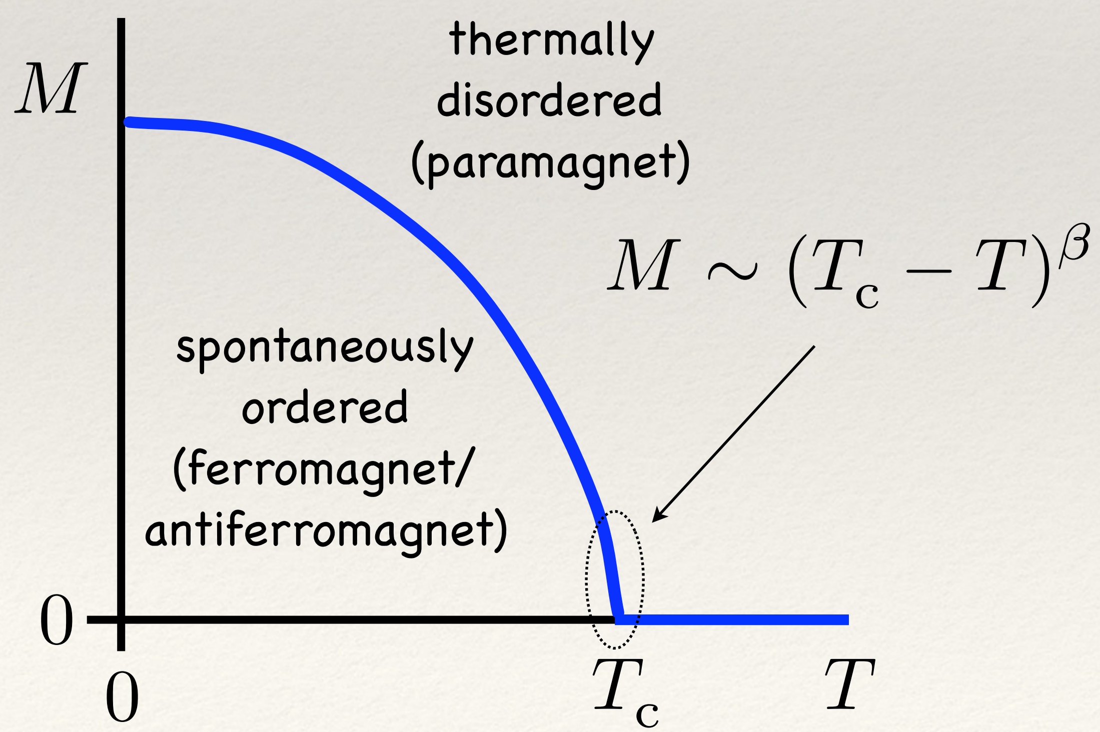
Below a *critical temperature* $T_{\rm c}$, long-ranged order develops :

$$\langle \hat{\boldsymbol{S}}_i \rangle = \boldsymbol{M} \cos(\boldsymbol{Q} \cdot \boldsymbol{R}_i + \varphi)$$

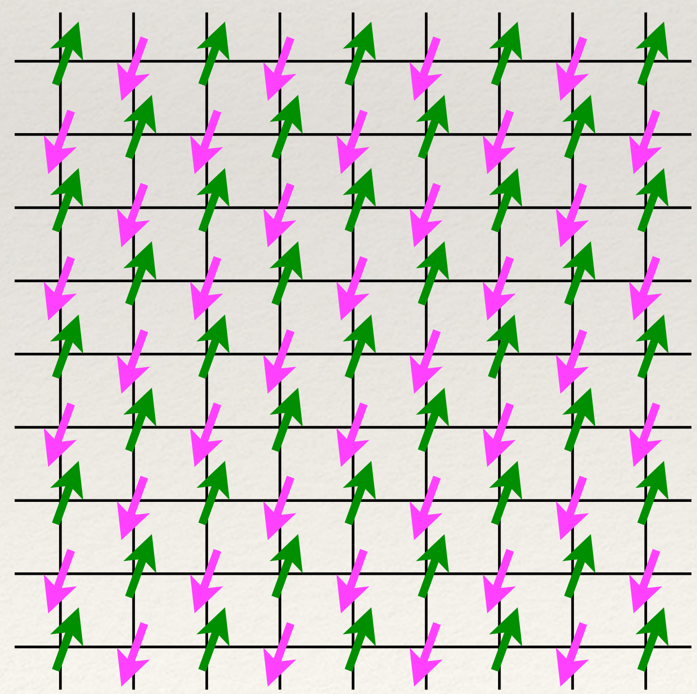
spontaneous moment (order parameter) ↗ $\boldsymbol{M}$

ordering wavevector ↖ $\boldsymbol{Q}$

Up through the 1980s, work on magnetism focused either on ordered phases of classical/quantum models (and their defects), or on special features of quantum models in one space dimension (solvable by Bethe's *Ansatz*).

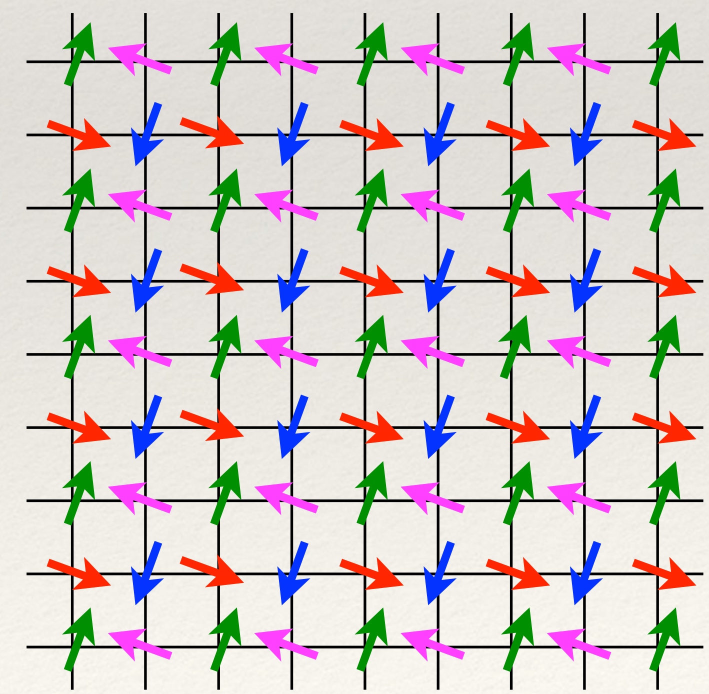


two sublattice Néel state



$\boldsymbol{Q} = \frac{\pi}{a}\,\hat{\boldsymbol{x}} + \frac{\pi}{a}\,\hat{\boldsymbol{y}}$

four sublattice Néel state

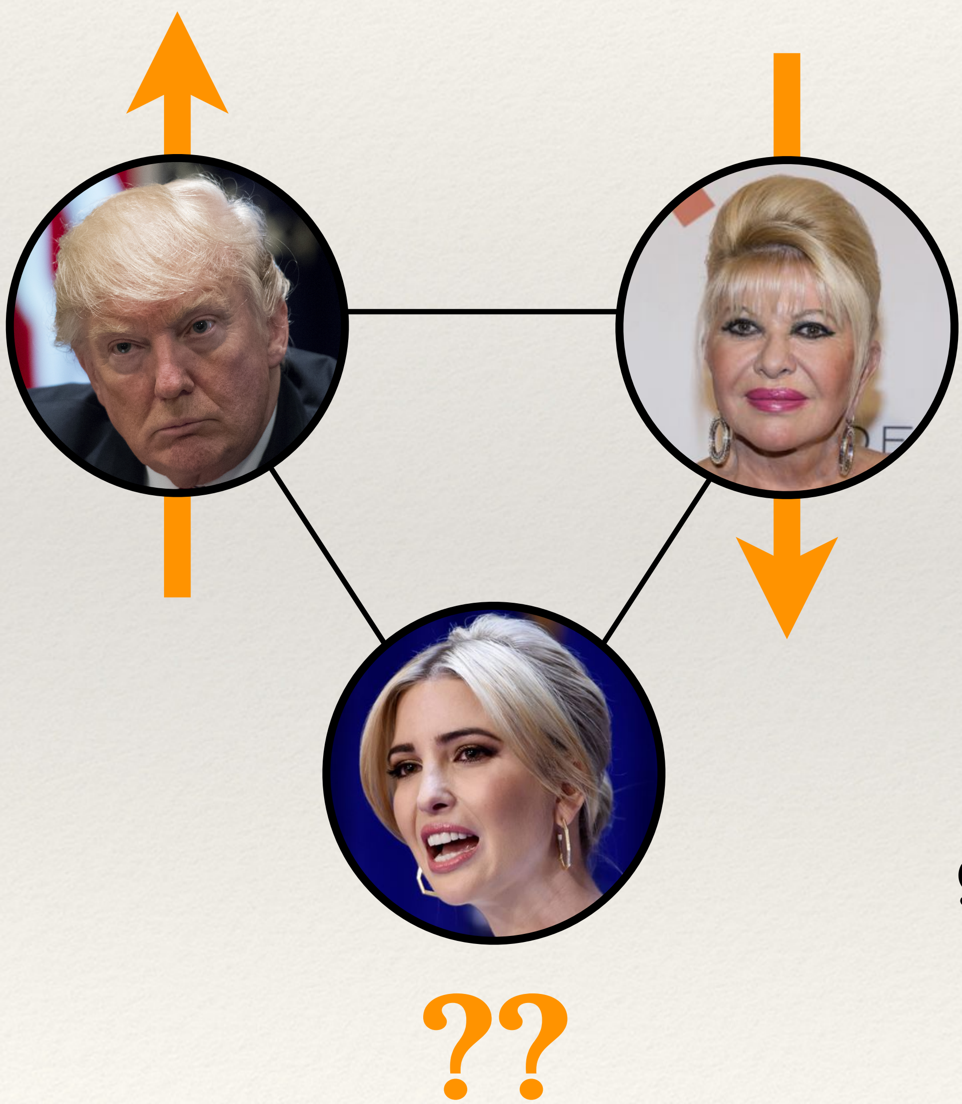


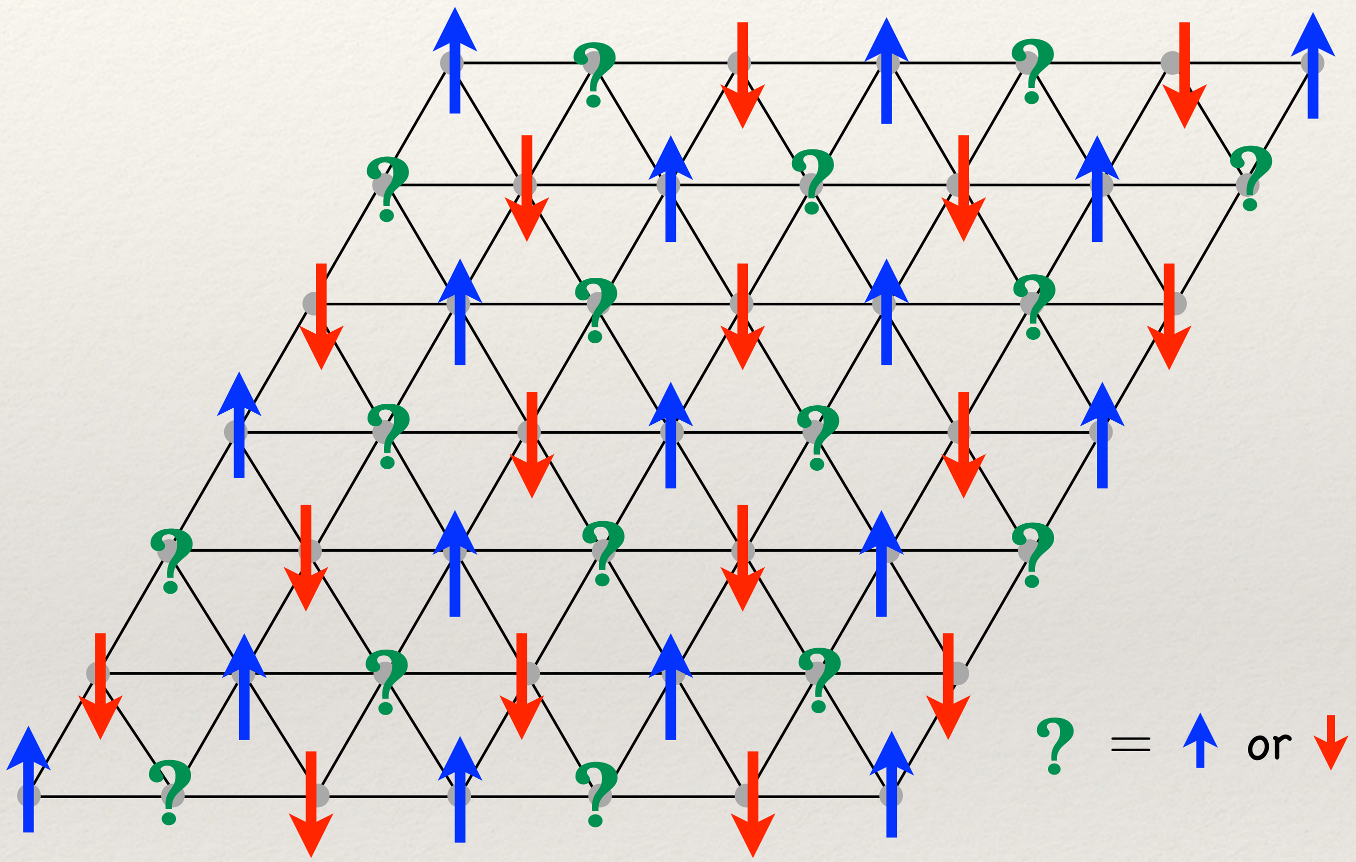
$\boldsymbol{Q}_1 = \frac{\pi}{a}\,\hat{\boldsymbol{x}}$, $\boldsymbol{Q}_2 = \frac{\pi}{a}\,\hat{\boldsymbol{y}}$

Frustration “You can’t always get what you want.”

Discrete symmetry : Ising model

$$H = |J| \left(\sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1\right)$$



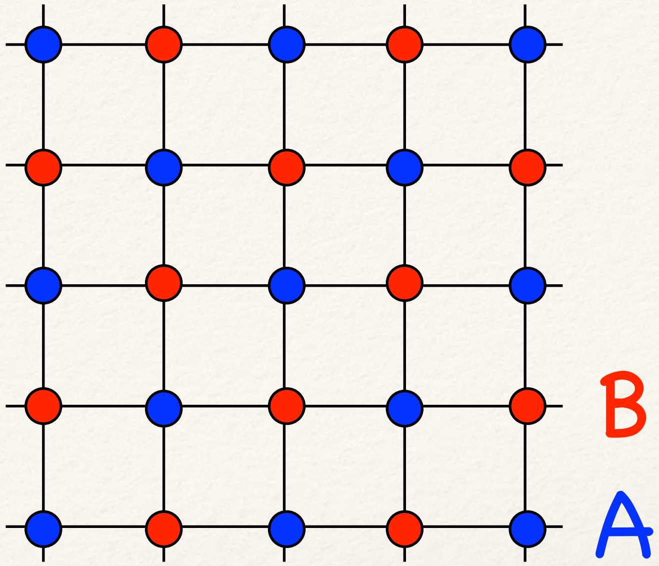


ground state entropy : $S_0/N \geq \frac{1}{3} \ln 2 = 0.2310$

$$S_0/N = \frac{3}{\pi} k_{\mathrm{B}} \int_0^{\pi/6} d\omega \, \ln(2\cos\omega) \simeq 0.3231$$

(G. S. Wannier, 1950)

Classical *vs.* quantum



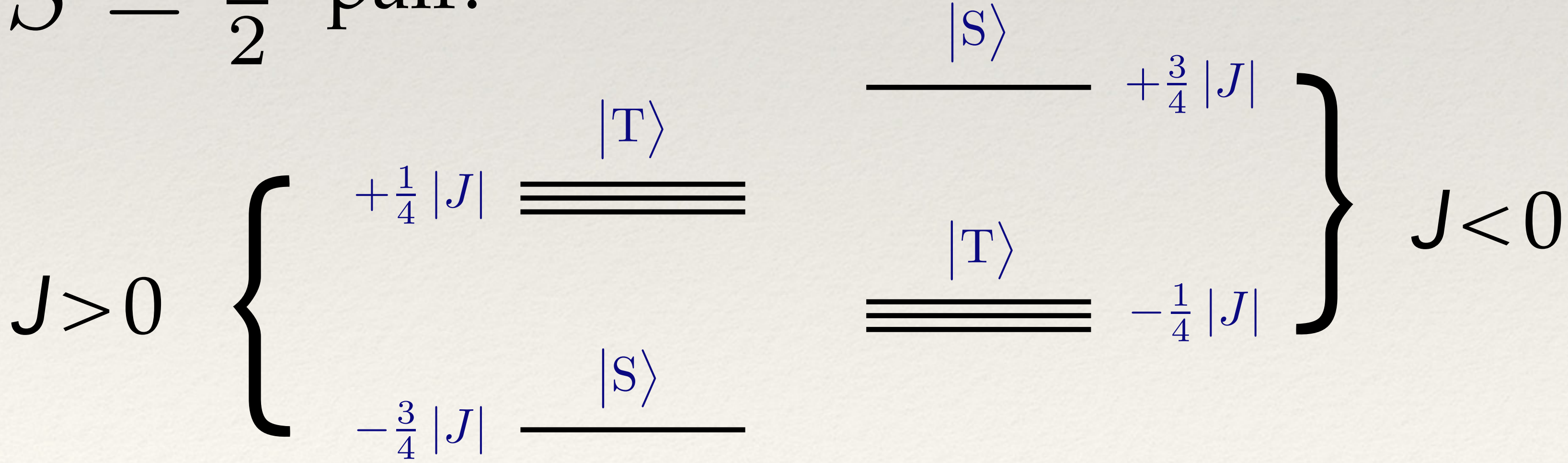
Heisenberg interaction : $H = J\boldsymbol{S}_i \cdot \boldsymbol{S}_j$

Redefining $\boldsymbol{S}_j \to -\boldsymbol{S}_j$ sends $J \to -J$

OK for classical spins on bipartite lattices! FM = AFM !!

NOT OK FOR QUANTUM SPINS! $\left[\boldsymbol{S}^x, \boldsymbol{S}^y\right] = i\hbar \boldsymbol{S}^z$

$S = \frac{1}{2}$ pair:



☞ Néel states

$$|\mathrm{N}\rangle = \begin{cases} |\uparrow\downarrow\rangle \\ |\downarrow\uparrow\rangle \end{cases}$$

not in spectrum!

Heisenberg interaction and ground states

$$\boldsymbol{S}_i \cdot \boldsymbol{S}_j = \underbrace{S_i^z S_j^z}_{\text{classical term}} + \underbrace{\tfrac{1}{2} S_i^+ S_j^- + \tfrac{1}{2} S_i^- S_j^+}_{\text{quantum fluctuations}}$$

Classical / quantum FM : $|\mathrm{F}\rangle = \left| \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ \uparrow & \uparrow & \uparrow & \uparrow \\ \uparrow & \uparrow & \uparrow & \uparrow \\ \uparrow & \uparrow & \uparrow & \uparrow \end{array} \right\rangle$ $S_i^+ S_j^- |\uparrow\uparrow\rangle = 0$

Classical Néel state: $|\mathrm{CN}\rangle = \left| \begin{array}{cccc} \downarrow & \uparrow & \downarrow & \uparrow \\ \uparrow & \downarrow & \uparrow & \downarrow \\ \downarrow & \uparrow & \downarrow & \uparrow \\ \uparrow & \downarrow & \uparrow & \downarrow \end{array} \right\rangle$ selected by $S_i^z S_j^z$ (A/B sublattice)

$m = S$

Quantum Néel state: $|\mathrm{QN}\rangle = \left| \begin{array}{cccc} \downarrow & \uparrow & \downarrow & \uparrow \\ \uparrow & \downarrow & \uparrow & \downarrow \\ \downarrow & \uparrow & \downarrow & \uparrow \\ \uparrow & \downarrow & \uparrow & \downarrow \end{array} \right\rangle - \dfrac{1}{4(z-1)S} \left| \begin{array}{cccc} \downarrow & \uparrow & \downarrow & \uparrow \\ \uparrow & \downarrow & \uparrow & \downarrow \\ \downarrow & \uparrow & \downarrow & \uparrow \\ \uparrow & \downarrow & \uparrow & \downarrow \end{array} \right\rangle + \dots$

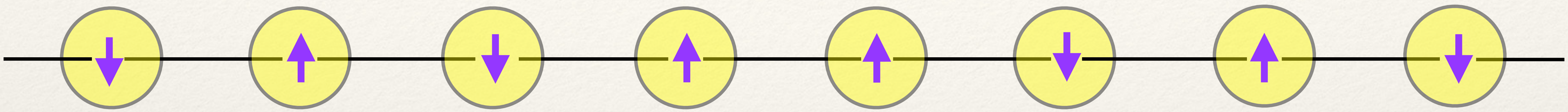
$m = S - 1$

local moment reduced by quantum fluctuations

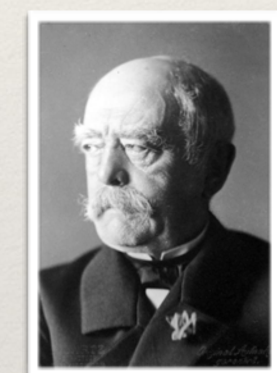
elementary excitations : FM/AFM *spin waves*

$S = \frac{1}{2}$ antiferromagnetic Heisenberg chain

$$H = +J \sum_n \boldsymbol{S}_n \cdot \boldsymbol{S}_{n+1}$$

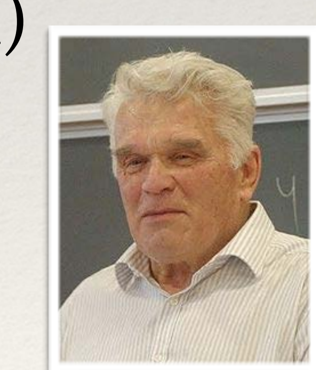


1931 : general form of eigenfunctions ("Bethe's *Ansatz*")



1938 : ground state energy (Hulthén) $E_0/NJ = \frac{1}{4} - \ln 2 > -\frac{3}{4}$

1962 : $S = 1$ excitation spectrum (des Cloiseaux and Pearson)

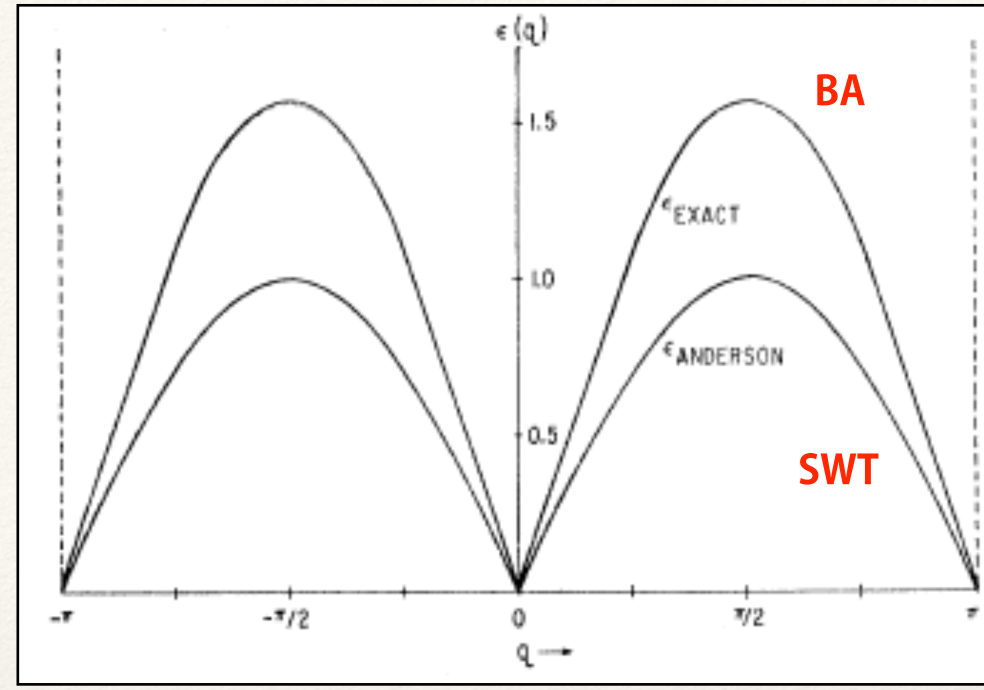


1981 : $S = ½$ excitation spectrum by Faddeev and Takhtajan

1998 : exact asymptotic correlation (Affleck) $\langle S_0^\alpha S_r^\beta \rangle \sim (-1)^r \delta^{\alpha\beta} \sqrt{\ln r} / [(2\pi)^{3/2} r]$

Triplet (S = 1) excitation branch:

Up to an overall factor $\pi/2$, this is the same as the spectrum from spin wave theory!

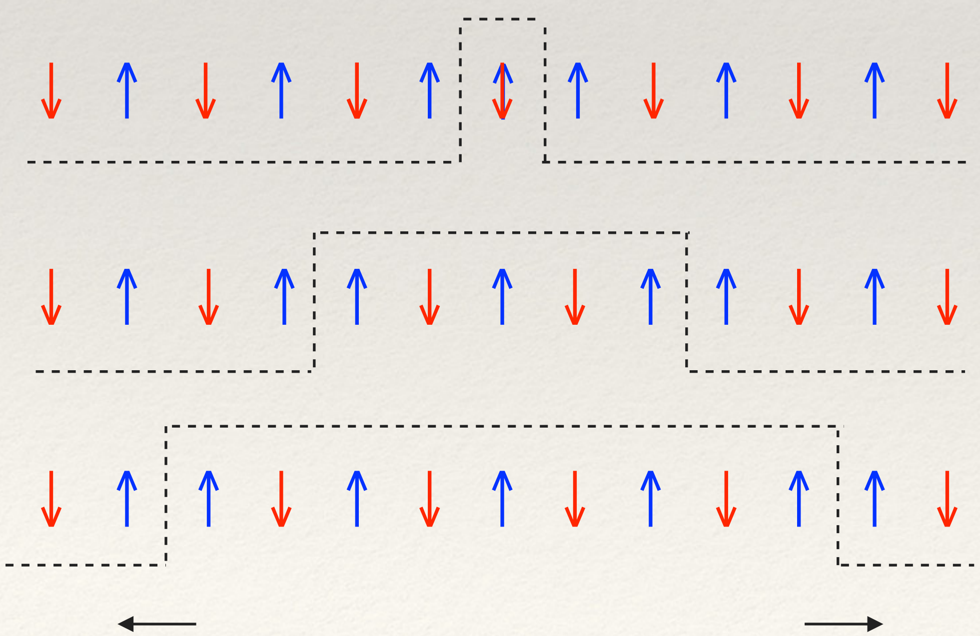


$$\varepsilon(q) = \tfrac{1}{2}\pi J \big|\sin(qa)\big|$$

However, these are ***composite*** excitations.

The ***true*** elementary excitations are S=½ doublets, which are **kinks**, with

$$\varepsilon(q) = \tfrac{1}{2}\pi J \sin(qa) \quad ; \qquad 0 \leq q \leq \frac{\pi}{a}$$



Triplet (and singlet) **continuum**:

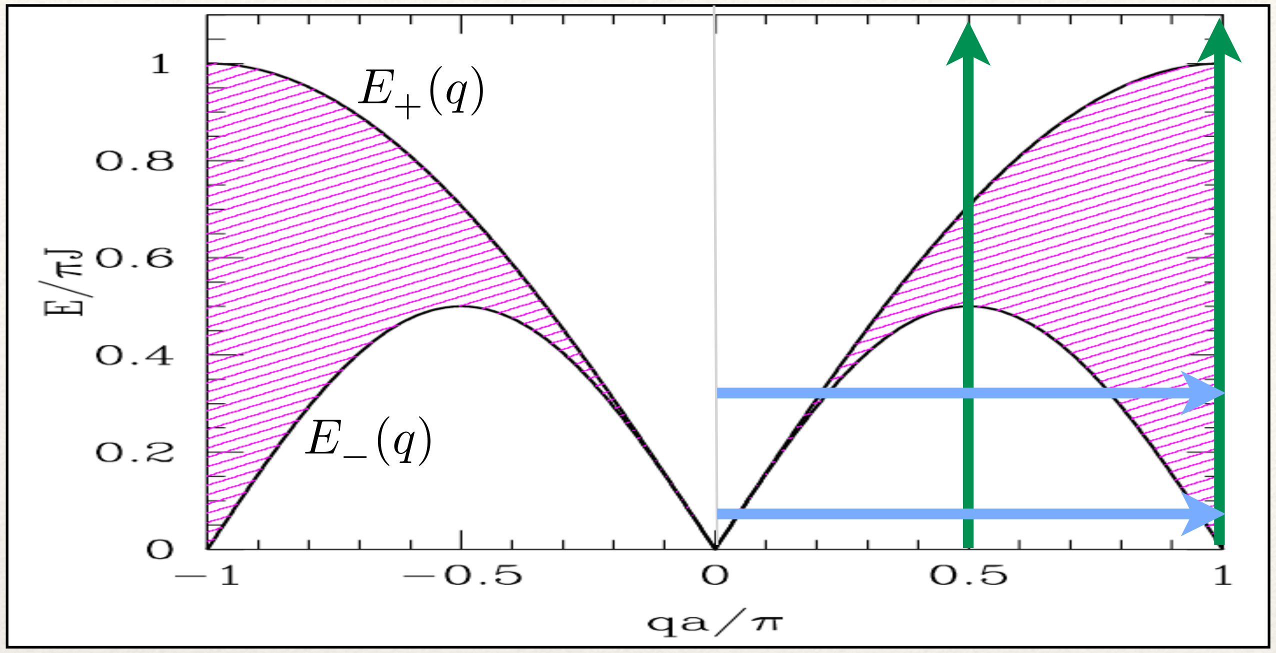
$$E_-(q) \leq \hbar\omega \leq E_+(q)$$

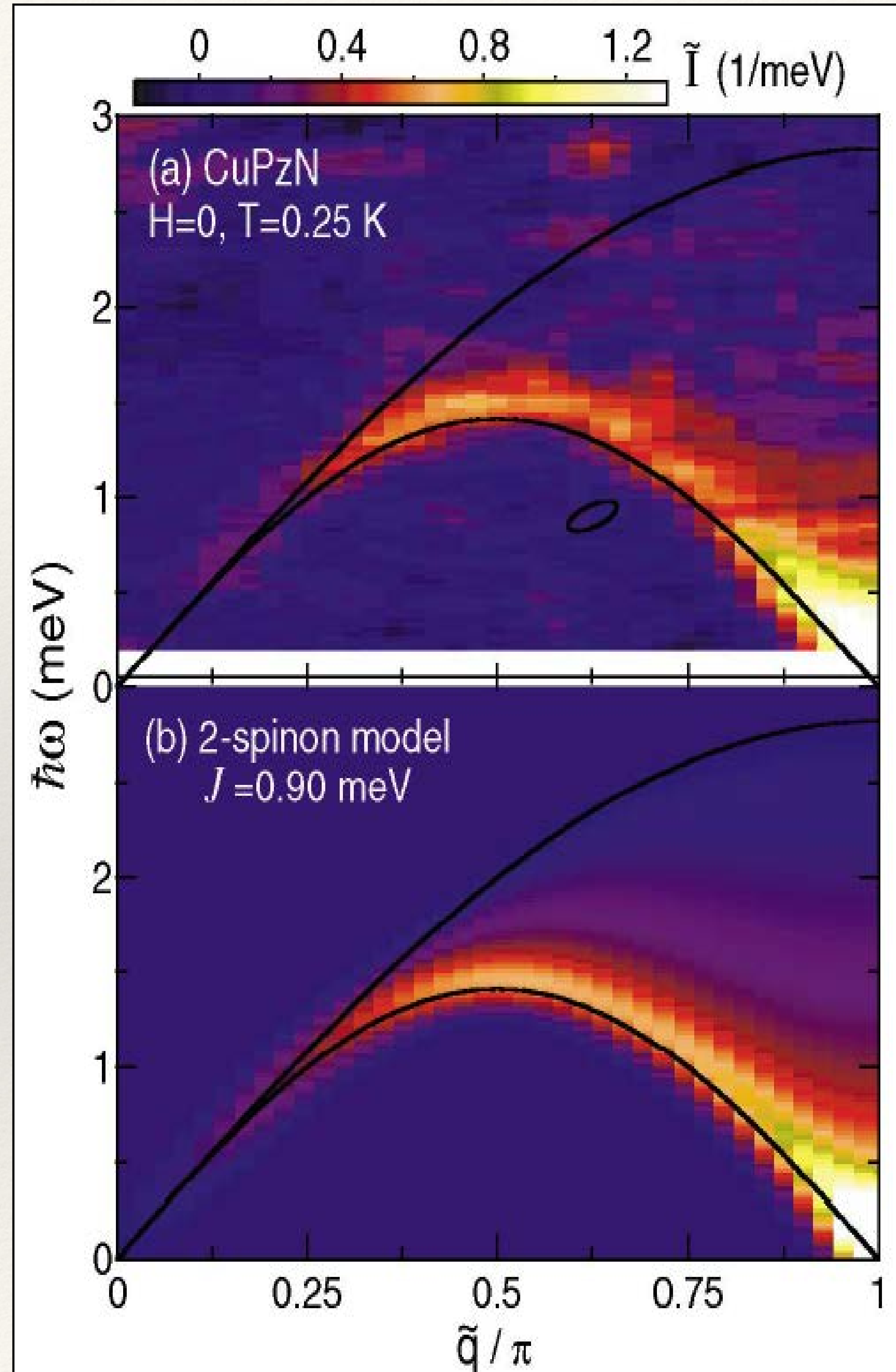
$$E_-(q) = \varepsilon(q) + \varepsilon(0) = \tfrac{1}{2}\pi J \big|\sin(qa)\big|$$

$$E_+(q) = \varepsilon(\tfrac{1}{2}q) + \varepsilon(\tfrac{1}{2}q) = \pi J \big|\sin(\tfrac{1}{2}qa)\big|$$

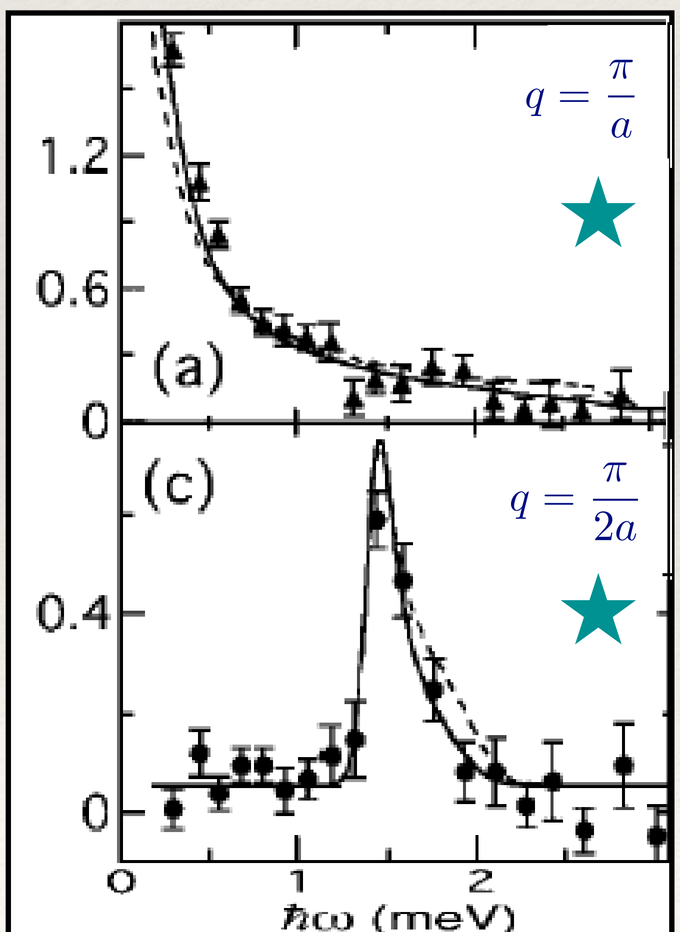
$\mathrm{Cu(C_4H_4N_2)(NO_3)_2} \equiv$ CuPzN $\quad J = 0.9\,\mathrm{meV}$

M. B. Stone *et al.*, PRL **91**, 037205 (2003)

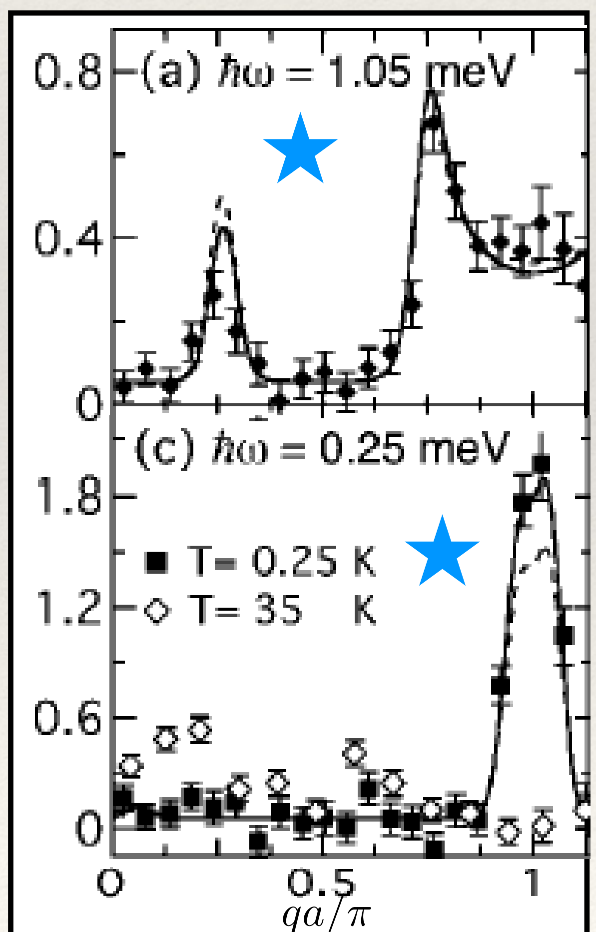




frequency scans



momentum scans



$S=1$ antiferromagnetic Heisenberg chain

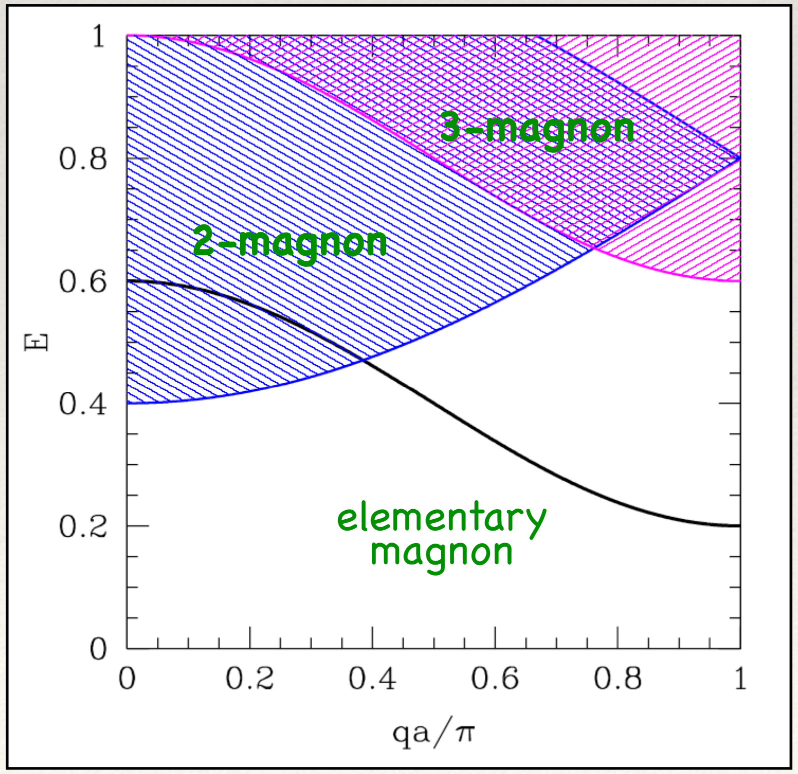
$$H = J\overbrace{\sum_n \boldsymbol{S}_n\cdot\boldsymbol{S}_{n+1}}^{\text{exchange interaction}} + D\sum_n \overbrace{\left(S^z_n\right)^2}^{\text{anisotropy: prefers } S^z_n=0}$$

The elementary excitation is a triplet (S=1) with dispersion ω(**q**), with an excitation gap at **q** = π, *i.e.* the **Haldane gap**.

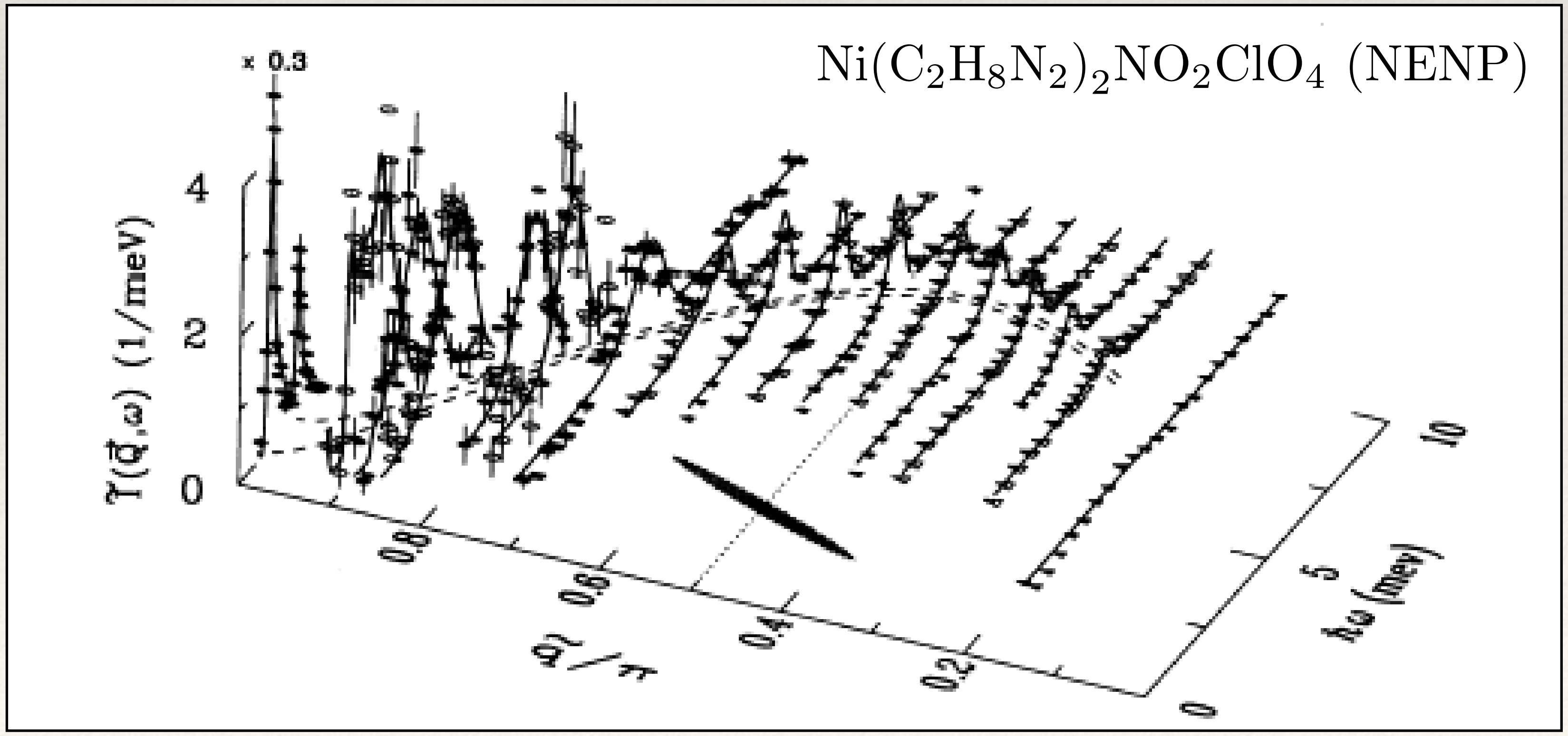
$$\langle \boldsymbol{S}_l\cdot\boldsymbol{S}_{l+n}\rangle \sim (-1)^n\,|n|^{-1/2}\exp\left(-|n|a/\xi\right)$$

F. D. M. Haldane, Phys. Lett. **93A**, 464 (1983)

elementary magnon / multimagnon continuua

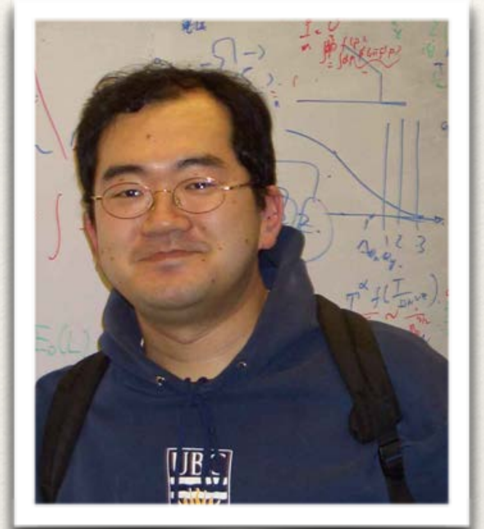


S. Ma et al., PRL **69**, 3571 (1992)



HOLSM Theorem

Lieb, Schultz, Mattis (1961)
Oshikawa (2000) / Hastings (2005)



Setting : lattice model with conserved U(1) charge, in any *d*.

$$\text{filling fraction: } \nu \equiv \frac{\text{total U(1) charge}}{\text{number of unit cells}}$$

"At fractional filling ν, a unique, gapped, featureless, insulating ground state is impossible."

$\nu \notin \mathbb{Z}$	unique	gapped	featureless	insulator	EXAMPLE
✓	✓	✓	✓	✓	NOT POSSIBLE
✓	✓	✗	✓	✗	METALLIC
✓	✓	✓	✗	✓	DENSITY WAVE
✓	✓	✗	✓	✓	SPIN-CHARGE SEPARATION
✓	✗	✓	✓	✓	TOPOLOGICAL ORDER
✗	✓	✓	✓	✓	BAND INSULATOR

One-dimensional Hubbard model : ν = # electrons per cell per spin DOF

$\nu \in \mathbb{Z}$: adiabatic connection to band insulator

$\nu \in \mathbb{Z}+\frac{1}{2}$: Mott phase preserves all symmetries, but with no adiabatic connection to band insulator

LSM theorem : flows from action of twist operator $U = \exp\left(\frac{2\pi i}{N}\sum_{j=1}^{N} j\,Q_j\right)$ which changes crystal momentum of ground state:

$$t\,|\Psi_0\rangle = e^{iK_0}|\,\Psi_0\,\rangle \qquad \text{and} \qquad |\Psi_1\rangle = U\,|\Psi_0\rangle$$

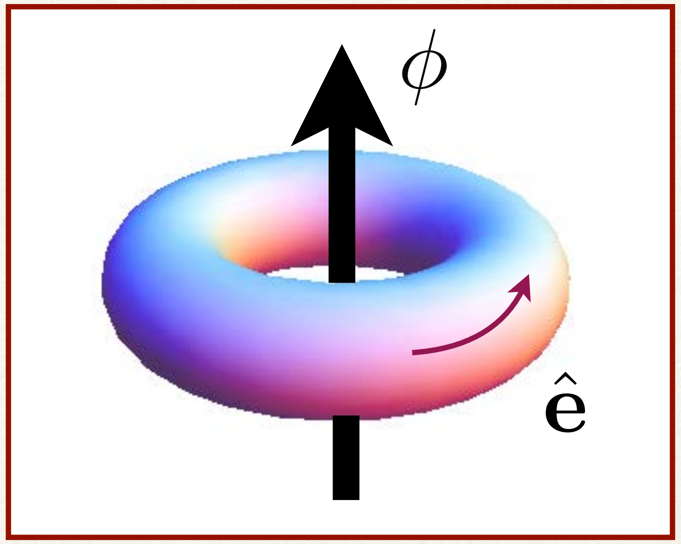
$$\Rightarrow \quad t\,|\Psi_1\rangle = e^{iK_1}|\Psi_1\rangle \qquad \text{where} \qquad \boldsymbol{K}_1 - \boldsymbol{K}_0 = -2\pi\nu$$

The LSM argument works only in d=1 because

$$\langle\Psi_0|U^\dagger H U|\Psi_0\rangle = E_0 + \frac{2\pi^2}{N^2}\langle\Psi_0|H_\perp^{\text{local}}|\Psi_0\rangle = E_0 + \mathcal{O}(N^{d-2})$$

⟹ gapless excitations or degenerate ground state for *S* = ½ HAFM

Oshikawa (2000) extended this argument to higher dimensions by considering the consequences of *adiabatic flux threading*. Place the system on a d-dimensional torus, and thread U(1) flux ϕ through one of its cycles, resulting in a translationally-invariant $H(\phi)$.



$\big[H(\phi), t\big] = 0 \quad \Rightarrow$ crystal momentum of $|\Psi_0(\phi)\rangle$ remains fixed!

$|\Psi_1\rangle \equiv U^\dagger|\Psi_0(2\pi)\rangle$ "pullback" from Hilbert space of $H(\phi = 2\pi)$ to that of $H(\phi = 0)$ — "large gauge transformation"

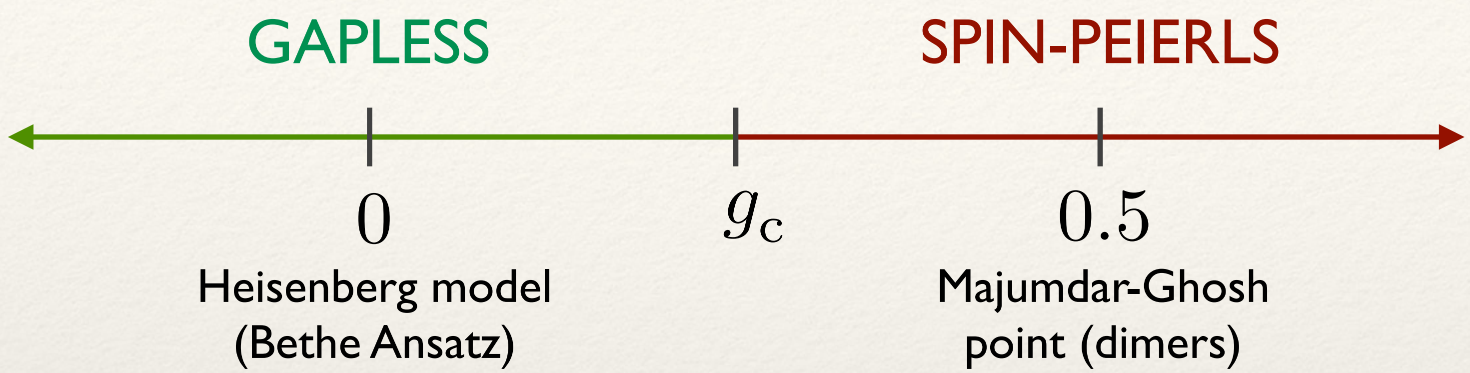
The difference in crystal momentum is then $\boldsymbol{K}_1 - \boldsymbol{K}_0 = 2\pi N_\perp \nu\, \hat{e}$, where $N_\perp$ is the number of sites in a hyperplane transverse to $\hat{e}$.

$\Delta\boldsymbol{K}$ not a reciprocal lattice vector $\Rightarrow \langle\Psi_0|\Psi_1\rangle = 0$

With ν = *p*/*q* this requires ($N_\perp$, *q*) relatively prime, *but not d* = 1!

Example : next-nearest neighbor Heisenberg chain

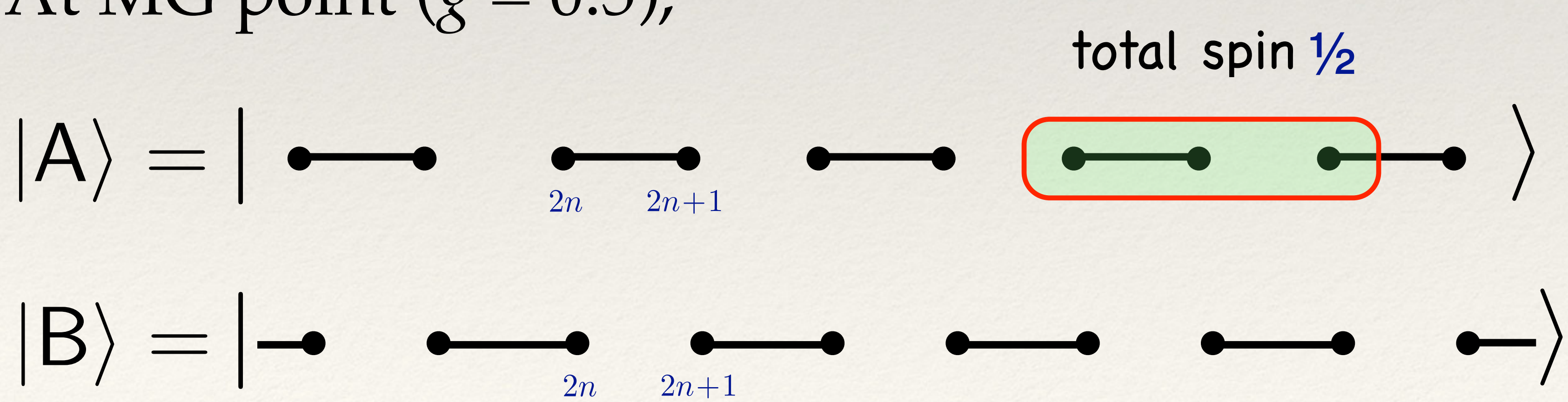
$$H = \sum_n \left[\boldsymbol{S}_n \cdot \boldsymbol{S}_{n+1} + g\, \boldsymbol{S}_n \cdot \boldsymbol{S}_{n+2} \right]$$



For $g \leq g_c \simeq 0.2411$, the spectrum is gapless.

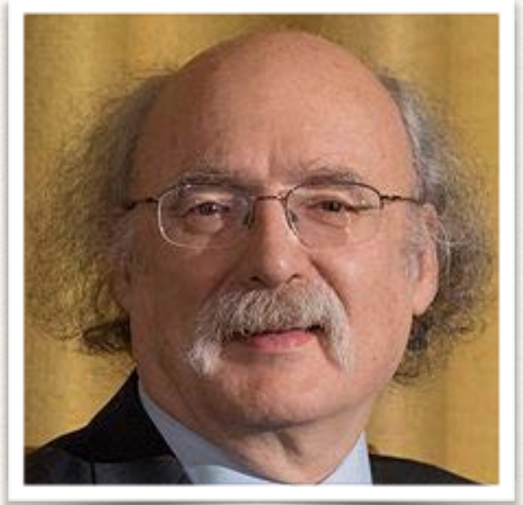
For $g > g_c$, the system is in a spin-Peierls phase (doubly degenerate ground state with excitation gap).

At MG point ($g = 0.5$),



$|A\rangle \pm |B\rangle$ has crystal momentum $0, \pi$

Historical interlude



F. D. M. Haldane (1983) argued that AFM Heisenberg chains with S=0,1,2,… should exhibit an excitation gap, while those with S=1/2, 3/2,… are gapless. He derived the O(3) nonlinear sigma model continuum field theory in the large S limit, based on classical equations of motion, and noted that the integer and half-odd-integer cases differed in their quantization.

Throughout the 1980s, this scenario, and gap for integer S, was called the "Haldane conjecture". But the real question should always have been why the half-odd-integer chains were gapless. Bethe's solution and wrongheaded notions of "quasi-LRO" misled many to presume that gaplessness was the natural state of affairs. Prior to Berry's seminal paper on geometric phases (1984), the essential differences between integer and half-odd-integer S chains at the quantum level were not widely appreciated, though Haldane clearly anticipated this distinction.

One aspect which gave Haldane pause was the existence of exactly solvable gapless integer S chains with bilinear-biquadratic interactions of the form

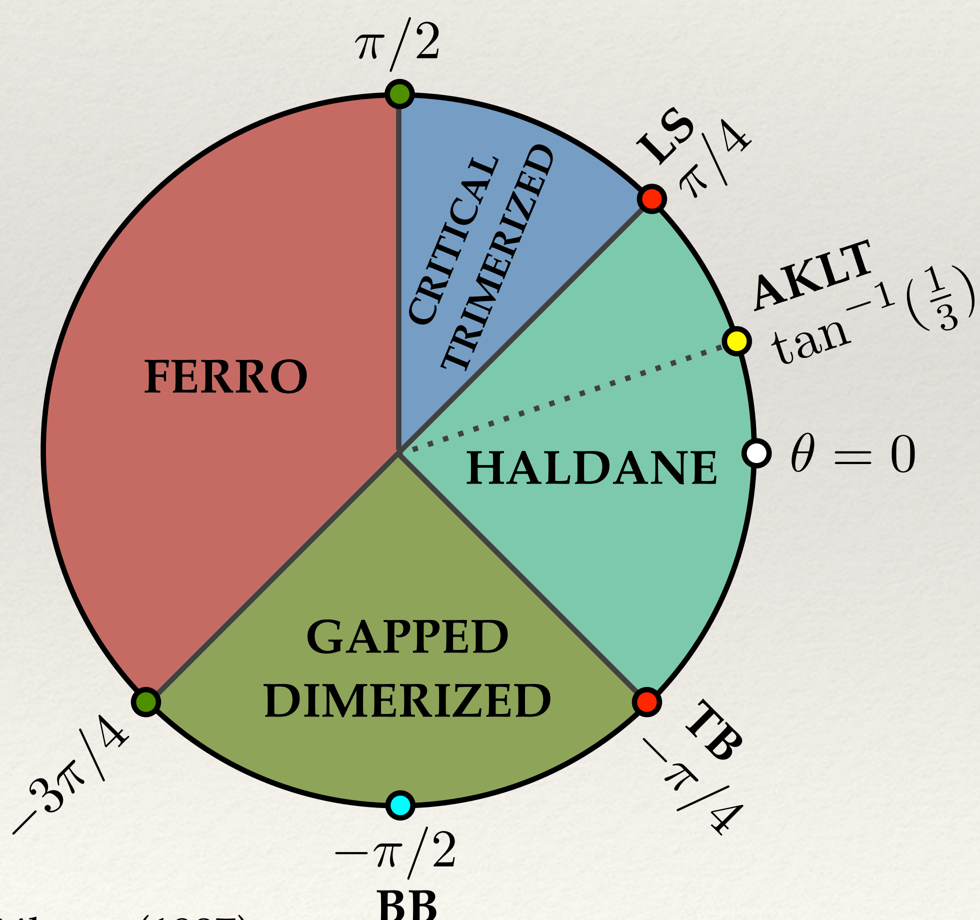
$$H_{n,n+1} = \cos\theta\, \boldsymbol{S}_n \cdot \boldsymbol{S}_{n+1} + \sin\theta\, (\boldsymbol{S}_n \cdot \boldsymbol{S}_{n+1})^2$$

$\theta = 0$: Heisenberg model

$\theta = +\pi/4$: Lai-Sutherland model (1975), gapless, **SU(3) symmetric**

$\theta = -\pi/4$: Takhtajan-Babujian model (1982), gapless

$\theta = -\pi/2$: Barber-Batchelor model (1989), **gapped, dimerized**



Sólyom (1987)

Läuchli, Schmid, Treibst (2006)

Haldane correctly reasoned that the sizable biquadratic terms must be responsible for the gap collapse. Of particular importance was the construction of a model at tan(*θ*)=*1*/*3* by Affleck, Kennedy, Lieb, and Tasaki (1987). Though non-integrable, it has a solvable ground state which demonstrably exhibits exponentially decaying correlations. AKLT's model would pave the way to matrix product states and to symmetry-protected topological phases.

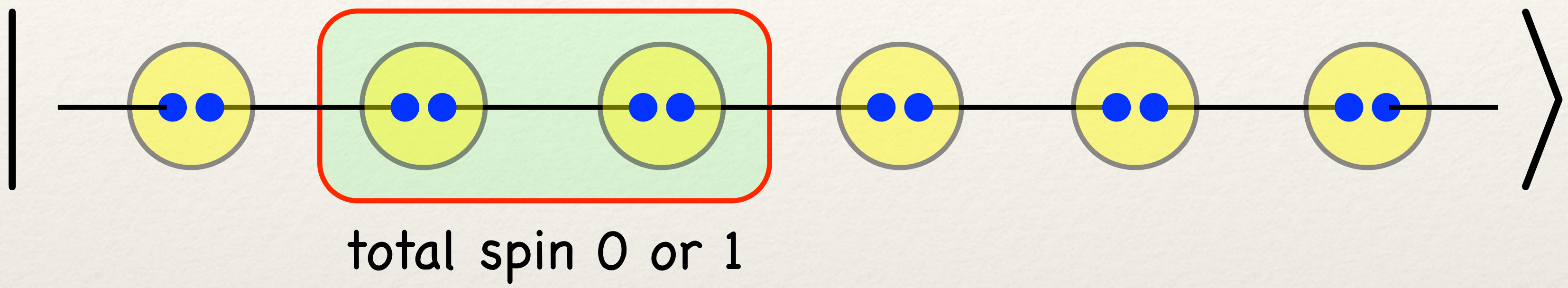
AKLT models for valence bond solids

Affleck *et al.* (1987)
DPA *et al.* (1988)

Spin *S* from symmetrized product of 2S spin-½ quanta :

$S\!=\!1$: $\quad \frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$ $\quad$ ● = symmetrizer

$|\Psi_0\rangle =$



Let $P_2(n,n+1)$ be the *projector* onto total spin $J_{n,n+1} = 2$.

Then $P_2(n,n+1)\,|\,\Psi_0\,\rangle = 0$ for every *n*, so $H\,|\,\Psi_0\,\rangle = 0$ with

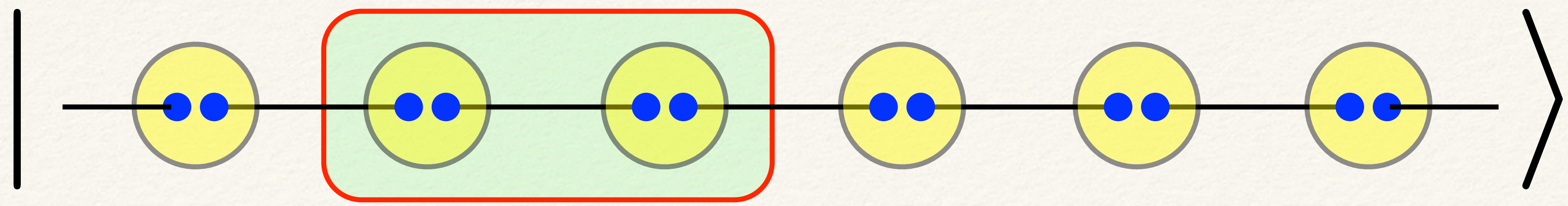
$$H = \sum_n P_2(n,n+1) = \tfrac{1}{2}\sum_n \Big[\boldsymbol{S}_n\cdot\boldsymbol{S}_{n+1} + \tfrac{1}{3}(\boldsymbol{S}_n\cdot\boldsymbol{S}_{n+1})^2\Big] + \tfrac{1}{3}N$$

General construction : $\;|\,\Psi_0(\mathcal{L},M)\,\rangle = \prod_{\langle ij\rangle\in\mathcal{L}} (a_i^\dagger b_j^\dagger - b_i^\dagger a_j^\dagger)^M\,|\,0\,\rangle$

$S = \tfrac{1}{2}Mz \quad , \quad J_{\max} = 2S - M$

Schwinger boson representation of SU(2)
(DPA, Auerbach, Haldane, 1988)

Finding a Hamiltonian ($S = 1$) :

$|\Psi_0\rangle =$ 

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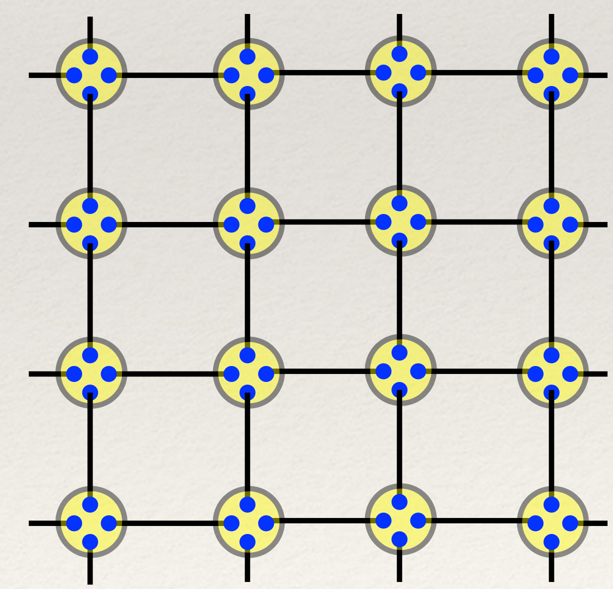
General construction :

Schwinger boson representation of SU(2)
(DPA, Auerbach, Haldane, 1988)

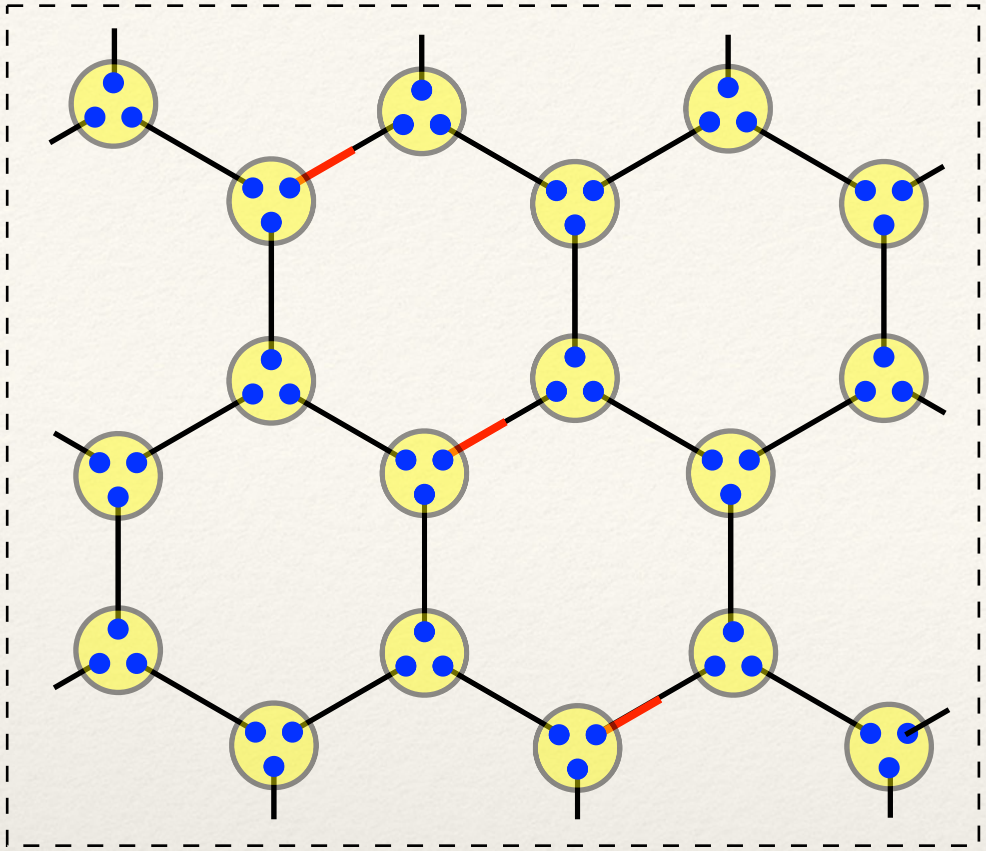
$$|\,\Psi_0(\mathcal{L}, M)\,\rangle = \prod_{\langle ij\rangle \in \mathcal{L}} (a_i^\dagger b_j^\dagger - b_i^\dagger a_j^\dagger)^M\,|\,0\,\rangle$$

- $S = \frac{1}{2}M z$ where z is number of neighbors
- $J_{\mathrm{max}} = 2S - M$ is maximum spin on each link

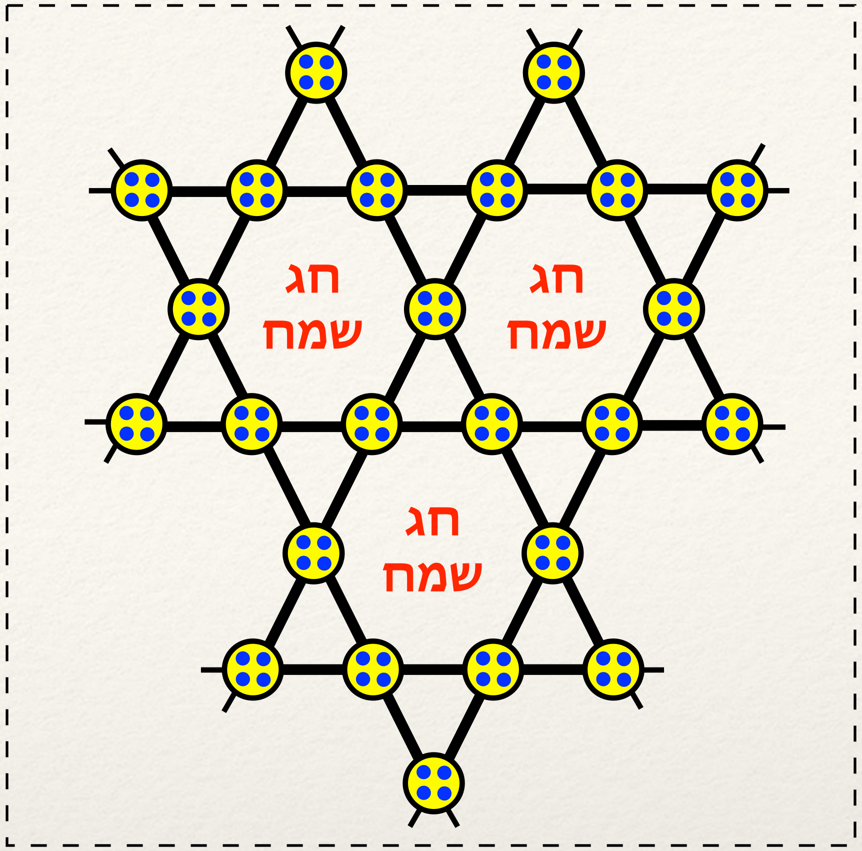


$S = 2$

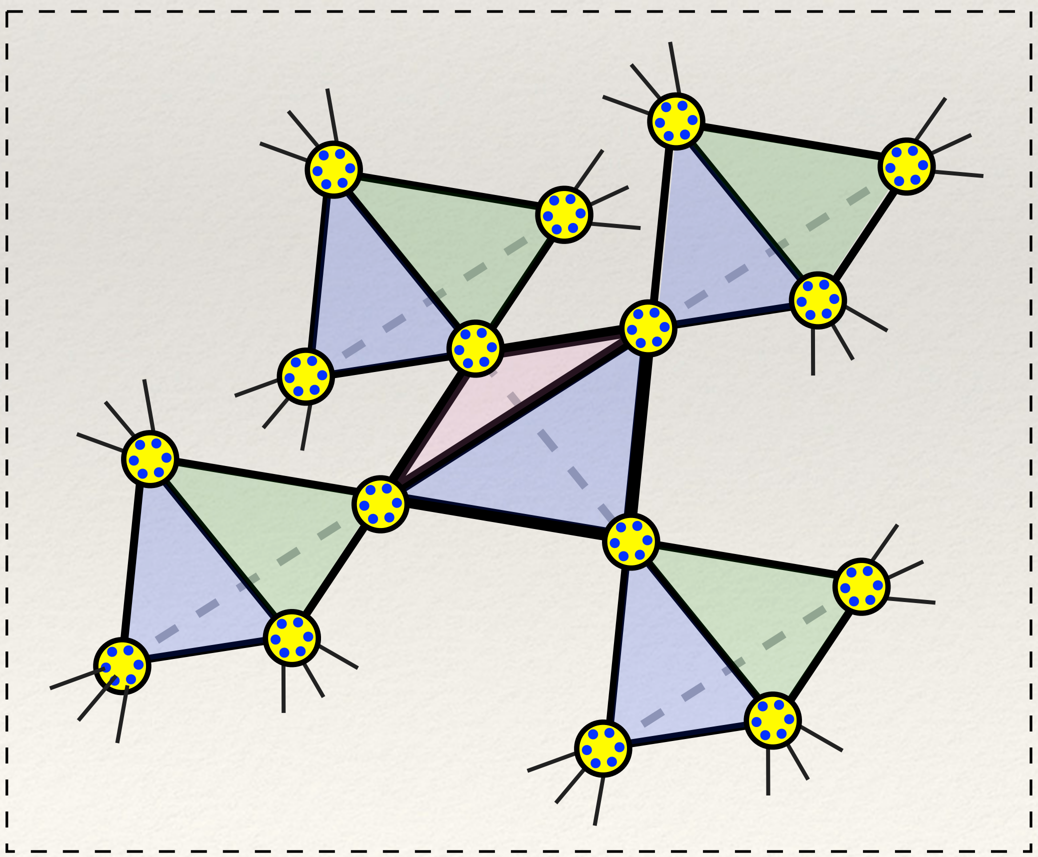
AKLT tensor network states



S = 3/2 honeycomb (z=3)

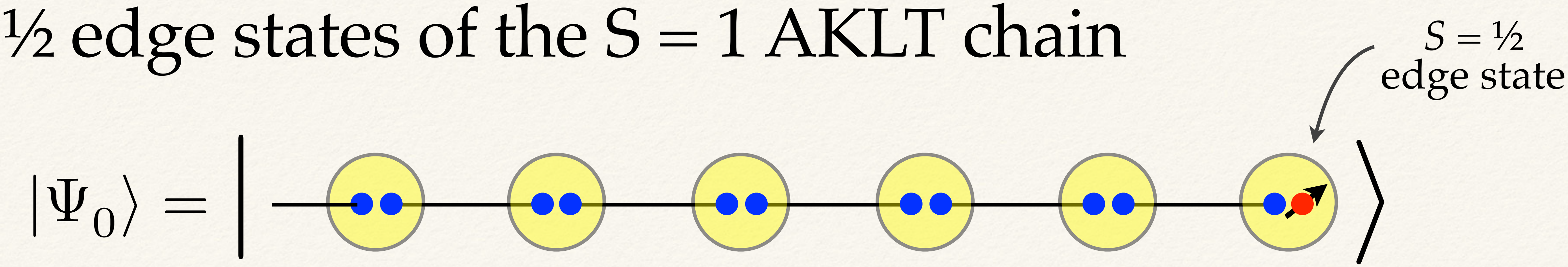


S = 2 Kagomé (z=4)

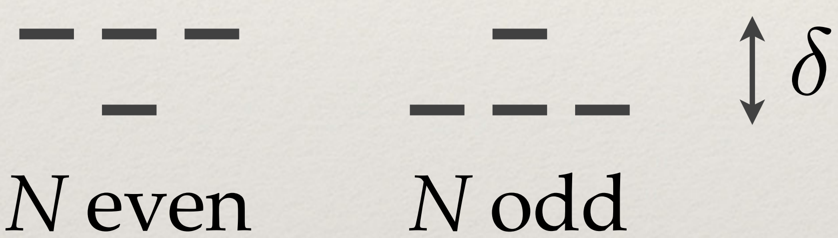


S = 3 pyrochlore (z=6)

S = ½ edge states of the S = 1 AKLT chain

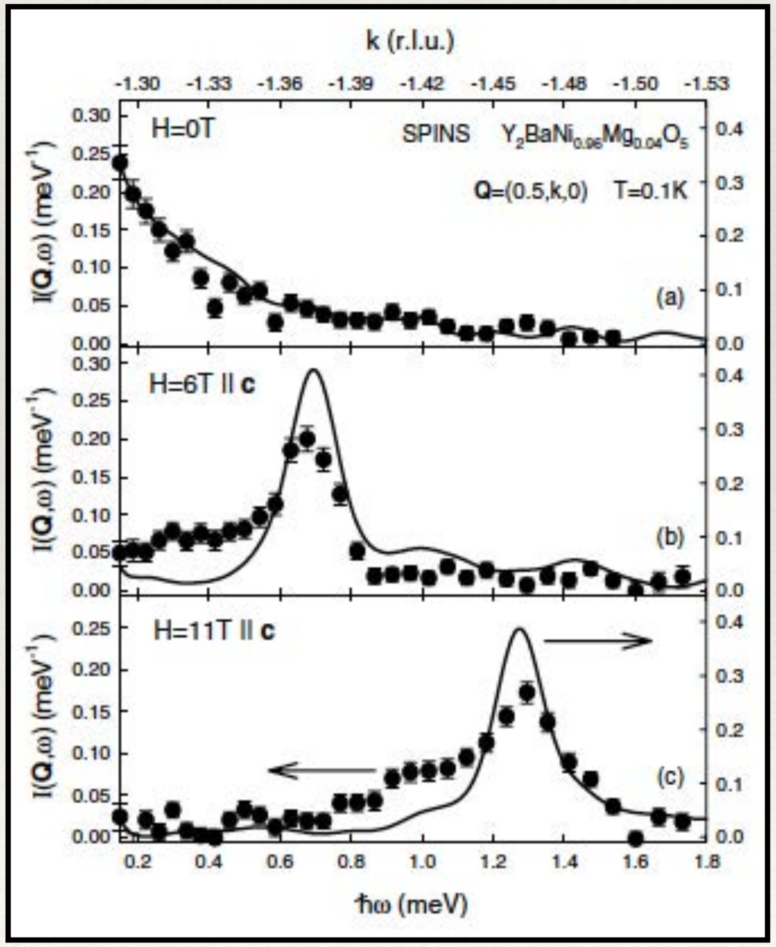


T. Kennedy (1990) : exact diagonalization of open $S = 1$ bilinear-biquadratic chains for $-\pi/4 < \theta < \pi/4$. Found that for $-\pi/4 < \theta < \theta_{\mathrm{AKLT}}$ the spectrum consisted of four low-lying states separated by a gap from the continuum, split into singlet and triplet :

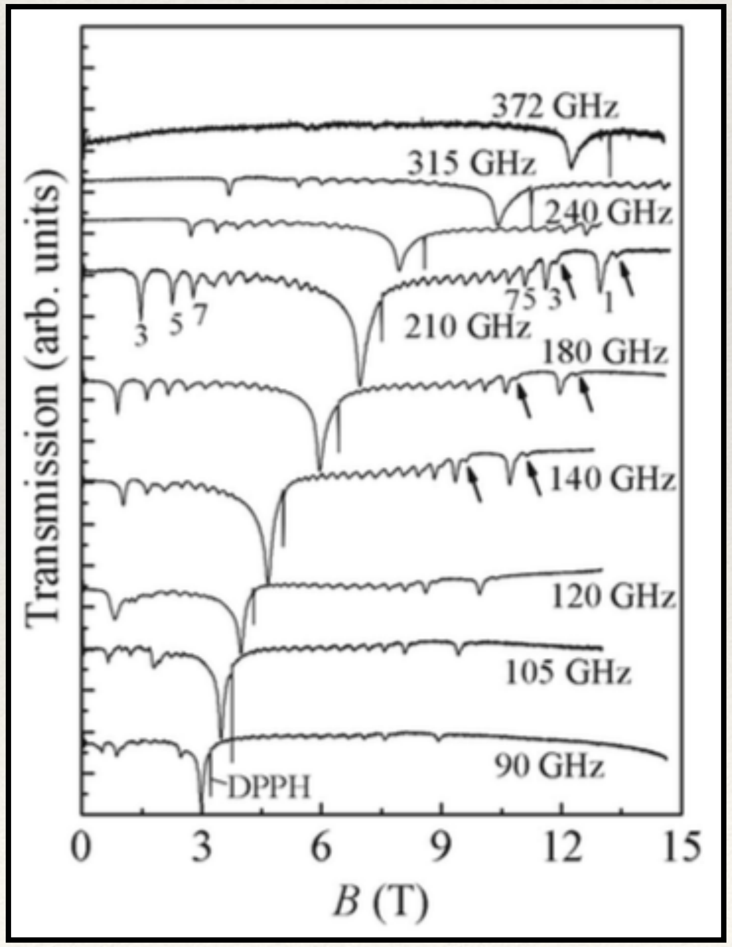


with gap $\delta \sim J\exp(-N/\xi)$, the signature of interacting S = ½ edge states! These edge states were experimentally observed in INS and ESR studies of Y_2BaNi$_{1-x}Mg_xO_5$.

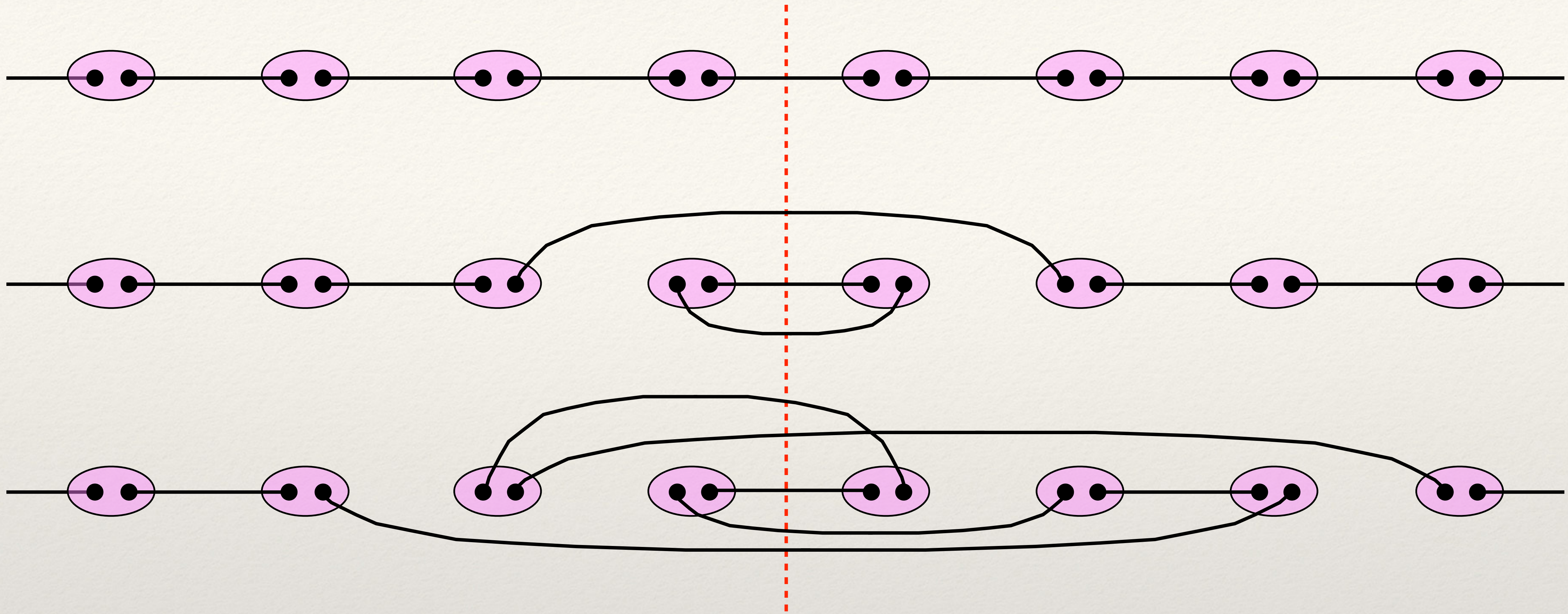
INS: Kenzelmann *et al.* (2003)



ESR: Yoshida *et al.* (2005)

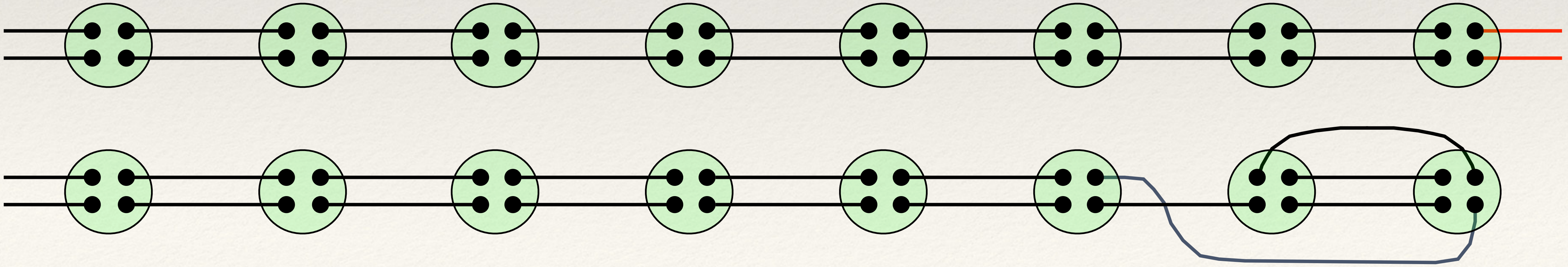


Away from AKLT point, ground state is a linear superposition:



Equivalent to "topological protection of edge states" for ***odd S***.

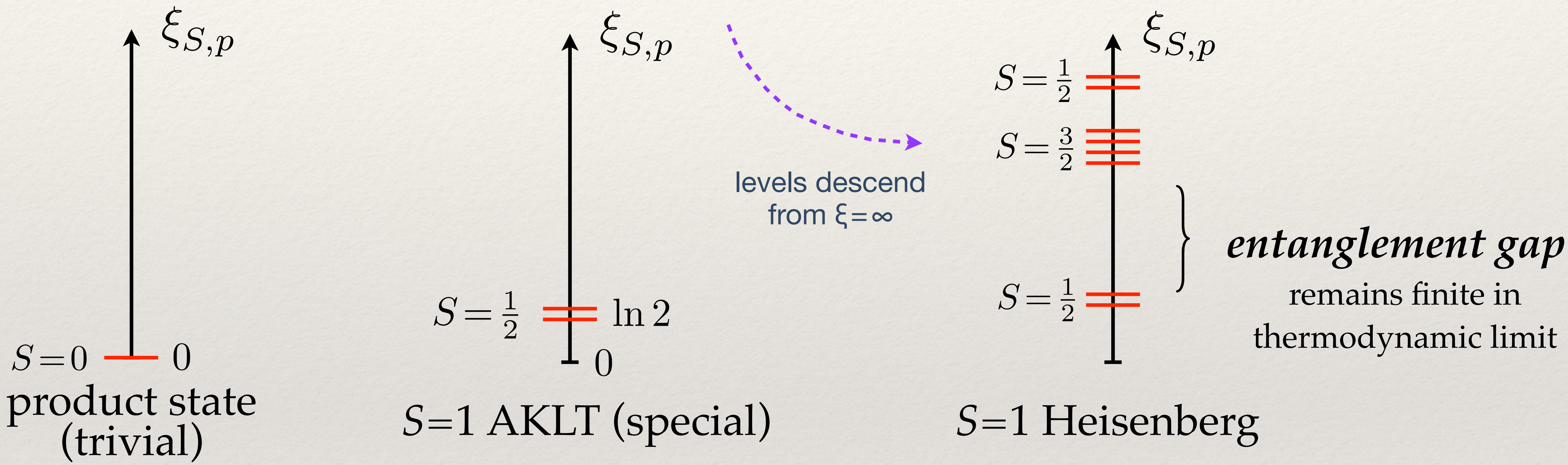
Pollman et al. (2010)

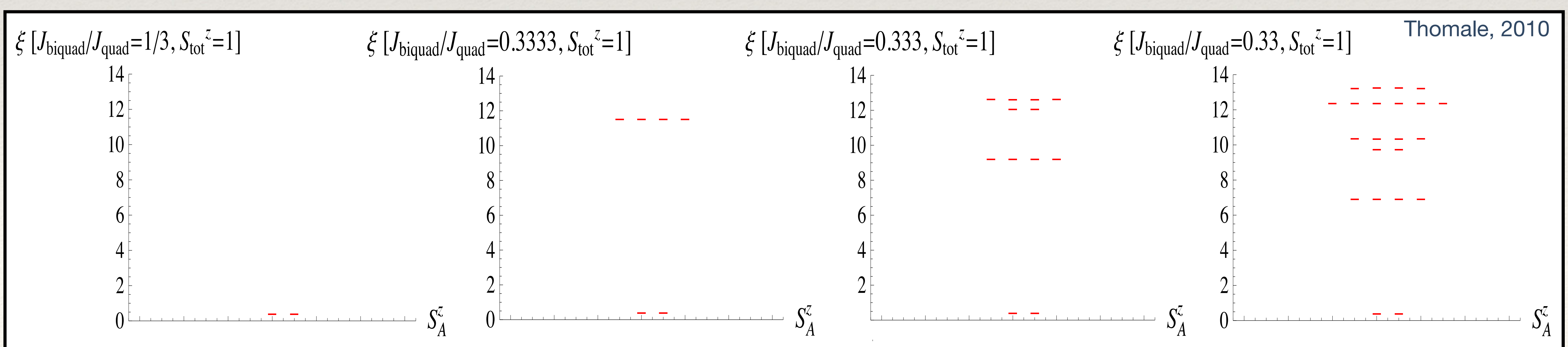


$S = 2$

For an AFM spin chain with SU(2) symmetry and a singlet ground state, the *entanglement eigenstates* may be classified by total spin :

$$|\Psi\rangle = \sum_{S} \sum_{p=1}^{N_S} e^{-\frac{1}{2}\xi_{S,p}} \sum_{m=-S}^{S} (-1)^{S-m} \, |\psi^{\mathrm{A}}_{S,m,p}\rangle \otimes |\psi^{\mathrm{B}}_{S,-m,p}\rangle$$

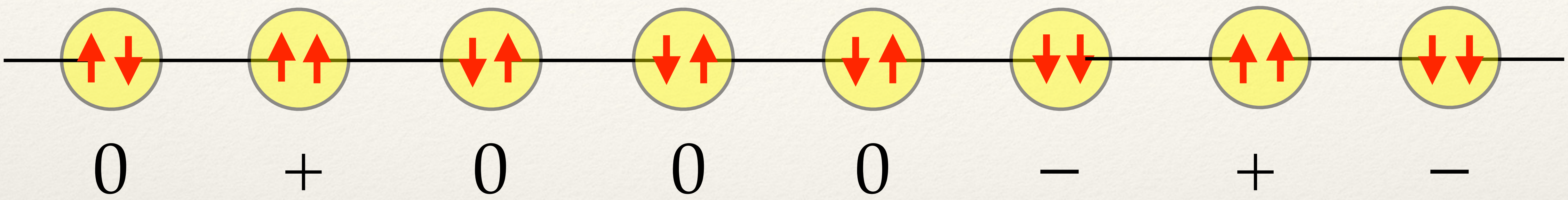




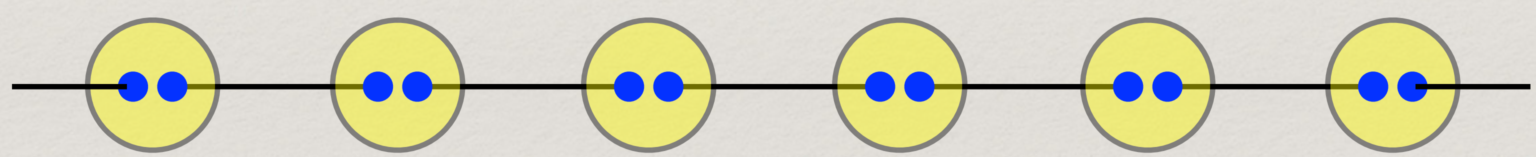
Hidden order and $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry

den Nijs and Rommelse (1989)
DPA and Girvin (1989)
Kennedy and Tasaki (1992)
Oshikawa (1992)
Tu, Zhang, and Xiang (2009)

Before projection, each link is a linear combination $|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$:



Write as $|\Psi\rangle = \sum_{\vec{m}} A_{m_1 \cdots m_N} |m_1, \ldots, m_N\rangle$ with $m_j =$ +1,0,-1

So $|\Psi_0\rangle = \Big|$  $\Big\rangle$ contains:

+ − + 0 0 0 0 0 0 − 0 0 0 0 + − + 0 ✓

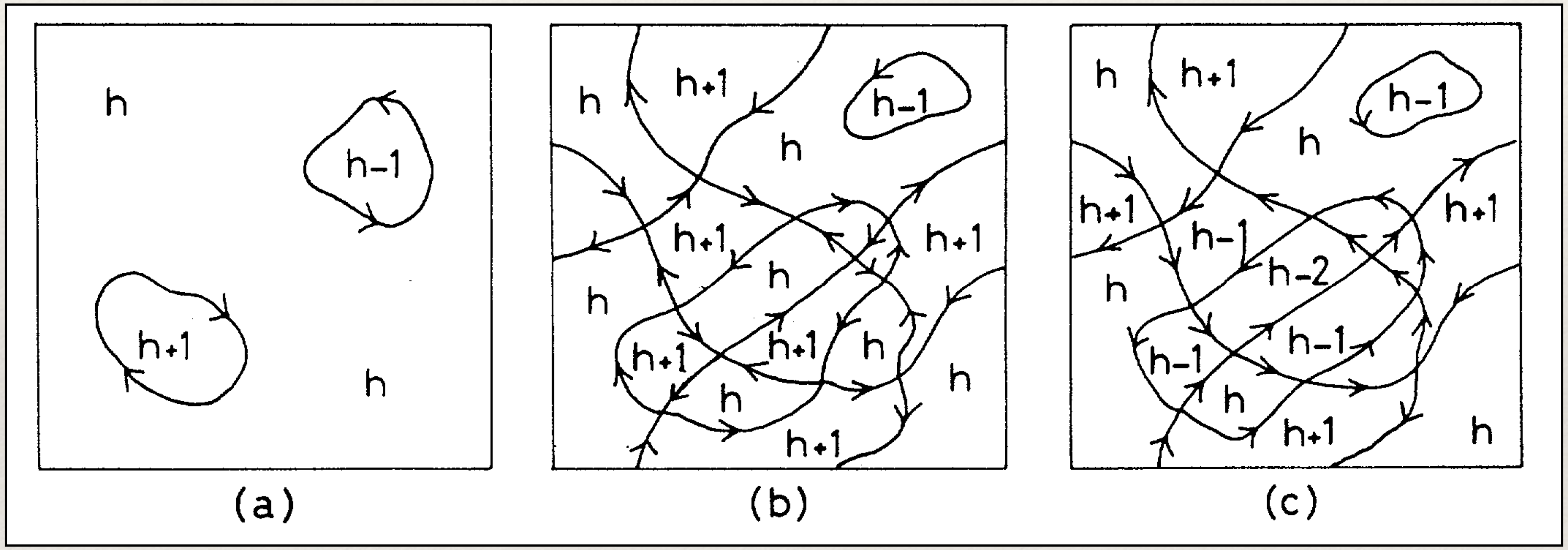
0 0 − 0 0 0 + 0 − 0 0 + − 0 0 0 0 + ✓

0 + 0 0 0 0 0 + 0 0 0 − 0 0 0 − 0 + ✗

Perfect antiferromagnetic order among the +/− states once the 0 states are removed! The system has a "string order parameter",

$$C(n) \equiv \langle S_0^z S_n^z \rangle = \tfrac{4}{3}\left(-\tfrac{1}{3}\right)^{|n|} \quad , \quad \widetilde{C}(n) \equiv \left\langle S_0^z \prod_{j=1}^{n-1} e^{i\pi S_j^z} \; S_n^z \right\rangle = -\tfrac{4}{9}$$

first derived by den Nijs and Rommelse (1989) in the context of classical models of the preroughening transition.



RSOS flat | disordered flat | RSOS rough

General S AKLT chains: $Q[n,n'] = \sum_{j=n}^{n'} S_j^z = \ell_{n-1} - \ell_{n'}$ with $l_j \in \{0,\ldots,S\}$

"Charge" Q is bounded : $-S \leq Q \leq S$ DPA and Girvin (1989)

Kennedy-Tasaki-Oshikawa nonlocal unitary transformation :

$$U = \prod_{j<k} e^{i\pi S_j^z S_k^x} \quad : \quad H \to \widetilde{H} = U H U^\dagger$$

If H is the $S = 1$ Heisenberg Hamiltonian with open boundaries,

$$\widetilde{H} = \sum_n \left[S_n^x \, e^{i\pi S_{n+1}^z} S_{n+1}^x + S_n^y \, e^{i\pi (S_n^z + S_{n+1}^z)} S_{n+1}^y + S_n^z \, e^{i\pi S_{n+1}^z} S_n^z \right]$$ still local!

This model has a $\mathbb{Z}_2$ x $\mathbb{Z}_2$ symmetry, i.e. global rotations by π about the x, y, or z axis (symmetry group D_2) which is *realized nonlocally* on H.

The string operator transforms as

$$U \Big(S_j^z \, e^{i\pi \sum_{l=j}^{k-1} S_l^z} S_k^z \Big) U^\dagger = S_j^z S_k^z$$

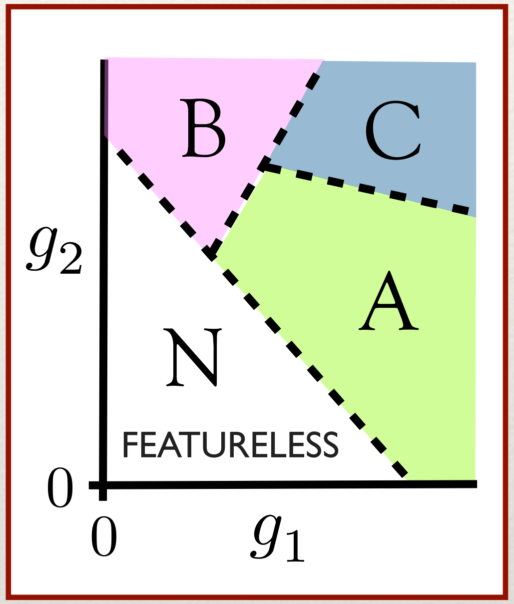
Spontaneous breaking of $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry then entails a fourfold ground state degeneracy for both $\widetilde{H}$ and H. The "hidden" string order for H is realized as *bulk ferromagnetism* for $\widetilde{H}$, where the moment points along $\pm\hat{\boldsymbol{x}} \pm \hat{\boldsymbol{z}}$.

AKLT chain as a **symmetry protected topological phase**

Big question : how to distinguish different phases of matter?

Conventional answer : Ordered phases of matter are classified by their patterns of spontaneous symmetry breaking. Disordered phases (liquid, gas) are essentially equivalent.

What about quantum (T=0) phases? Same story?



phase diagram

Modern perspective : *"beyond the Landau paradigm"*

Hastings and Wen (2005) Chen, Gu, and Wen (2010)

Two *gapped* quantum phases are distinct if their WFs are *adiabatically disconnected*. A quantum phase which cannot be adiabatically connected, by some sequence of local unitary transformations, to a (trivial) product state is ***topologically ordered***.

Examples : FQHE states, Kitaev toric code ($\mathbb{Z}_2$ spin liquid)

For local Hamiltonians in $d=1$, the ground state is always adiabatically connected to a trivial product state. Verstraete (2005)

However, imposing a *symmetry* G can result in an obstruction to this adiabatic connection. In this case, either:

- SSB : ground state $|\Psi_0\rangle$ state breaks G.

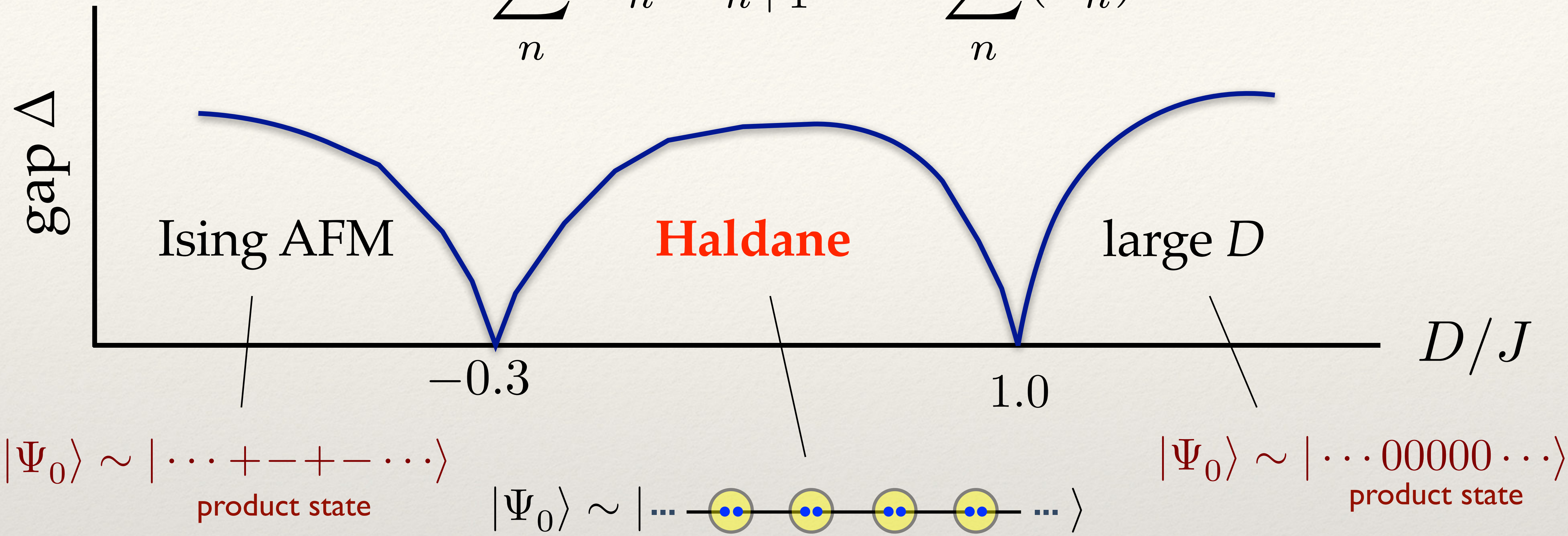
- G remains unbroken, and $|\Psi_0\rangle$ adiabatically connected to trivial product state by local G-preserving unitaries.

- G remains unbroken, but $|\Psi_0\rangle$ adiabatically disconnected from a trivial product state via local G-preserving unitaries. $|\Psi_0\rangle$ is then a symmetry-protected topological (SPT) phase.

Back to $S=1$ chain :

single ion anisotropy

$$H = J\sum_n \boldsymbol{S}_n\cdot\boldsymbol{S}_{n+1} + D\sum_n (S_n^z)^2$$



PHYSICAL REVIEW B 81, 064439 (2010)

Entanglement spectrum of a topological phase in one dimension

Pollmann, Turner, Berg, and Oshikawa

Haldane phase is an SPT protected by any of :

(i) time-reversal, (ii) space inversion, or (iii) broken $\mathbb{Z}_2\times\mathbb{Z}_2$ symmetry

Explicit symmetry-breaking adiabatic trivialization

Schwinger boson representation of $S=1$ AKLT chain :

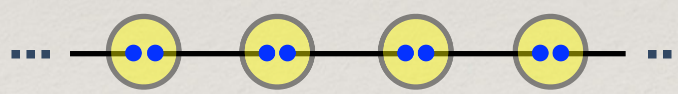
$$|\Psi_0\rangle = \prod_j \left(a_j^\dagger\, b_{j+1}^\dagger - b_j^\dagger\, a_{j+1}^\dagger\right)|0\rangle$$

↓ break time reversal, space inversion

$$|\Psi_0\rangle = \prod_n \left(\cos\theta\, a_n^\dagger b_{n+1}^\dagger - \sin\theta\, b_n^\dagger a_{n+1}^\dagger\right)|0\rangle$$

↓ break lattice translation

$$|\Psi_0\rangle = \prod_n \left(\cos\theta_{\rm e}\, a_{2n}^\dagger\, b_{2n+1}^\dagger - \sin\theta_{\rm e}\, b_{2n}^\dagger\, a_{2n+1}^\dagger\right)\left(\cos\theta_{\rm o}\, a_{2n+1}^\dagger\, b_{2n+2}^\dagger - \sin\theta_{\rm o}\, b_{2n+1}^\dagger\, a_{2n+2}^\dagger\right)|0\rangle$$

AKLT : $|\Psi_0\rangle = |$  $\rangle$, $\theta_{\rm e} = \frac{\pi}{4}$, $\theta_{\rm o} = \frac{\pi}{4}$

$D=\infty$: $|\Psi_0\rangle = |\cdots 00000 \cdots\rangle$, $\theta_{\rm e} = 0$, $\theta_{\rm o} = 0$

$D=-\infty$: $|\Psi_0\rangle = |\cdots +-+- \cdots\rangle$, $\theta_{\rm e} = 0$, $\theta_{\rm o} = \frac{\pi}{2}$

$S=2$ AKLT chain : adiabatically connected to large-D state

not topologically protected! Pollmann et al. (2012)

Continuum field theories of quantum magnetism

Using spin coherent state path integral, one can derive :

Quantum ferromagnet (naive continuum limit of CSPI) : Read and Sachdev (1995)

$$\mathcal{S} = \int\! d^d\!x \int\! dt \left[-\,\hbar M_0\, \mathbf{A}(\hat{\mathbf{\Omega}}) \cdot \frac{\partial \hat{\mathbf{\Omega}}}{\partial t} - \tfrac{1}{2}\rho_{\rm s} \left|\boldsymbol{\nabla}\hat{\mathbf{\Omega}}\right|^2 \right]$$

magnetization density

$\boldsymbol{\nabla} \times \mathbf{A} = \hat{\mathbf{\Omega}}$

spin stiffness

Quantum antiferromagnet (requires some work) : Haldane (1983, 1988) Affleck (1985)

susceptibility

Néel vector

spin stiffness

solid angle

$$\mathcal{S} = \int\! d^d\!x \int\! dt \left[\tfrac{1}{2}\chi\, (\partial_t \hat{\boldsymbol{n}})^2 - \tfrac{1}{2}\rho_{\rm s} \left|\boldsymbol{\nabla}\hat{\boldsymbol{n}}\right|^2 \right] - \hbar S \sum_i \eta_i\, \omega[\hat{\boldsymbol{n}}_i]$$

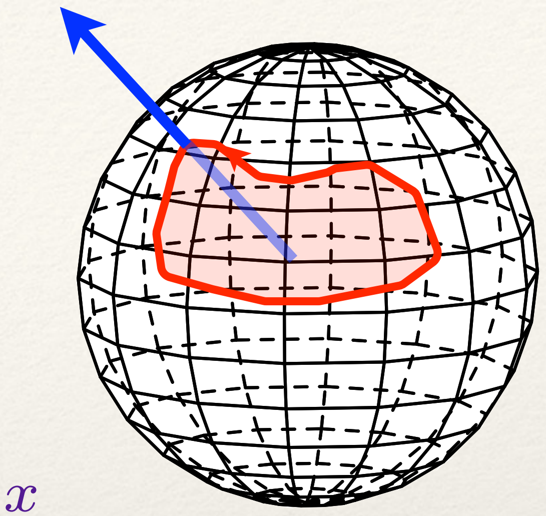
nonlinear sigma model in (d+1)-dimensions

from Berry phase

Lorentz invariance with $c^2 = \rho_{\rm s}/\chi$

Goldstone's theorem precludes SSB for $(d+1) \leq 2$, but expect mass gap with exponential decay of correlations. How to understand S = ½ chain?

Berry phase in d=1 : spin liquid vs. Haldane gap



Néel vector

$$\mathcal{S}_{\text{Berry}}/\hbar = -S\int dt\sum_j (-1)^j\,\omega[\hat{\boldsymbol{n}}_j]$$

$$= \tfrac{1}{2}S\int dx\int dt\,\hat{\boldsymbol{n}}\cdot\frac{\partial\hat{\boldsymbol{n}}}{\partial t}\times\frac{\partial\hat{\boldsymbol{n}}}{\partial x} = 2\pi S\,Q_{tx}$$

Q_{tx} is an integer topological invariant (Pontrjagin number)

$$e^{i\mathcal{S}_{\text{Berry}}/\hbar} = e^{2\pi i S Q_{tx}}$$

2S even : topological term is invisible ☞ conventional NLσM
Haldane gap, exponential decay of correlations
all Heisenberg chains with 2S even qualitatively the same

2S odd : destructive interference between topological sectors
spin liquid behavior, “quasi-LRO”, power law decays
all Heisenberg chains with 2S odd qualitatively the same

Quantum AFM in $d > 2$: Néel order vs. quantum disorder

Euclidean action functional of the Néel field :

Chakravarty, Halperin, Nelson (1988)
DPA and Auerbach (1988)
Read and Sachdev (1989)
Sachdev (1999)

$$\frac{1}{\hbar}\mathcal{S}_{\rm E} = \frac{1}{2g}\int d^d x \int_0^{L_\tau} dx^0 \, (\partial_\mu n^a)^2$$

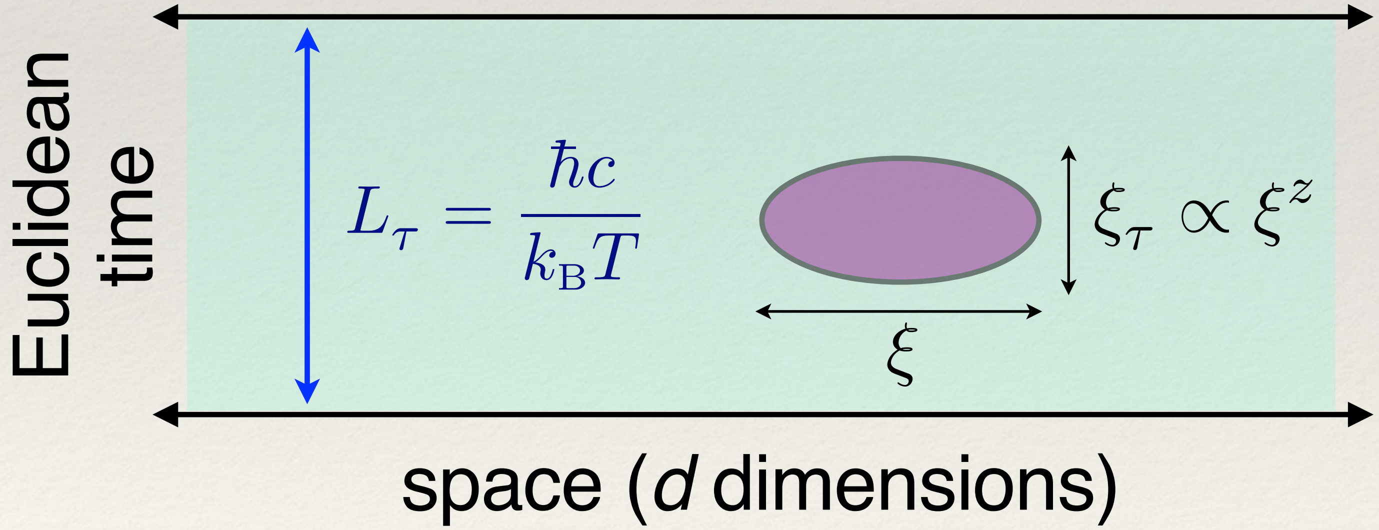
$x^0 = c\tau$

quantum parameter:

$$g = \hbar c/\rho_{\rm s}$$

Berry phase term cancels for smooth Néel field configurations.

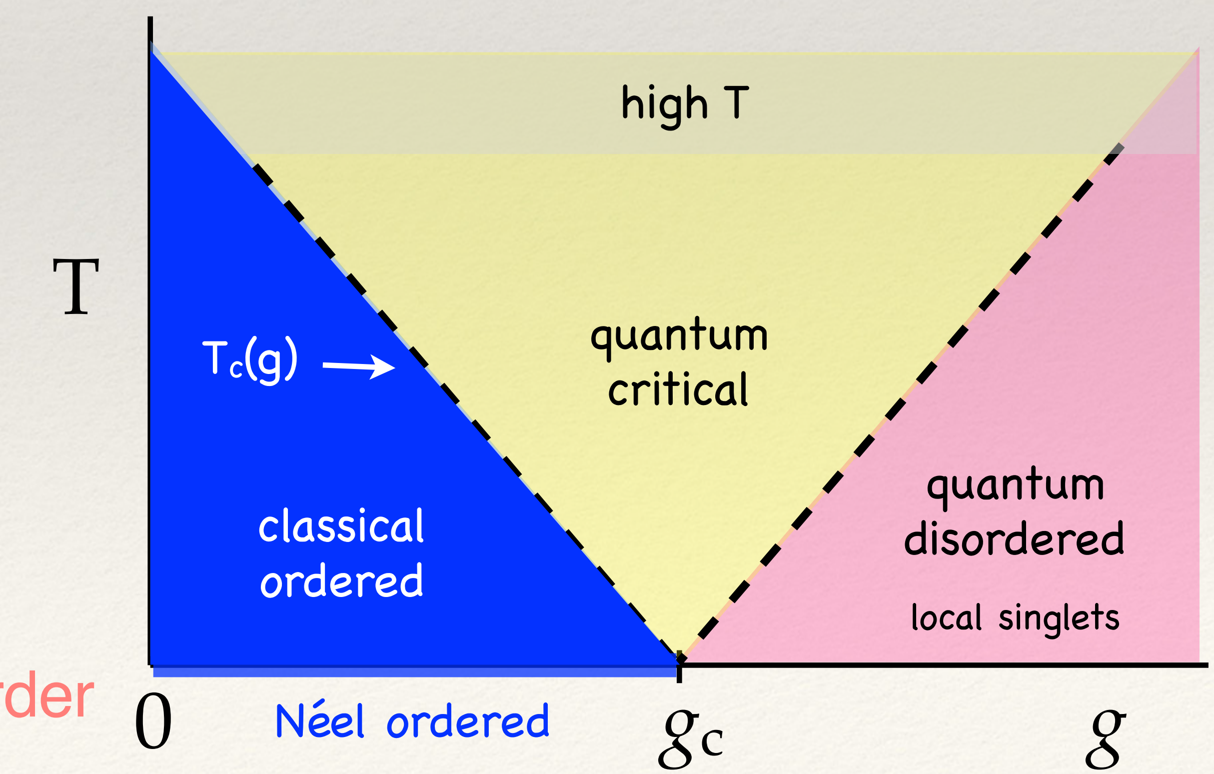
QM in d dimensions ↔ statistical mechanics in $(d+1)$ dimensions



T=0: $(d+1)$-dimensional NLσM with "temperature" g.

$g<g_c$: Néel order

$g>g_c$: quantum disorder



On all d=2 Bravais lattices, even for "best case" S=½ , the nearest neighbor Heisenberg antiferromagnet possesses Néel order at $T=0$.

What do we need to get a quantum disordered ground state?

- Extension to different algebras of quantum spin:

  $SU(N)$, $Sp(N)$, …

  Affleck and Marston (1987)
  DPA and Auerbach (1988)
  Read and Sachdev (1990)

- Reduction to space of local singlet coverings:

  quantum dimer model

  Rokhsar and Kivelson (1988)
  Moessner and Sondhi (2001)

- Further neighbor couplings, frustrated lattices

  geometrical frustration

  Misguich and Lhuillier (2003)
  Moessner (2001)

  depleted lattices

  e.g. CaV_4O_9, $SrCu_2(BO_3)_2$
  Taniguchi et al., (1995)
  Kageyama et al. (1999)

- Models with gauge symmetries:

  Kitaev's models

  Kitaev (2005, 2006)

Large N extensions of SU(2)

DPA and Auerbach, (1988)
Read and Sachdev (1990)

Schwinger representation of SU(2):

$$S^+ = a^\dagger b \qquad S^z = \tfrac{1}{2}(n_a - n_b)$$
$$S^- = a\, b^\dagger \qquad 2S = n_a + n_b$$

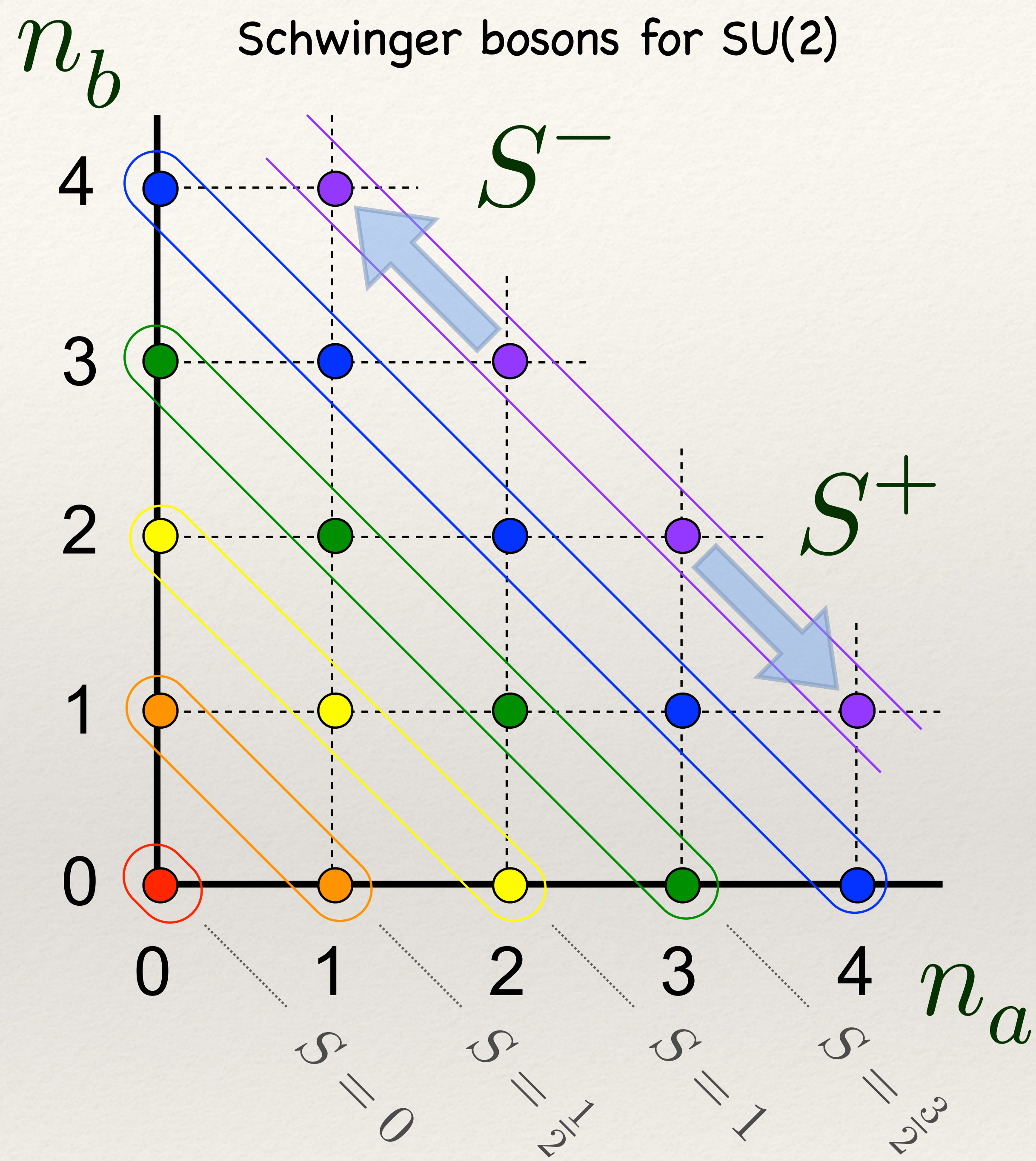
Heisenberg interaction:

$$\boldsymbol{S}_i \cdot \boldsymbol{S}_j = S^2 - \tfrac{1}{2}\mathcal{A}_{ij}^\dagger \mathcal{A}_{ij}$$
$$\mathcal{A}_{ij} = a_i b_j - b_i a_j$$

$\downarrow$ N copies

$$\mathcal{A}_{ij} = \sum_{m=1}^{N} (a_{im} b_{jm} - b_{im} a_{jm})$$

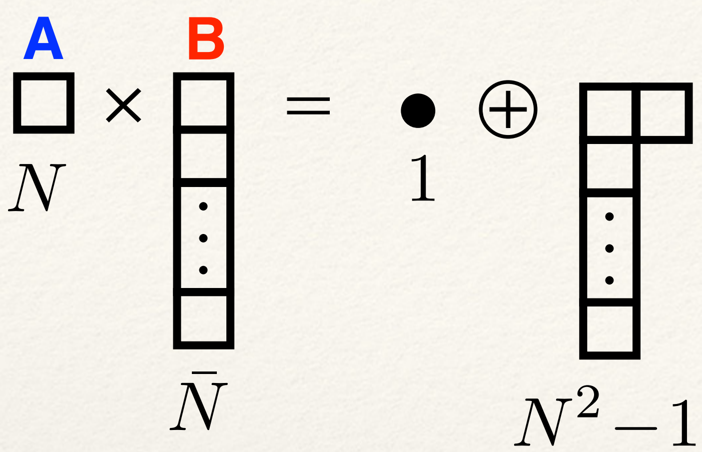
$$n_c = \sum_{m=1}^{N} (a_{im}^\dagger a_{im} + b_{im}^\dagger b_{im}) \equiv \kappa N$$



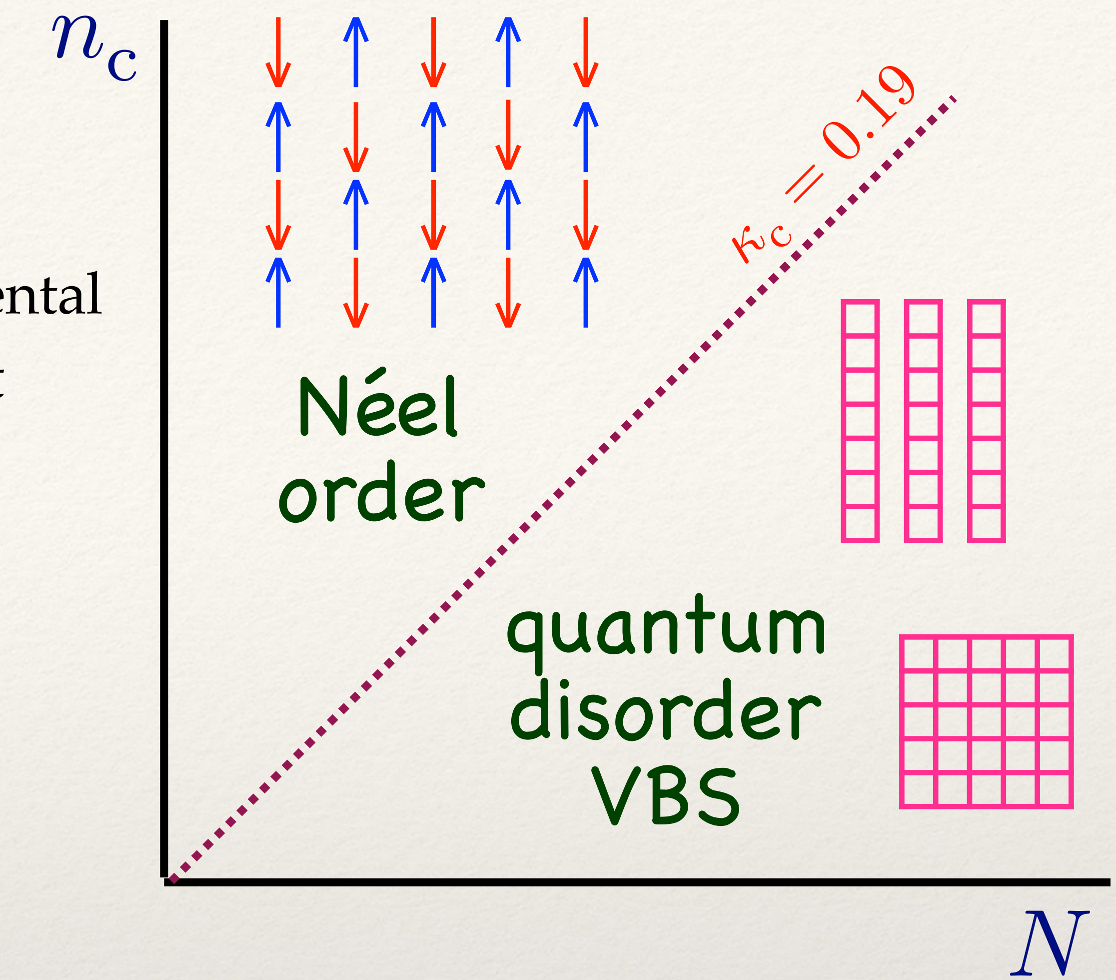
which is the Sp(N) extension

$$\mathrm{SU}(2) \cong \mathrm{Sp}(1) \quad \Rightarrow \quad \kappa = S$$

Original large-N model (AA 1988, RS 1989) :



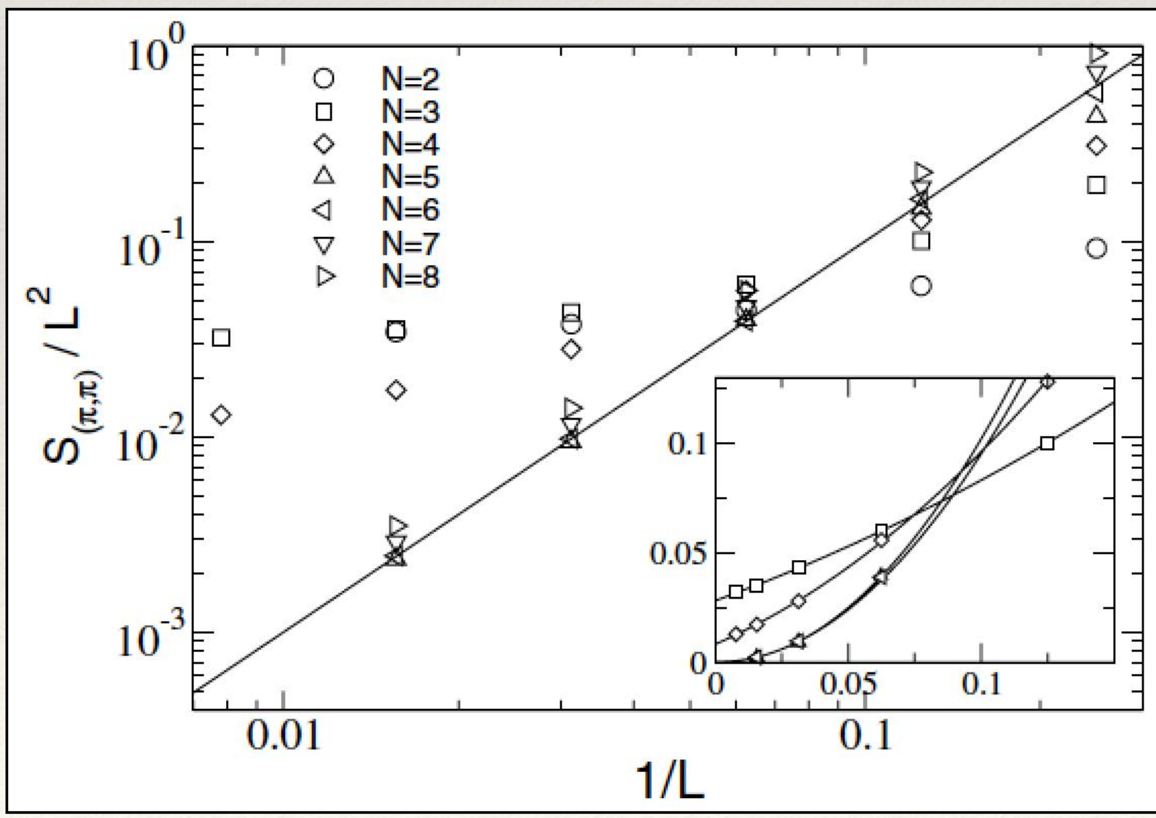
bipartite lattice :
fundamental × antifundamental
yields singlet + adjoint
representations



$$\mathcal{A}_{ij} = \sum_{\mu=1}^{N} b_{i\mu}\, b_{j\mu} \quad , \quad n_c = \sum_{\mu=1}^{N} b^\dagger_{i\mu} b_{i\mu} \equiv \kappa N$$

$$H = -\frac{1}{2N} \sum_{i<j} J_{ij}\, \mathcal{A}^\dagger_{ij} \mathcal{A}_{ij}$$

Harada, Kawashima, and Troyer (2003):
SU(N) antiferromagnet with $n_c = 1$ (square lattice)
via quantum Monte Carlo method.



$N \leq 4$: Néel order

$N \geq 5$: quantum disorder
(columnar valence bond crystal)

Spin liquids

review : Savary and Balents, *Rep. Prog. Phys.* (2017)

1992 : surprisingly difficult to avoid magnetic order via frustration

2017 : embarrassment of riches! several experimental candidates!
trivial to elicit model spin liquids via Kitaev constructions

What is a spin liquid? No precise definition, but general desiderata:

- quantum disordered ground state
- no broken lattice translational symmetries
- half-odd integer spin per unit cell

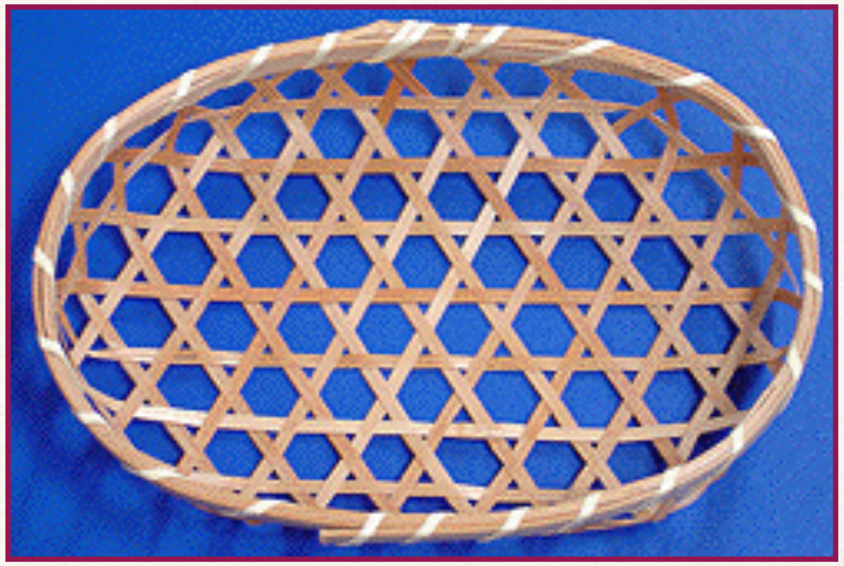
Three broad classes :

topological : gapped, local singlets, short-ranged correlations -
FQHE, $J_{1,2,3}$ kagome, triangular lattice QDM, toric code

U(1) **gapless** : gapless charged fermions + gauge field - Heisenberg kagome?
or gapped charges + gapless photon - quantum spin ice

$\mathbb{Z}_2$ **gapless** : gapless Majorana fermions, local spin correlations -
Kitaev honeycomb and generalizations

Kagome lattice antiferromagnet



kagome = “basket weave”

Classical kagome HAFM has infinite number of soft modes

Kagome QAFM @ T=0 : nonmagnetic, $\langle \boldsymbol{S}_0 \cdot \boldsymbol{S}_\mathbf{r} \rangle$ short-ranged

Number of low-lying singlet excitations ~exp(0.14·N)

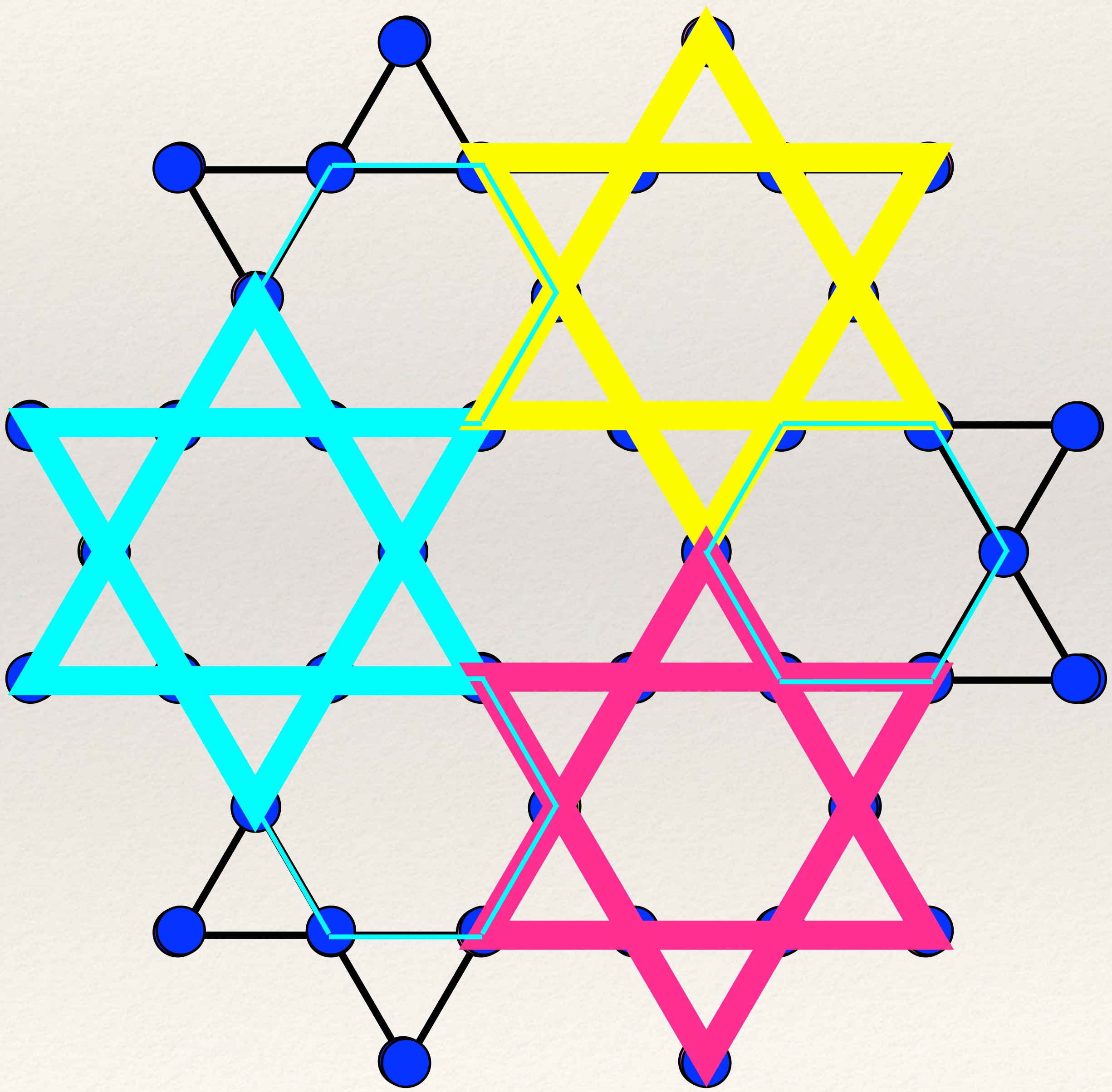
Early experiments on quasi-kagome QAFM showed:

No long-ranged magnetic order down to 50 mK

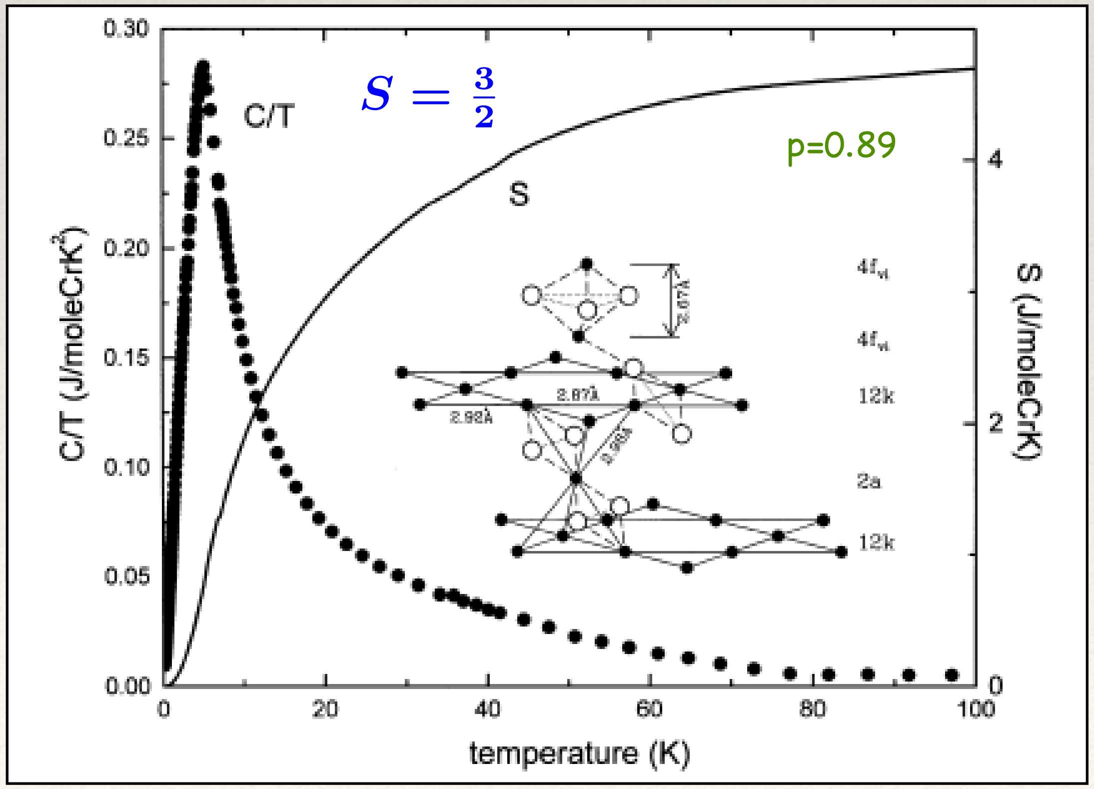
Heat capacity $C_V \propto T^2$: gapless excitations < 0.1

Weak dependence of C_V on field H (*singlets!*)

S=½ material : herbertsmithite $ZnCu_3(OH)_6Cl_2$



Ramirez *et al.* (2000) : $SrCr_{9p}Ga_{12-9p}O_{19}$



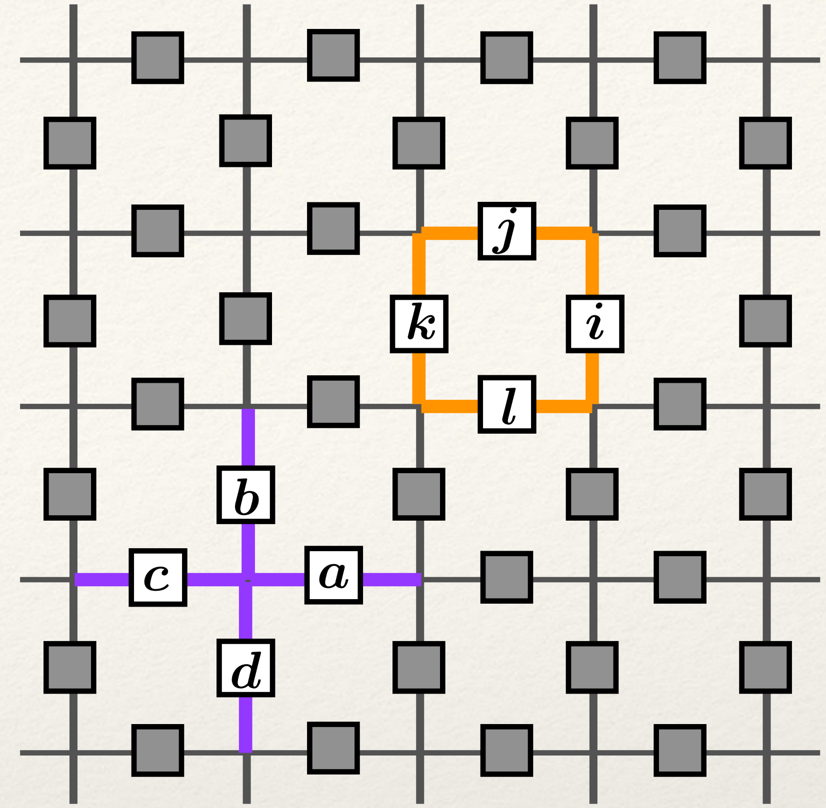
Kitaev's models

Toric code (2003):

$$\mathcal{H} = -J_{\mathrm{E}} \sum_{+} \sigma_a^z \sigma_b^z \sigma_c^z \sigma_d^z - J_{\mathrm{M}} \sum_{\square} \sigma_i^x \sigma_j^x \sigma_k^x \sigma_l^x$$

electric charges magnetic vortices

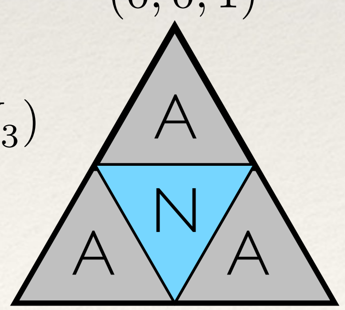
Topologically degenerate ground state (4x on torus) with gap to electric (e) and magnetic (m) excitations that have nontrivial mutual statistics and form composite (e-m) fermions.



Honeycomb lattice model (2006):

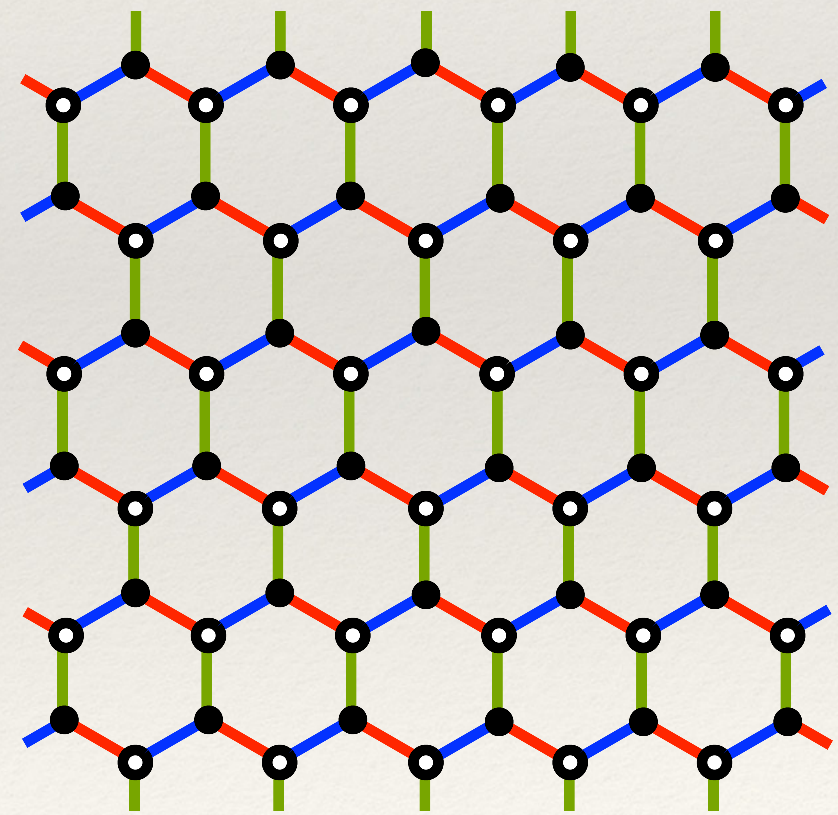
$$\mathcal{H} = J_1 {\sum_{\langle ij \rangle}}' \sigma_i^x \sigma_j^x + J_2 {\sum_{\langle ij \rangle}}' \sigma_i^y \sigma_j^y + J_3 {\sum_{\langle ij \rangle}}' \sigma_i^z \sigma_j^z$$

(J_1, J_2, J_3)



A : gapped abelian phases

N : gapless nonabelian phase



"Topological phases and quantum computation" : Kitaev and Laumann (2009)