PHYSICS 211C : CONDENSED MATTER PHYSICS
HW ASSIGNMENT #6

(1) Express the following operators in terms of Jordan-Wigner fermions:

(a) $X_n\, X_{n+1}$

(b) $Y_n\, Y_{n+1}$

(c) $X_n\, Y_{n+1}$

(d) $Y_n\, Y_{n+1}$

(e) $X_n\, Z_{n+1}\, X_{n+2}$

(f) $Y_n\, Z_{n+1}\, Y_{n+2}$

(g) $X_n\, Z_{n+1}\, Y_{n+2}$

(h) $Y_n\, Z_{n+1}\, X_{n+2}$

(i) Z_n

(2) For an infinitely long chain, find the diagonalized fermion Hamiltonian in momentum space resulting from

$$H = \sum_{n=-\infty}^{\infty} \Big\{ J_{xx}\, X_n\, X_{n+1} + J_{yy}\, Y_n\, Y_{n+1} + J_{xy}\, X_n\, Y_{n+1} + J_{yx}\, Y_n\, X_{n+1} + J_{xzx}\, X_n\, Z_{n+1}\, X_{n+2} + J_{yzy}\, Y_n\, Z_{n+1}\, Y_{n+2} + J_{xzy}\, X_n\, Z_{n+1}\, Y_{n+2} + J_{yzx}\, Y_n\, Z_{n+1}\, X_{n+2} + h\, Z_n \Big\} \quad ,$$

where the $J_{\ldots}$ are constants.

(3) Consider the square-octagon lattice Kitaev model depicted in Fig. 1. Show that the spin Hamiltonian

$$H = J_x {\sum_{\langle ij \rangle}}' X_i\, X_j + J_y {\sum_{\langle ij \rangle}}'' Y_i\, Y_j + J_z {\sum_{\langle ij \rangle}}''' Z_i\, Z_j \quad ,$$

where the primes on the sums denote XX, YY, and ZZ links, respectively, may be written in the form

$$H = \sum_{\langle ij \rangle} J_{ij}\, u_{ij}\, i\, \theta_i^0\, \theta_j^0 \quad ,$$

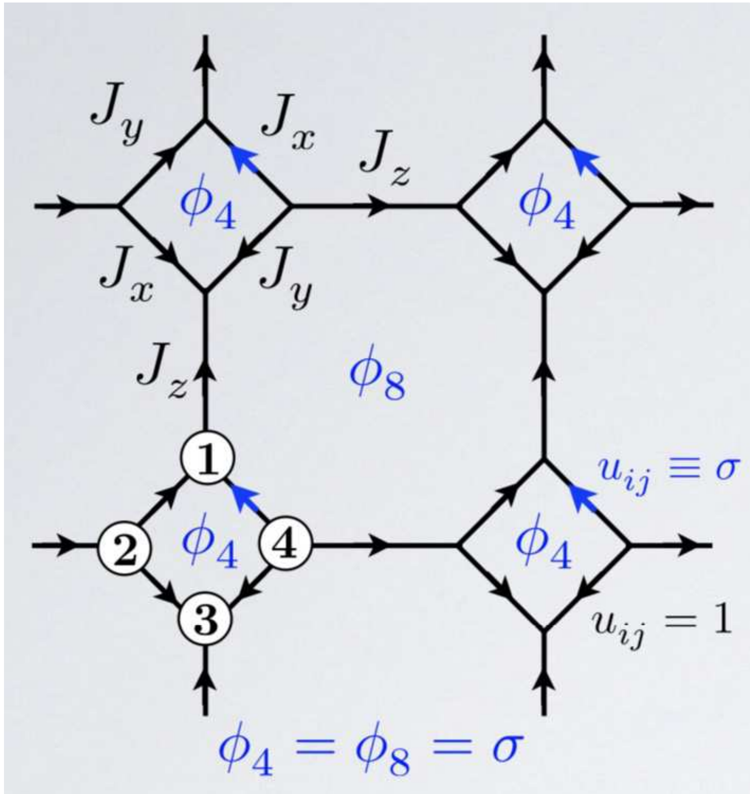

Figure 1: The square-octagon lattice model, with interactions $J_x X_i X_j$, $J_y Y_i Y_j$, and $J_z Z_i Z_j$. The $\mathbb{Z}_2$ gauge fields u_{ij} are given by $u_{ij} = +1$ along the links denoted by black arrows and as $u_{ij} = \sigma$ along the links denoted by blue arrows, where $\sigma \in \{+1, -1\}$. The $\mathbb{Z}_2$ plaquette fluxes ϕ_4 and ϕ_8 for the squares and octagons, respectively, are then $\phi_4 = \phi_8 = \sigma$. The magnetic unit cell is equivalent to the structural unit cell, and contains four sites.

where $u_{\langle ij \rangle} = \pm 1$ is the $\mathbb{Z}_2$ gauge field along the link directed from site i to site j, and J_{ij} is one of $\{J_x, J_y, J_z\}$ for each link. Show that this may be written in the form

$$H = i \sum_{\boldsymbol{k}}{}' A_{st}(\boldsymbol{k}) \left(c^\dagger_{\boldsymbol{k}s}\, c_{\boldsymbol{k}t} - \tfrac{1}{2}\, \delta_{s,t} \right) + H_{\text{TRIM}}$$

where the first sum is taken over half the Brillouin zone corresponding to the underlying (square) Bravais lattice, s and t range over the four basis elements $\{1, 2, 3, 4\}$, and H_{TRIM} is the contribution from the time-reversal-invariant momenta (which may be ignored in the thermodynamic limit since it contributes an amount $\mathcal{O}(1)$ to the total free energy). Find the 4×4 matrix $A_{st}(\boldsymbol{k})$.

(4) How would you construct a toric code Hamiltonian on the body centered cubic lattice? Describe the star and plaquette operators.