(1) Express the following operators in terms of Jordan-Wigner fermions:

(a) \( X_n X_{n+1} \)
(b) \( Y_n Y_{n+1} \)
(c) \( X_n Y_{n+1} \)
(d) \( Y_n Y_{n+1} \)
(e) \( X_n Z_{n+1} X_{n+2} \)
(f) \( Y_n Z_{n+1} Y_{n+2} \)
(g) \( X_n Z_{n+1} Y_{n+2} \)
(h) \( Y_n Z_{n+1} X_{n+2} \)
(i) \( Z_n \)

(2) For an infinitely long chain, find the diagonalized fermion Hamiltonian in momentum space resulting from

\[
H = \sum_{n=-\infty}^{\infty} \left\{ J_{xx} X_n X_{n+1} + J_{yy} Y_n Y_{n+1} + J_{xy} X_n Y_{n+1} + J_{yx} Y_n X_{n+1} + J_{xzx} X_n Z_{n+1} X_{n+2} + J_{yzy} Y_n Z_{n+1} Y_{n+2} + J_{xyy} X_n Z_{n+1} Y_{n+2} + J_{yxz} Y_n Z_{n+1} X_{n+2} + h Z_n \right\},
\]

where the \( J_{\ldots} \) are constants.

(3) Consider the square-octagon lattice Kitaev model depicted in Fig. 1. Show that the spin Hamiltonian

\[
H = J_x \sum_{\langle ij \rangle}' X_i X_j + J_y \sum_{\langle ij \rangle}'' Y_i Y_j + J_z \sum_{\langle ij \rangle}''' Z_i Z_j,
\]

where the primes on the sums denote \( XX \), \( YY \), and \( ZZ \) links, respectively, may be written in the form

\[
H = \sum_{\langle ij \rangle} J_{ij} u_{ij} i \theta_i^0 \theta_j^0,
\]
Figure 1: The square-octagon lattice model, with interactions $J_x X_i X_j$, $J_y Y_i Y_j$, and $J_z Z_i Z_j$. The $\mathbb{Z}_2$ gauge fields $u_{ij}$ are given by $u_{ij} = +1$ along the links denoted by black arrows and as $u_{ij} = \sigma$ along the links denoted by blue arrows, where $\sigma \in \{+1, -1\}$. The $\mathbb{Z}_2$ plaquette fluxes $\phi_4$ and $\phi_8$ for the squares and octagons, respectively, are then $\phi_4 = \phi_8 = \sigma$. The magnetic unit cell is equivalent to the structural unit cell, and contains four sites.

where $u_{ij} = \pm 1$ is the $\mathbb{Z}_2$ gauge field along the link directed from site $i$ to site $j$, and $J_{ij}$ is one of $\{J_x, J_y, J_z\}$ for each link. Show that this may be written in the form

$$H = i \sum_k A_{st}(k) (c_{ks}^\dagger c_{kt} - \frac{1}{2} \delta_{s,t}) + H_{\text{TRIM}}$$

where the first sum is taken over half the Brillouin zone corresponding to the underlying (square) Bravais lattice, $s$ and $t$ range over the four basis elements $\{1, 2, 3, 4\}$, and $H_{\text{TRIM}}$ is the contribution from the time-reversal-invariant momenta (which may be ignored in the thermodynamic limit since it contributes an amount $O(1)$ to the total free energy). Find the $4 \times 4$ matrix $A_{st}(k)$.

(4) How would you construct a toric code Hamiltonian on the body centered cubic lattice? Describe the star and plaquette operators.