(1) Compute the moments $\langle z | x^n | z \rangle$ and $\langle z | p^n | z \rangle$ for $p \in \{0, 1, 2, 3, 4\}$, where $|z\rangle$ is the coherent state defined in Eqn. 15.10 of the lecture notes. Express your answer in terms of $Q$ and $P$, where $Q = 2\ell \text{Re} z$ and $P = (\hbar/\ell) \text{Im} z$.

*Hint: Compute $\langle z | \exp(\lambda x) | z \rangle$ and $\langle z | \exp(\lambda p) | z \rangle$ and differentiate with respect to $\lambda$ as needed.*

(2) Consider the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{k}{2d^2} (x^2 - d^2)^2,$$

where $d$ is a length scale. The potential $V(x)$ represents a double well.

(a) Using your results from problem (1), obtain the Euclidean Lagrangian

$$L_E = i\hbar \text{Im} (\bar{z} \partial_\tau z) + H(\bar{z}, z)$$

but express $L_E$ in terms of $\{Q, P, \dot{Q}, \dot{P}\}$, and show that

$$L_E(Q, P, \dot{Q}, \dot{P}) = iQ\dot{P} + H(Q, P).$$

(b) Where are the minima of $H(Q, P)$ located? Under what conditions are there two minima at $Q = \pm Q_0$?

(c) Consider the tunneling problem in the case when there are two minima in $H(Q, P)$. Compute the tunneling path between the minima by solving the Euler-Lagrange equations of motion derived from $L_E$, i.e.

$$i \frac{\partial P}{\partial \tau} = -\frac{\partial H}{\partial Q}, \quad i \frac{\partial Q}{\partial \tau} = +\frac{\partial H}{\partial P}.$$ 

Analytically continue from $P$ to $P = iP$ and find the equations governing the instanton path in the $(Q, P)$ plane.

(d) Show that $H(Q, P = -iP)$ is constant along the instanton path. Then find the difference in the action between the instanton path and the trivial path where $P(\tau) = 0$ and $Q(\tau) = Q_0$ and compute the tunnel splitting between symmetric and antisymmetric states, discussed in §15.4.2 of the lecture notes.

(3) Verify Eqn. 15.54 of the lecture notes by finding the $O((z_1^{2S} z_2^{2S})$ term of the matrix element in the (unnormalized) generalized coherent state

$$| z, \hat{\Omega} \rangle \equiv e^{zu \hat{a}^\dagger} e^{zv \hat{b}^\dagger} | 0 \rangle,$$

where $z \in \mathbb{C}$. Show that $a | z, \hat{\Omega} \rangle = zu | z, \hat{\Omega} \rangle$ and $b | z, \hat{\Omega} \rangle = zv | z, \hat{\Omega} \rangle$, and

$$\langle z, \hat{\Omega} | z', \hat{\Omega}' \rangle = \exp [\bar{z}z'(\bar{u}u' + \bar{v}v')] .$$

(1)