PHYSICS 211C : CONDENSED MATTER PHYSICS
HW ASSIGNMENT #5

(1) Compute the moments $\langle\, z \,|\, x^n \,|\, z \,\rangle$ and $\langle\, z \,|\, p^n \,|\, z \,\rangle$ for $p \in \{0, 1, 2, 3, 4\}$, where $|\, z \,\rangle$ is the coherent state defined in Eqn. 15.10 of the lecture notes. Express your answer in terms of Q and P, where $Q = 2\ell\, \mathsf{Re}\, z$ and $P = (\hbar/\ell)\, \mathsf{Im}\, z$.

Hint : Compute $\langle\, z \,|\, \exp(\lambda x) \,|\, z \,\rangle$ *and* $\langle\, z \,|\, \exp(\lambda p) \,|\, z \,\rangle$ *and differentiate with respect to* λ *as needed.*

(2) Consider the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{k}{2d^2}\,(x^2 - d^2)^2 \quad ,$$

where d is a length scale. The potential $V(x)$ represents a double well.

(a) Using your results from problem **(1)**, obtain the Euclidean Lagrangian

$$L_{\mathsf{E}} = i\hbar\, \mathsf{Im}\,(\bar{z}\, \partial_\tau z) + H(\bar{z}, z) \quad ,$$

but express L_{E} in terms of $\{Q, P, \dot{Q}, \dot{P}\}$, and show that

$$L_{\mathsf{E}}(Q, P, \dot{Q}, \dot{P}) = iQ\dot{P} + H(Q, P) \quad .$$

(b) Where are the minima of $H(Q, P)$ located? Under what conditions are there two minima at $Q = \pm Q_0$?

(c) Consider the tunneling problem in the case when there are two minima in $H(Q, P)$. Compute the tunneling path between the minima by solving the Euler-Lagrange equations of motion derived from L_{E}, *i.e.*

$$i\,\frac{\partial P}{\partial \tau} = -\frac{\partial H}{\partial Q} \qquad , \qquad i\,\frac{\partial Q}{\partial \tau} = +\frac{\partial H}{\partial P} \quad .$$

Analytically continue from P to $\mathcal{P} \equiv iP$ and find the equations governing the instanton path in the $(Q, \mathcal{P})$ plane.

(d) Show that $H(Q, P = -i\mathcal{P})$ is constant along the instanton path. Then find the difference in the action between the instanton path and the trivial path where $\mathcal{P}(\tau) = 0$ and $Q(\tau) = Q_0$ and compute the tunnel splitting between symmetric and antisymmetric states, discussed in §15.4.2 of the lecture notes.

(3) Verify Eqn. 15.54 of the lecture notes by finding the $\mathcal{O}\big(\bar{z}_1^{2S} z_2^{2S}\big)$ term of the matrix element in the (unnormalized) generalized coherent state

$$|\, z, \hat{\boldsymbol{\Omega}} \,\rangle \equiv e^{zua^\dagger}\, e^{zvb^\dagger}\, |\, 0 \,\rangle \quad ,$$

where $z \in \mathbb{C}$. Show that $a\,|\, z, \hat{\boldsymbol{\Omega}} \,\rangle = zu\,\big|\, z, \hat{\boldsymbol{\Omega}} \,\big\rangle$ and $b\,\big|\, z, \hat{\boldsymbol{\Omega}} \,\big\rangle = zv\,|\, z, \hat{\boldsymbol{\Omega}} \,\rangle$, and

$$\langle\, z, \hat{\boldsymbol{\Omega}} \,|\, z', \hat{\boldsymbol{\Omega}}' \,\rangle = \exp\!\big[\bar{z}z'(\bar{u}u' + \bar{v}v')\big] \quad . \tag{1}$$