(1) Consider an ion with a partially filled shell of angular momentum \( J \), and \( Z \) additional electrons in filled shells. Show that the ratio of the Curie paramagnetic susceptibility to the Larmor diamagnetic susceptibility is

\[
\frac{\chi_{\text{para}}}{\chi_{\text{dia}}} = -\frac{g_L^2 J(J + 1)}{2Zk_B T} \frac{\hbar^2}{m\langle r^2 \rangle}.
\]

where \( g_L \) is the Landé \( g \)-factor. Estimate this ratio at room temperature.

(2) In an ideal paramagnet, the spins are noninteracting and the Hamiltonian is

\[
\mathcal{H} = \sum_{i=1}^{N_p} \gamma_i J_i \cdot \mathbf{H},
\]

where \( \gamma_i = g_i \mu_i / \hbar \) and \( J_i \) are the gyromagnetic factor and spin operator for the \( i \)\textsuperscript{th} paramagnetic ion, and \( \mathbf{H} \) is the external magnetic field.

(a) Show that the free energy \( F(H, T) \) can be written as

\[
F(H, T) = T \Phi(H/T).
\]

If an ideal paramagnet is held at temperature \( T_i \) and field \( H_i \hat{z} \), and the field \( H_i \) is adiabatically lowered to a value \( H_f \), compute the final temperature. This is called “adiabatic demagnetization”.

(b) Show that, in an ideal paramagnet, the specific heat at constant field is related to the susceptibility by the equation

\[
c_H = T \left( \frac{\partial s}{\partial T} \right)_H = \frac{H^2 \chi}{T}.
\]

Further assuming all the paramagnetic ions to have spin \( J \), and assuming Curie’s law to be valid, this gives

\[
c_H = \frac{1}{3} n_p k_B J(J + 1) \left( \frac{g \mu_B H}{k_B T} \right)^2,
\]

where \( n_p \) is the density of paramagnetic ions. You are invited to compute the temperature \( T^* \) below which the specific heat due to lattice vibrations is smaller than the paramagnetic contribution. Recall the Debye result

\[
c_V = \frac{12}{5} \pi^4 n k_B \left( \frac{T}{\Theta_D} \right)^3,
\]

where \( n = 1/\Omega \) is the inverse of the unit cell volume (\( i.e. \) the density of unit cells) and \( \Theta_D \) is the Debye temperature. Compile a table of a few of your favorite insulating solids, and tabulate \( \Theta_D \) and \( T^* \) when 1% paramagnetic impurities are present, assuming \( J = \frac{5}{2} \).