Jackson 2nd Ed: Chap 17
Garg: secs 18, 39, 67

RADIATION REACTION

A charge is the source of EM field. A field produces a force on a charge. What are the effects of a field produced by a charge on the charge itself?

Start with a simple question: what is the energy in the field of a point charge? (Say the electron, which, as best we know, is a point particle).

If q is at $\vec{x}'$, then $\vec{E}(\vec{x}) = \frac{q}{|\vec{x}-\vec{x}'|^3}(\vec{x}-\vec{x}')$

(For continuous distribution ρ, $\vec{E}(\vec{x}) = \int d^3x' \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|^3}(\vec{x}-\vec{x}')$)

Then $u = \frac{1}{8\pi}\vec{E}^2$ ($+\vec{B}^2$, but $\vec{B}=0$ for stationary charge, assumed here for simplicity). Then

$$\mathcal{E}\,(\text{energy}) = \int d^3x \frac{1}{8\pi}\vec{E}^2 = \frac{q^2}{8\pi}\int d^3x \frac{1}{|\vec{x}-\vec{x}'|^4} = \infty$$

the divergence arising from the region $\vec{x} = \vec{x}'$.

Is this a problem? An additive but constant (infinite) energy does not affect dynamics — which depends on energy differences. BUT an <u>accelerated</u> charge looses (radiates) energy, so it does matter.

If instead we replace point charge by smeared distribution of charge, we get a finite self-energy:

For a continuum $\mathcal{E} = \frac{1}{8\pi}\int d^3x \int d^3x' d^3x'' \frac{\rho(x')\rho(x'')}{|\vec{x}-\vec{x}'|^3|\vec{x}-\vec{x}''|^3}(\vec{x}-\vec{x}')\cdot(\vec{x}-\vec{x}'')$

$$= \frac{1}{2}\int d^3x' \int d^3x'' \frac{\rho(x')\rho(x'')}{|\vec{x}'-\vec{x}''|}$$ is finite (except for $\rho(\vec{x}) \sim \delta^{(3)}(\vec{x})$).

Mathematical digression: (not for class)

We have used our knowledge from electrostatics that

$$\mathcal{E} = \frac{1}{2}\int d^3x' d^3x'' \frac{\rho(x')\rho(x'')}{|\vec{x}' - \vec{x}''|}$$

which is usually derived from $\mathcal{E} = \frac{q_1 q_2}{r}$ for two charges, so

$$\mathcal{E} = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{q_i q_j}{r_{ij}} = \frac{1}{2} \sum_{i,j=1}^{N} \frac{q_i q_j}{r_{ij}} \quad \text{(introducing infinite self energies)}$$

then replacing $\sum_i q_i \to \int d^3x' \rho(x')$.

Question: can we obtain this directly from the integral of $\frac{1}{8\pi}\int \vec{E}^2$?

Answer: yes! We need to show

$$I \equiv \int d^3x \frac{(\vec{x}-\vec{x}')\cdot(\vec{x}-\vec{x}'')}{|\vec{x}-\vec{x}'|^3 |\vec{x}-\vec{x}''|^3} = 4\pi \cdot \frac{1}{|\vec{x}' - \vec{x}''|}$$

Writing $\vec{x}' = \vec{R} + \vec{r}$, $\vec{x}'' = \vec{R} - \vec{r}$ so $\vec{r} = \frac{1}{2}(\vec{x} - \vec{x}')$, we have (after shifting $\vec{x} \to \vec{x} - \vec{R}$)

$$I \equiv \int d^3x \frac{x^2 - r^2}{\left((x^2+r^2)^2 - 4x^2r^2\cos^2\theta\right)^{3/2}} = 2\pi \int_0^\infty x^2 dx \int_{-1}^{1} d\xi \frac{x^2 - r^2}{\left[(x^2+r^2)^2 - 4x^2r^2\xi^2\right]^{3/2}}$$

Now $\frac{d}{d\xi} \frac{1}{a} \frac{\xi}{\sqrt{a - b\xi^2}} = \frac{1}{(a - b\xi^2)^{3/2}}$ so

$$I = 4\pi \int_0^\infty dx \frac{x^2(x^2-r^2)}{(x^2+r^2)^2} \cdot \frac{1}{|x^2-r^2|} = \frac{4\pi}{r}\left[\int_1^\infty dx \frac{x^2}{(x^2+1)^2} - \int_0^1 dx \frac{x^2}{(x^2+1)^2}\right] = \frac{2\pi}{r} = \frac{4\pi}{|\vec{x}' - \vec{x}''|}$$

End digression.

Generally a difficult integral. But we can gain insight by dimensional analysis. If ρ is for total charge e ($q=e$ for electron) and is for a spherical distribution with radius r_e, then we expect $E \sim \frac{e^2}{r_e}$ and this should be no larger than the rest energy $m_e c^2$. The lower bound on r_e is the classical electron radius (I use the same symbol)

$$\frac{e^2}{r_e} = m_e c^2 \quad \Rightarrow \quad r_e = \frac{e^2}{m_e c^2} \simeq 2.8 \times 10^{-13}\,\text{cm}$$

The time it takes light to traverse this is $\tau_e = \frac{r_e}{c} = \frac{e^2}{m_e c^3} \simeq 10^{-23}\,\text{s}$.

Therefore: we should not trust classical E&M at length scales shorter than r_e - or time scales shorter than τ_e. Note that this is just from internal consistency of Maxwell's theory, regardless of quantum limitations: we should not trust it either for distance scales shorter than the Compton wave-length $\lambda_e = \frac{\hbar}{m_e c}$. Note that $\frac{\lambda_e}{r_e} = \frac{\hbar c}{e^2} = 137$, the breakdown of classical E&M occurs well before the self-inconsistency at $|\vec{x}| \sim r_e$ kicks in!

If no point charges are allowed in classical E&M, what is the electron? We need a mathematical model of a charge distribution that consistently accounts for it. (See digression next page)

Alternatively redefine "bare" mass to also be infinite and precisely cancel the infinity from self-energy. This can be done consistently - the result is independent of the precise manner in which the calculation is regulated (meaning, modified to make intermediate steps finite $\rightarrow$ discussed later).

Historical digression:

Abraham & Lorentz proposed the electron's structure is purely electromagnetic. In particular that its mass/energy and momentum are completely due to the EM field. There is a charge density $\rho(\vec{x})$ localized to $\sim r_e$, and we know from above this should give an energy $m_e c^2$ for the electron.

To explore the relation between $\mathcal{E}$ & $\vec{p}$, consider electron in motion, with vel $\vec{\beta}$. In the rest frame fields are $\vec{E}_0$ and $\vec{B}_0 = 0$. Boosting to frame with vel $-\vec{\beta}$, we have $\vec{E}$ & $\vec{B}$ related by $\vec{B} = \vec{\beta} \times \vec{E}$

Exercise: check this!

$\left(\vec{B} = \gamma \vec{\beta} \times \vec{E}_0 \quad \vec{E} = \gamma \left(\vec{E}_0 - \frac{\gamma}{\gamma+1} \vec{\beta} \cdot \vec{E}_0 \, \vec{\beta} \right), \gamma \vec{E}_0 = \vec{E} + \frac{\gamma^2}{\gamma+1} \vec{\beta} \cdot \vec{E}_0 \vec{\beta} \Rightarrow \vec{B} = \vec{\beta} \times (\gamma \vec{E}_0) = \vec{\beta} \times \vec{E} \right)$

Now $\mathcal{E} = \frac{1}{8\pi} \int d^3x \left(E^2 + B^2 \right) = \frac{1}{8\pi} \int d^3x \left[E^2 + \left(\beta^2 E^2 - (\vec{\beta} \cdot \vec{E})^2 \right) \right]$

The momentum is $\vec{p} = \int d^3x \, \vec{g} = \frac{1}{4\pi c} \int d^3x \; \vec{E} \times \vec{B} = \frac{1}{4\pi c} \int d^3x \, \vec{E} \times (\vec{\beta} \times \vec{E}) = \frac{1}{4\pi c} \int d^3x \left[\vec{\beta} E^2 - (\vec{\beta} \cdot \vec{E}) \vec{E} \right]$

In the NR limit $(\beta \ll 1)$ $\vec{E} = \vec{E}_0 + \mathcal{O}(\beta^2)$ is spherically symmetric so

$$\int d^3x \, E^i E^j = \frac{1}{3} \delta^{ij} \int d^3x \, E^2 .$$

Hence $\vec{p} = \frac{\vec{\beta}}{4\pi c} \int d^3x \, E^2 \left(1 - \frac{1}{3} \right) = \frac{2}{3} \frac{\vec{\beta}}{4\pi c} \cdot (8\pi \mathcal{E}) = \frac{4}{3} \vec{\beta} \left(\mathcal{E}/c \right)$

Oops! Is the 4/3 (which should be 1) an algebra mistake?

No! It is an inherent failure of the model.

Poincaré proposed a solution, with two important ingredients:

(i) There must be additional non-EM forces holding the charge together.

Then $T^{\mu\nu} = T^{\mu\nu}_{EM} + T^{\mu\nu}_{X}$ where "X" = other mysterious force field.

(ii) Proper treatment of Lorentz covariance:

$$p^\mu = \int d^3x \, T^{0\mu}$$

is a 4-vector <u>if</u> $\partial_\mu T^{\mu\nu} = 0$

He showed $T^{\mu\nu}_X$ contributes $-\frac{1}{3}$ to $\mathcal{E}/c$ so that $\vec{p} = \vec{\beta}\,\mathcal{E}$, and gives fully covariant results (this came later: Fermi, Kwal, Rohrlich, Wilson, ...).

None of this addresses successfully other issues with radiation-reaction that we turn to next.

Digression:

If $\partial_\mu T^{\mu\nu} = 0$ then $P^\mu = \int d^3x\, T^{0\mu}$ is a 4-vector.

Proof: $T^{\mu\nu}(x)$ is a tensor: $T'^{\mu\nu}(x) = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta\, T^{\alpha\beta}(\Lambda^{-1}x)$.

We aim at showing that $P'^\mu = \Lambda^\mu{}_\nu P^\nu$ where P'^μ is constructed as above by an observer in a frame K', and P^μ in frame K.

$$P'^\mu = \int d^3x'\, T'^{0\mu}(x') \qquad P^\mu = \int d^3x\, T^{0\mu}(x)$$

Note that $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ does <u>not</u> imply $d^3x' = d^3x$

We do have $d^3x' = d^4x'\, \delta(x'^0) = d^4x\, \delta(\Lambda^0{}_\mu x^\mu)$

So $P'^\mu = \int d^4x\, \delta(\Lambda^0{}_\mu x^\mu)\, \Lambda^0{}_\alpha \Lambda^\mu{}_\beta\, T^{\alpha\beta}(x)$

We have two issues to contend with: (i) the integral is not over d^3x at $x^0 = 0$ (or any constant), and (ii) we have $T^{\alpha\beta}$ rather than $T^{0\beta}$.

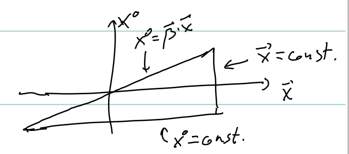
Now $\Lambda^0{}_\mu x^\mu = \gamma(x^0 - \vec{\beta}\cdot\vec{x})$. So we have

$$P'^\mu = \Lambda^\mu{}_\nu \int d^3x\, \frac{1}{\gamma}\cdot\gamma\left[T^{0\nu} - \beta^i T^{i\nu}\right] \quad \text{with } T^{\mu\nu}(x) = T^{\mu\nu}(x^0 = \vec{\beta}\cdot\vec{x}, \vec{x})$$

We'd like to show that this does not depend on the choice of spacelike hypersurface. Consider

$$0 = \int_V d^4x\, \partial_\mu T^{\mu\nu} = \int_{\partial V} d^3x\, n_\mu T^{\mu\nu}$$

where n_μ is the normal to the surface ∂V at element d^3x. Now, we want V to be



We assume $T \to 0$ as $|\vec{x}| \to \infty$ so the $|\vec{x}| = \text{const} \to \infty$ surface does not contribute.

The other two (spacelike) surfaces have $n_\mu = (1,0,0,0)$ and $n_\mu = \gamma(1, -\beta^i)$

so $\displaystyle \int d^3x \frac{1}{\gamma}\gamma(T^{0\nu} - \beta^i T^{i\nu})\Big|_{x^0 = \vec{x}\cdot\vec{\beta}} = \int d^3x\, T^{0\nu}\Big|_{x^0=0}$

End Digression

Going back to question on trusting classical E&M, we may ask more pointedly: under what conditions can we neglect the reactive effects of radiation?

Qualitatively: simple criterion $\mathcal{E}_{radiated} \ll \mathcal{E}$ (typical energy relevant to problem)

By the same token $\mathcal{E}_{rad} \sim \mathcal{E}$ should give us an estimate of when radiation reaction cannot be neglected.

Now $P = \frac{2}{3}\frac{q^2}{c^3}a^2$ So $\mathcal{E}_{rad} \sim \frac{q^2}{c^3}a^2\Delta t$

where Δt is time over which particle is accelerated.

For $\mathcal{E}$ we have to look case by case, and be judicious (after all $\mathcal{E}$ is defined up to additive constant; eg, we can swamp all energies in the NR case if we write $\mathcal{E} = mc^2 +$ kinetic).

Two typical cases:

(i) Accelerate particle from rest $\rightarrow$ $\mathcal{E} = \frac{1}{2}mv_{final}^2 \sim ma^2(\Delta t)^2$

RR non-neglible: $\frac{q^2}{c^3}a^2\Delta t \sim ma^2(\Delta t)^2 \Rightarrow \Delta t \sim \frac{q^2}{mc^3}$

For $q=e$ and $m=m_e$ this is $\Delta t \sim \tau_e$. And RR can be neglected for $\Delta t \gg \tau_e$

A bit of a surprise: $\tau_e = cr_e$ was introduced above as the minimum size of a charge distribution so that mass does not exceed $m_e c^2$. This was only from static fields. If we get around that somehow (by subtraction?) the scales τ_e, r_e reappear!

(ii) Circular motion:

acceleration: $\omega_0^2\rho$ energy: $\frac{1}{2}mv^2 = \frac{1}{2}m(\omega_0\rho)^2$

Criterial $\frac{q^2}{c^3}(\omega_0^2\rho)^2\,\Delta t \sim m(\omega_0\rho)^2$ $\left[\frac{q^2}{mc^3}\right]\omega_0^2\,\Delta t \sim 1$

For electron $\Delta t\,\tau_e\omega_0^2 \sim 1$

We cannot use such long Δt that ρ changes appreciably (then no longer circular and formulae are incorrect). So we need this to work for at least one period, or $\Delta t \sim \frac{1}{\omega_0}$. This condition is then $\tau_e\omega_0 \sim 1$ For angular frequencies $\omega_0 \gtrsim \frac{1}{\tau_e} \sim 10^{23}$ Hz RR cannot be neglected.

Note that this argument applies equally to any (quasi)-periodic motion with angular frequency ω_0 (and, irrelevant, amplitude ρ).

In both cases we see τ_e sets the relevant scale. And it is so small) ($\frac{1}{\tau_e}$ so large) that for most (but not all!) practical purposes RR can be safely neglected (justifying the success of all you've learned in EM that neglects this).

Quantifying RR: baby version.

If q looses energy by radiation it must decelerate but $F=ma$ says there must be a force acting on it. Of course in order to radiate it has to accelerate so there already was an external force ($\vec{F}_{ext}$) acting on it.

$$m\ddot{\vec{x}} = \vec{F}_{ext} + \vec{F}_{RR} \qquad \text{RR = "radiative reaction"}$$

How to determine $\vec{F}_{RR}$? We'll see below a derivation, fully covariant, using retarded Green's function. Abraham and Lorentz's method is similar but (i) NR and (ii) r_e (small but non-zero) used as cut-off.

For now, cheat a little. Set

$$\begin{pmatrix}\text{energy radiated}\\ \text{in } \Delta t\end{pmatrix} = -\begin{pmatrix}\text{work done by}\\ \vec{F}_{RR} \text{ in } \Delta t\end{pmatrix}$$

$$\int_{\Delta t} P\,dt = \frac{2}{3}\frac{q^2}{c^3}\int_{\Delta t}|\ddot{\vec{x}}|^2 dt \qquad\qquad -\int_{\Delta t}\vec{F}_{RR}\cdot\dot{\vec{x}}\,dt$$

Now the LHS has $\int \ddot{\vec{x}}\cdot\ddot{\vec{x}}\,dt = \underbrace{\dot{\vec{x}}\cdot\ddot{\vec{x}}\Big|_{t_i}^{t_f}}_{\text{ignore}} - \int_{\Delta t}\dddot{\vec{x}}\cdot\dot{\vec{x}}\,dt$

(because $\ddot{\vec{x}}=0$ at t_i & t_f or periodic).

Then we identify $\boxed{\vec{F}_{RR} = \frac{2}{3}\frac{q^2}{c^3}\dddot{\vec{x}}}$

Since it is T-odd ($t\to -t \Rightarrow \dddot{x}\to-\dddot{x}$) we expect dissipation (much like for $\vec{F}=\gamma\dot{\vec{x}}$, air drag). Of course dissipation is precisely what we expect.

We can now write

$$m(\ddot{\vec{x}} - \tau \dddot{\vec{x}}) = \vec{F}_{ext}$$

where $\tau = \frac{2}{3}\frac{q^2}{mc^3}$ is just as before, but now including the factor of $\frac{2}{3}$ so that we do not have to carry it around.

WEIRD!

For $\vec{F}_{ext} = 0 \Rightarrow \frac{d}{dt}\ddot{x} = \frac{1}{\tau}\ddot{x} \Rightarrow \ddot{\vec{x}} = \vec{a}_0 e^{t/\tau}$

Clearly only $\vec{a}_0 = 0$ is physical. But useful lesson: when we turn on $\vec{F}_{ext}$ there will also be un-physical, as well as physical, solutions.

In preparation for that consider the case of a general time dependent $\vec{F}_{ext}(t)$. We use the method of Green functions:

$$\left(\frac{d^2}{dt^2} - \tau\frac{d^3}{dt^3}\right)G(t) = \delta(t) \Rightarrow \vec{x}(t) = \int_{-\infty}^{\infty} dt'\, G(t-t')\vec{F}_{ext}(t')$$

To find G consider its Fourier transform $\tilde{G}(\omega)$:

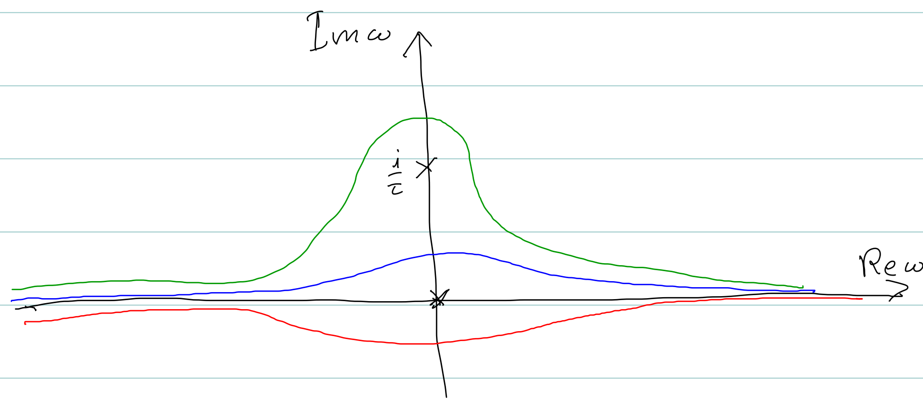
$$G(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t}\tilde{G}(\omega)$$

$$\Rightarrow \left(\frac{d^2}{dt^2} - \tau\frac{d^3}{dt^3}\right)G = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi} e^{-i\omega t}(-\omega^2 - i\omega^3\tau)\tilde{G}(\omega)$$

and set this equal to $\delta(t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}$

$$\Rightarrow \tilde{G}(\omega) = -\frac{1}{\omega^2(1+i\omega\tau)} \quad \text{and} \quad G(t) = -\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{e^{-i\omega t}}{\omega^2(1+i\omega\tau)}$$

As with other Green functions we encounter poles and we need to choose a contour of integration about them. We should expect our choice will determine behavior appropriate for various boundary conditions, to which we now turn.

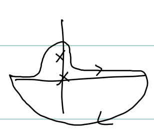


Poles of $\tilde{G}(\omega)$ in complex ω-plane (and possible contours of integration)

Consider each possibility:

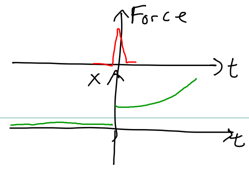
- Top (green): Can close contour on upper half and get $G(t)=0$

provided the argument of $\exp(-i\omega t)$ has negative real part $\Leftrightarrow$ $\mathrm{Re}\left[-i\left(\mathrm{Re}(\omega)+i\,\mathrm{Im}(\omega)\right)t\right] = \mathrm{Im}(\omega)\,t$ is negative for $t<0$. This is the analogue of a retarded Green function: it is causal. For $t>0$ we get contribution from $\omega = i\tau$ and $\omega = 0$.



$$\oint \frac{d\omega\, e^{-i\omega t}}{\omega^2(1+i\omega\tau)} = -2\pi i\left[i\tau e^{t/\tau} - i(t+\tau)\right] \Rightarrow G(t) = -\tau e^{t/\tau} + (t+\tau)$$

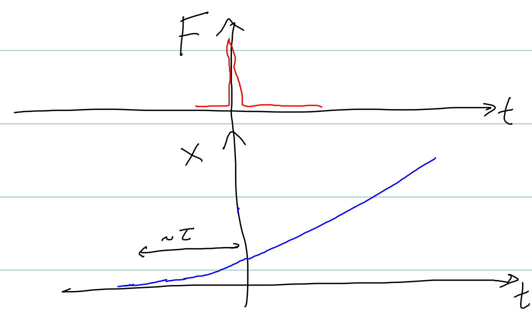
This is unphysical: an impulsive force at time $t=0$ sets the charge in accelerated motion for all $t>0$ times



- Bottom (red): we can close the contour to get $G(t)=0$ for $t>0$. Analogous to advanced Green function for radiation. Useful only if we know that $\vec{x}=\vec{x}_{free}$ for $t>0$.
- Middle (blue): neither advanced nor retarded: for $t<0$ close contour above, and for $t>0$ close it below: in both cases pick up poles: neither advanced nor retarded:

$$G = \begin{cases} \tau e^{t/\tau} & t<0 \\ t+\tau & t>0 \end{cases}$$

In this case



The charge moves at constant velocity after hit by a hammer. But it starts moving $\tau \simeq \frac{2}{3}\frac{q^2}{mc^3}$ before contact. This is acausal behavior. This is not a concern because on those time (and distance) scales, quantum mechanical effects take over. As best we know QED (quantum electrodynamics) is fully causal - in the quantum mechanical sense.

Line Breadth and shift of oscillator.

Consider a harmonic oscillator $\vec{F}_{ext} = -m\omega_0^2 \vec{x}$

$$\Rightarrow \quad \ddot{\vec{x}} - \tau \dddot{\vec{x}} + \omega_0^2 \vec{x} = 0$$

For simplicity, do 1-dim only:

Look for solutions $A\, e^{-i\alpha t}$: $\quad -\alpha^2 - i\alpha^3\tau + \omega_0^2 = 0$

There are 3 solutions. Two of them survive even if we take $\tau \to 0$ and correspond to the solutions in the absence of RR. The 3rd solution gives the unphysical exponential growth and we must discard it.

Since τ is small, solve perturbatively (in powers of $\omega_0\tau$).

$\alpha_i = \alpha_i^{(0)} + \alpha_i^{(1)} + \alpha_i^{(2)} + \cdots$ where $\alpha_i^{(n)} \sim (\omega_0\tau)^n$ and $i = +/-$ labels the sols.

Clearly $\alpha^{(0)2} = \omega_0^2 \Rightarrow \alpha_+^{(0)} = \omega_0 \quad \alpha_-^{(0)} = -\omega_0$

Then $-(\pm\omega_0 + \alpha_\pm^{(1)})^2 - i(\pm\omega_0)^3\tau + \omega_0^2 = 0 \Rightarrow \mp 2\alpha_\pm^{(1)} \mp i\omega_0^2\tau = 0 \Rightarrow \alpha_\pm^{(1)} = -\frac{i}{2}\omega_0^2\tau$

and we can write the solution to this order

$$x = A_+ e^{-i[\omega_0 - \frac{i}{2}\omega_0^2\tau]t} + A_- e^{-i[-\omega_0 - \frac{i}{2}\omega_0^2\tau]t}$$

This is a damped oscillator, with $x \propto e^{-\frac{1}{2}\Gamma t} \quad \Gamma = \omega_0^2\tau$

exactly as expected from energetic considerations early on.

This charge is radiating. The field oscillates with the same frequency.

More precisely the spectral analysis of the radiation is

$$\frac{dI}{d\omega} \propto \left| \text{Fourier transform of } \vec{x}(t) \right|^2$$

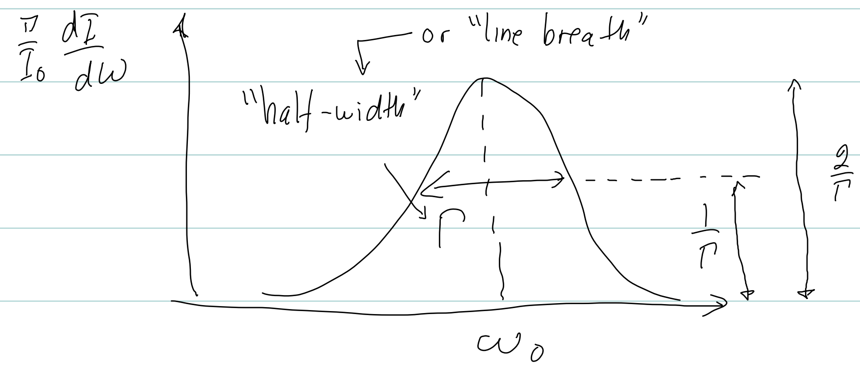
Now $\displaystyle\int_0^\infty dt\, e^{i\omega t}\left[e^{-i(\omega_0-\frac{i}{2}\Gamma)t}\right] = i\,\frac{1}{\omega-\omega_0+\frac{i}{2}\Gamma}$

where we have assumed the oscillator is excited from $t=0$ onward and neglect the process of turning on. (One can clearly do a better, cleaner job).

Then $$\frac{dI}{d\omega} \propto \left|\frac{1}{\omega-\omega_0+\frac{i}{2}\Gamma}\right|^2$$
$$= \frac{1}{(\omega-\omega_0)^2+(\Gamma/2)^2}$$

($\displaystyle\int_0^\infty d\omega\,\frac{1}{(\omega-\omega_0)^2+(\Gamma/2)^2} \simeq \int_{-\infty}^\infty d\omega\,\frac{1}{\omega^2+(\Gamma/2)^2} = 2\pi i\cdot\frac{1}{i\Gamma} = \frac{2\pi}{\Gamma}$ done by contour integration)

or $$\frac{dI}{d\omega} = \frac{1}{\pi}I_0\left[\frac{\Gamma/2}{(\omega-\omega_0)^2+(\Gamma/2)^2}\right] \quad \text{with} \int\frac{dI}{d\omega}d\omega = I_0$$



If the spectrum is given as function of wavelength, $\omega\lambda = 2\pi c$ then a small interval in λ is $\delta\lambda \simeq \frac{2\pi c}{\omega^2}\delta\omega$. Since the half-width is small, $\Gamma/\omega_0 = \omega_0\tau \ll 1$ (by assumption), the half-width in λ-space is $\Delta\lambda = c\frac{2\pi}{\omega_0^2}\Gamma = 2\pi c\tau$; for the lectron $c\tau_e = r_e$ the classical electron radius (up to $\times\frac{2}{3}$), independent of λ.

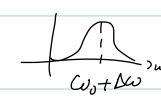
Final comments:

- Quantum mechanics: for atomic transitions photons are not monochromatic. Same phenomenon. QM line widths depend on strength of transition, and are related to partial lifetimes of levels $T_{1/2}^{-1} = \sum_{\substack{i \\ \text{levels}}} \Gamma_i$

- Had we retained $\mathcal{O}(\omega_0\tau)^2$ in our solution for α (ie $\alpha^{(2)}$) we would've obtained

$$\alpha = (\omega_0 + \Delta\omega_0) - \frac{i}{2}\Gamma$$

with Γ as before and $\Delta\omega_0 = -\frac{5}{8}\omega_0^3\tau^2$

This is the line-shift. It moves the center of the curve 

But very little $\frac{\Delta\omega}{\Gamma} \approx \mathcal{O}(\omega_0\tau) << 1$. QM effects have $\Delta\omega \sim \Gamma$.
In atoms this is called a Lamb shift (Lamb 1st observed).

$\frac{\Delta\omega_0}{\omega_0} \sim (\omega_0\tau)\ln\frac{m_e c^2}{\hbar\omega_0}$ vs classical $\frac{\Delta\omega_0}{\omega_0} \sim (\omega_0\tau)^2$

Coleman, Acausality
Barut, Chaps V, VI

Self-Field of Electron & the Force of Radiation Reaction

I believe this amounts to a modern (and relativistic) version of Abraham-Lorentz.

Rather than infering the self-force on the electron (due to its radiation field) by matching its rate of kinetic energy loss to the power radiated, which only gives us a time average so the inference of $\vec{F}_{RR}$ ("RR" = radiation reaction) is not completely justified, can we obtain $\vec{F}_{RR}$ directly?

The program should be clear:

(i) Compute $A_\mu \to F_{\mu\nu}$ due to electron

(ii) Compute $\vec{F}_{RR}$ due to $F_{\mu\nu}$ → give motion of electrons

Of course, there is no ordering here (which is first, the chicken of the egg? Ans: both/neither). These are simultaneous equations. As often done with simultaneous equations, solve for one variable in terms of the other, then plug into second equation.

To keep the aim clear, let's list expectations:

* The static component of self-force should give an infinite self-energy (ie, mass). We should regulate this (ie, cut-off the integral near $\vec{x} = \vec{x}_{electron}$), then subtract it using a "bare" mass (i.e, a contribution to the energy which is not of electromagnetic origin). Call this m_0.
* The radiation field should give rise to a force responsible for energy loss: it should be T-odd (dissipation! think air drag $\vec{F} \propto \vec{v}$) and we expect $\vec{F}_{RR} \propto \dddot{\vec{x}}$.

The two equations are

- Field due to point charge (electron):

$$A_\mu(x) = 4\pi q \int d\lambda\, u_\mu\, G_{ret}(x - y(\lambda))$$

(Notes from PHYS 203A, p.1 of chapter 4 "Fields of Moving charges")

and

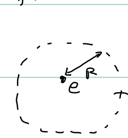
- Equation of motion

$$\frac{dp_\alpha}{d\lambda} = \frac{q}{c} F_{\alpha\beta} U^\beta$$

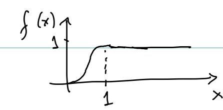
(PHYS 203A, chap 2., p.6)

And we take $p^\alpha = m_0 U^\alpha$ with m_0 as explained above.

The integral giving A_μ will diverge at the electron. To get around this problem we introduce a cut-off. How it's introduced should not matter provided (i) It only affects $|\vec{x} - \vec{x}_{el}| < r_e$ and (ii) I has a paramter that removes the cutting-off in some limit. For example



$\to A_\mu \to f\left(\frac{r}{R}\right) A_\mu^{unregulated}$



Remove cut-off by $R \to 0$.

Our choice of cut-off is in wave-number space: recall

$$G(x) = -\int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot x}}{k^2}$$

(PHYS 203A chap. 2, p. 13 of revised notes).

The $x \to 0$ region corresponds to $k \to \infty$. So we take

$$G_\Lambda(x) = -\int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \left[\frac{1}{k^2} - \frac{1}{k^2 - \Lambda^2}\right]$$

(cut-off removed by $\Lambda \to \infty$).

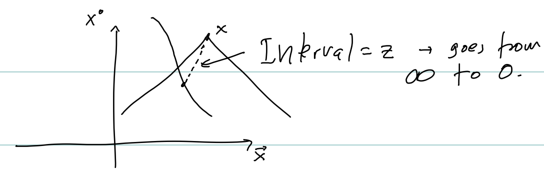
It is $k^2 - \Lambda^2$ rather than $k^2 + \Lambda^2$ so that poles are at $k^0 = \pm\sqrt{\vec{k}^2 + \Lambda^2}$, real.

We are ready to compute. We need $F_{\mu\nu}$ so take $\partial_\mu A_\nu$ above:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 4\pi q \int d\lambda \, U_\nu \, \partial_\mu G^{ret}_\Lambda(x - y(\lambda)) \; - (\mu \leftrightarrow \nu)$$

Here $U^\nu = \frac{dy^\nu}{d\lambda}$. The integral runs over $-\infty < \lambda < \lambda_0$, where λ_0 solves the retarded condition $x - y(\lambda_0) = 0$. We choose as parameter

$$\lambda = z^2 = (x-y)^2$$



This is useful because $G^{ret}_\Lambda(x)$ is a scalar function with dimensions of L^{-2}, ie, of wave-vector, so it depends on x and Λ only through the combination $\Lambda^2 x^2$ which is dimensionless, and to get dimensions right we write

$$G^{ret}_\Lambda(x-y) = \Lambda^2 f(\Lambda z)$$

for some function f. This function can be computed by performing the integral explicitly; this is difficult and not illuminating, and not necessary.

Use $\partial^y_\mu G^{ret}_\Lambda(x-y) = \Lambda^2 \partial^y_\mu z \frac{df}{dz}$ and $2z \partial^y_\mu z = 2(x-y)^\lambda \partial_\mu (x-y)_\lambda$

Recall

$$\partial^y_\mu y_\lambda = \frac{(x-y)_\mu U_\lambda}{(x-y)\cdot U} \qquad \text{(PHYS703A, chap 4, p.2)}$$

so $z \partial^y_\mu z = (x-y)^\lambda \left[\frac{(x-y)_\mu U_\lambda}{(x-y)\cdot U} \right] = (x-y)_\mu$ so we have

$$\partial_\mu G^{ret}_\Lambda = \Lambda^2 \frac{(x-y)_\mu}{z} \quad \text{and} \quad F_{\mu\nu} = 4\pi q \Lambda^2 \int_0^\infty dz \, \frac{dy_\nu}{dz} \frac{(x-y)_\mu}{z} \frac{df}{dz} \; - (\mu \leftrightarrow \nu)$$

Integrate by parts

$$F_{\mu\nu}(x) = -4\pi q \Lambda^2 \int_0^\infty dz\, f(\Lambda z) \frac{d}{dz}\left[\frac{(x-y)_\mu}{z}\frac{dy_\nu}{dz}\right] - (\mu\leftrightarrow\nu)$$

$$= 4\pi q \Lambda^2 \int_0^\infty dz\, f(\Lambda z) \left[\frac{(x-y)_\mu}{z}\frac{d^2y_\nu}{dz^2} - \frac{(x-y)_\mu}{z^2}\frac{dy_\nu}{dz}\right] - (\mu\leftrightarrow\nu)$$

Now, we are interested in $F_{\mu\nu}(x)$ for x = location of charge q.
So at some time x^0, we want $X = y(\lambda_*)$ with λ_* determined by $x^0 = y^0(\lambda_*)$. In terms of z, λ_* is $z=0$. So $x-y = y(0)-y(z)$.

Since the divergences are associated with the field at $x = x_{electron} = y(0)$, we expand the integrand in powers of z.

Note that $$\int_0^\infty dz\, f(\Lambda z)\, z^n = \frac{1}{\Lambda^{n+1}} \underbrace{\int_0^\infty d\zeta\, f(\zeta)\, \zeta^n}_{\text{some pure number}} \equiv c_n \frac{1}{\Lambda^{n+1}}$$

So only a finite number of terms need be retained: beyond some power the expansion terms vanish as $\Lambda\to\infty$. This will leave us with some divergent terms (expected, like self-energy), and some Λ-independent terms, the big pay-off of this long computation.

In fact, since there is a Λ^2 in front we need include only $n=1$ above.

So we have

$$F_{\mu\nu}(x) = 4\pi q \Lambda^2 \int_0^\infty dz\, f(\Lambda z) \left[\frac{(x-y)_\mu}{z} \frac{d^2 y_\nu}{dz^2} - \frac{(x-y)_\mu}{z^2} \frac{dy_\nu}{dz} \right] - (\mu \leftrightarrow \nu)$$

Use $y^\mu(z) = y^\mu(0) + z \left.\frac{dy^\mu}{dz}\right|_0 + \frac{z^2}{2} \left.\frac{d^2 y^\mu}{dz^2}\right|_0$ and $\frac{d^2 y_\nu}{dz^2} = \left.\frac{d^2 y_\nu}{dz^2}\right|_0 + z \left.\frac{d^3 y_\nu}{dz^3}\right|_0$

and let dots denote derivatives at current time, ie, $\dot{y}^\mu = \left.\frac{dy^\mu}{dz}\right|_0$, etc.

$$F_{\mu\nu}(y(0)) = -4\pi q \Lambda^2 \int_0^\infty dz\, f(z\Lambda) \left[(\dot{y}_\mu + \tfrac{1}{2} z \ddot{y}_\mu)(\ddot{y}_\nu + z \dddot{y}_\nu) - \mu \leftrightarrow \nu \right] + \mathcal{O}(1/\Lambda)$$

↳ ignore henceforth

$$= -4\pi q \left[c_0 \Lambda (\dot{y}_\mu \ddot{y}_\nu - \dot{y}_\nu \ddot{y}_\mu) + c_1 (\dot{y}_\mu \dddot{y}_\nu - \dot{y}_\nu \dddot{y}_\mu) \right]$$

Postpone determination of c_0 & c_1. Instead, we are ready to compute $\vec{F}_{RR}$

$$\frac{d}{dz} m_0 u_\alpha = \frac{q}{c} F_{\alpha\beta} u^\beta \quad \text{or}$$

$$m_0 \ddot{y}_\mu = \frac{q}{c} F_{\mu\nu} \dot{y}^\nu = -\frac{4\pi q^2}{c} \left[c_0 \Lambda (\dot{y}_\mu \ddot{y} \cdot \dot{y} - \dot{y}^2 \ddot{y}_\mu) + c_1 (\dot{y}_\mu \dddot{y} \cdot \dot{y} - \dot{y}^2 \dddot{y}_\mu) \right]$$

Now, as $z \to 0$, $\frac{dz}{ds} \to 1$, so we can interpret the derivatives as w.r.t s

so $\dot{y}^2 = 1$ and $\dot{y} \cdot \ddot{y} = 0$ (and $\dot{y} \cdot \dddot{y} + \ddot{y}^2 = 0$). So

$$\left(m_0 - \frac{4\pi q^2}{c} c_0 \Lambda \right) \ddot{y}_\mu = \frac{4\pi q^2}{c} c_1 \left(\dddot{y}_\mu + \dot{y}_\mu \ddot{y}^2 \right)$$

The divergent self-energy can be combined with a divergent bare mass $m_0(\Lambda)$ to leave a finite mass, the physical electron mass $m_e = m_0 - \frac{4\pi q^2}{c} c_0 \Lambda$ (so we don't much care what c_0 is). So we have

$$m_e \ddot{y}^\mu = \frac{4\pi q^2}{c} c_1 \left(\dddot{y}^\mu + \dot{y}^\mu \ddot{y}^2 \right)$$

In the nonrelativistic limit, $\dot{\vec{y}} \ll 1$ and we recognize the NR version of $\vec{F}_{RR}$, proportional to $\dddot{\vec{y}}$. Comparing with our simplistic energy conservation-on-average argument we can read off the constant c_1:

$\vec{F}_{RR} = \frac{2}{3} \frac{q^2}{c^3} \frac{d^3 \vec{y}}{dt^3} \quad \Rightarrow \quad 4\pi c_1 = \frac{2}{3} \quad \left(c_1 = \frac{1}{6\pi} \right).$ So finally

$$\boxed{m \ddot{y}^\mu = \frac{2}{3} \frac{q^2}{c} \left(\dddot{y}^\mu + \dot{y}^\mu \ddot{y}^2 \right)}$$