PHYS 203B - Spring 2020

Homework #3:

All problems are open book and open notes. However you are not to search the web for answers. You can discuss with your class peers (meaning, other students taking this course). You may discuss the content, but do not work out the details jointly. **Your solution of the problem is individual.** Some of the problems ask to show a particular expression for some quantity is correct. As such getting the “correct” expression is not a useful criterion in demonstrating your mastery of the material. For that reason, grading will focus on evidence that a complete logical reasoning, and a coherent and consistent mathematical derivation, starting from the main results obtained in class, is presented. Advice: (i) use words to justify your work, (ii) be judicious in what steps to include in your work (even if you are very smart and could mentally derive the result in your mind in one step, you should take the time to write down sufficiently many intermediate steps that you would convince a reasonable TA that you understand and the work is yours.

1. Calculate, up to the terms of order $1/r^3$ the long-range electric field induced by a cut and voltage-biased conducting sphere. The figure below shows the sphere cut by a (horizontal) plane parallel to the $xy$-plane, at an arbitrary distance $z = d(< R)$ from the center. The electric potential $\phi$ equals $V$ on the cap above the cut plane, and zero in the rest of the sphere (below the cut plane).

![Diagram of a conducting sphere cut by a horizontal plane](image)

2. Calculate the potential distribution above the plane surface of a conductor, with a strip of width $w$ separated by very thin cuts, and biased with voltage $V$ — see the figure below.

![Diagram of a conductor with a strip and cuts](image)
1. This is a boundary value problem: outside the sphere there are no charges ⇒ \( \nabla^2 \phi = 0 \).

And we look for solutions subject to the b.c.'s. Since these are specified on a sphere, we choose to work with spherical coordinates. And we know the general solution:

\[
\phi(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} C_{l,m} \frac{1}{r^{l+1}} Y_{l,m}(\theta, \phi) + C
\]

we have not included the \( r^0 \) solutions since we will \( \rho = 0 \) is \( r \to \infty \). Note that we should allow the \( r^0 \) with \( \rho \to 0 \) term, which is just a constant (denoted by \( C \)).

Now a spherical symmetry ⇒ \( \rho \) independent of \( \phi \) only \( r, \theta \). So we write (matching coefficients):

\[
\phi(r, \theta) = \sum_{l=0}^{\infty} C_l \frac{1}{r^{l+1}} P_l(\cos \theta) + C.
\]

The boundary conditions are:

\[
\phi(R, \theta) = \begin{cases} 
  +V & \theta > \theta_0 \\
  0 & \theta < \theta_0 
\end{cases}
\]

\( C_l \) determined from:

\[
\int_0^{\pi \cos \theta} \sin \theta d\theta = C_l \frac{1}{2^{l+1}}
\]

Using orthonormality of \( P_l \): \( \int_{-1}^{1} P_l(x) P_m(x) dx = \frac{1}{2} \delta_{l,m} \)

Multiply \((1)\) by \( P_l(\cos \theta) \) and integrate over \( x = \cos \theta \) from \(-1 \leq 1\) : Do this at \( r=0 \):

\[
\int_{-1}^{1} \int_{0}^{\pi} \frac{2 \sin \theta}{2^{l+1}} = \int_{-1}^{1} dx P_l(x) \phi(x, \phi) = V \int_{0}^{\pi} \frac{1}{2^{l+1}} d\sin \theta \]

and for \( r=0 \) term replace LHS by \( \frac{2}{2^{l+1}} C_l \)

Now, instead of doing the integral in \((1)\) in general, since we are asked to obtain \( \phi(r, \theta, \phi) \)

only to order \( \frac{1}{r^3} \) we need only up to \( l = 2 \). So do them using explicit form of \( P_l(x) \):
\[ \Theta_i(x) = 1, \quad \Theta_i(x) = x \quad \rho_i(x) = \frac{1}{2} (3x^2 - 1) \]

\[ D ('.2 + \sigma') = \mathcal{V} \int_{\mathcal{A}_k} dx \cdot 1 = \mathcal{V} (1 - \frac{d}{r}) \]

\[ \frac{\sigma}{3} C_1 \int_{\mathcal{A}_k} dx \cdot x = \mathcal{V} \frac{1}{2} (1 - \frac{d}{r}) \]

\[ \frac{\sigma}{3} C_2 \int_{\mathcal{A}_k} dx \cdot \frac{1}{2} (3x^2 - 1) = \mathcal{V} \frac{1}{2} \left[ 1 - \left( \frac{d}{r} \right)^3 - (1 - \frac{d}{r}) \right] = \frac{\sigma}{3} \mathcal{V} \frac{d}{r} \left( 1 - \frac{d}{r} \right)^3 \]

\[ c_0 = \frac{\mathcal{V} A \left( 1 - \frac{d}{r} \right)}{r} \quad c_i = \frac{3}{4} \mathcal{V} A \left( 1 - \frac{d}{r} \right)^2 \quad c_2 = \frac{5}{4} \mathcal{V} A \frac{d}{r} \left( 1 - \frac{d}{r} \right)^2 \]

\[ \Phi = \frac{1}{2} \mathcal{V} \left( 1 - \frac{d}{r} \right) \left[ \frac{r}{2} + \frac{3}{2} \left( \frac{d}{r} \right)^2 \log r + \frac{5}{2} \left( \frac{d}{r} \right)^3 \frac{1}{2} (3 \log -1) + \cdots \right] \]

\[ + \frac{\sigma}{3} \left( 1 - \frac{d}{r} \right) \]

Clearly, we can set \( c = 0 \). The charge is all localized, and \( \Phi = \frac{\mathcal{V} A \log d}{r^2} \sim \frac{1}{r^2} \) at long distances.
2. Use cylindrical coordinates, \( \rho \theta z \rightarrow x \).

By symmetry, \( \phi \) is \( z \)-independent. No charge \( \frac{\partial \rho}{\partial y} = 0 \Rightarrow \int \phi = 0 \) hence.

Solve in cylindrical coordinates, Separation of variables:

\[
\phi(x, y) = X(x)Y(y), \quad \frac{\partial^2 \phi}{\partial y^2} = 0 \Rightarrow \frac{1}{X} \frac{d}{dx} \left( \frac{dX}{dx} \right) + \frac{1}{Y} \frac{d^2Y}{dy^2} = 0
\]

\[
\Rightarrow \frac{d^2X}{dx^2} + \omega^2 X = 0 \quad \text{and} \quad \frac{d^2Y}{dy^2} + \omega^2 Y = 0 \Rightarrow X = e^{\pm i\theta x}, \quad Y = e^{\pm \omega y}
\]

For \( Y \), since we region of interest has \( y > 0 \), and \( \phi \) does not grow exponentially as \( y \to \infty \), we have only one sign: \( Y \propto e^{-\omega y} \) \((\omega > 0)\).

Also, the setup is invariant under \( x \to -x \), so we will have \( \phi(-x, y) = \phi(x, y) \).

So our solution is of the form

\[
\phi(x, y) = \sum_{n=0}^{\infty} C_n e^{-\omega_n y} \phi_n(x)
\]

We need to impose boundary conditions: \( \phi(x, 0) = \begin{cases} V & \text{if } |x| < \frac{a}{2} \\ 0 & \text{otherwise} \end{cases} \)

Since this is not periodic, the \( \phi_n \) above is not a Fourier series, but rather a Fourier transform.

\[
\phi(x, y) = \int_{-\infty}^{\infty} \phi(x) e^{-2\pi i y \xi} d\xi, \quad \text{with} \quad \phi(x) = \int_{-\infty}^{\infty} \phi_n(x) e^{2\pi i \xi} d\xi = \begin{cases} V & \text{if } |x| < \frac{a}{2} \\ 0 & \text{otherwise} \end{cases}
\]

So, we have

\[
\phi(x, y) = \int_{-\infty}^{\infty} \int_{0}^{\infty} C_n e^{-\omega_n y} \phi_n(x) e^{-2\pi i y \xi} d\omega_n d\xi
\]

but also

\[
\int_{-\infty}^{\infty} e^{-2\pi i y \xi} d\xi = \frac{1}{2\pi i} \left( e^{2\pi i y \xi} - e^{-2\pi i y \xi} \right) = \frac{2V}{\beta} \sin \left( \frac{\pi y}{a} \right)
\]

Hence

\[
\phi(x, y) = \frac{2V}{\beta} \int_{0}^{\infty} \sin \left( \frac{\pi y}{a} \right) \phi_n(x) e^{-\omega_n y} d\omega_n
\]

\( \omega = \frac{2\pi n}{a} \)

The integral can be done analytically

\( \text{(not required for your solution):} \quad \frac{V}{\pi} \left[ \tan^{-1} \left( \frac{x + \frac{a}{2}}{y} \right) - \tan^{-1} \left( \frac{x - \frac{a}{2}}{y} \right) \right] \)
3. Consider a model of a linear center fed antenna of length $a$ with current density that varies sinusoidally along its length: if the $z$-axis is placed along the antenna with the origin at its center, the model of the current is

$$I(z, t) = I_m \cos(\pi z/a) \cos(\omega_0 t)$$

Find the double differential angular and frequency spectrum of radiation, the spectrum of radiation, and the radiation resistance:

(a) using the dipole approximation
(b) exactly (not using the dipole approximation).

4. An insulating spherical shell of radius $R$ carries surface charge density $\sigma(\theta) = \sigma_0 \cos \theta$, where $\sigma_0$ is a constant. The shell is sent spinning around the $x$-axis ($\mathbf{\omega} = \omega \mathbf{\hat{x}}$). Calculate the power radiated.

5. Consider a pure electric quadrupole (E2) radiator.

(a) Find the angular distribution of electric quadrupole radiation (E2), for each of $m = 2, 1 \text{ and } 0$ sources. Sketch patterns for each case.

(b) Use this to find the angular distribution of radiation from a source for which the only non-vanishing components are

i. $D_{xy}$ and $D_{yz}$;

ii. $D_{xx}$ and $D_{yy}$;

iii. $D_{xx} = D_{yy}$ and $D_{zz}$. 
3. We can use the formalism developed in the text/notes by translating I into $\vec{F}$:

$$
\vec{F} = \mathcal{I} \delta(x) \delta(y) \delta(t)
$$

$$
\mathcal{I} = \int_{-\infty}^{\infty} \cos(\omega t) \cos(\omega t) \, dt = \frac{1}{2} \left( \delta(\omega - \omega) + \delta(\omega + \omega) \right)
$$

Dipole approximation:

$$
\vec{F}_d(\omega) = \int d^3 \vec{r} \vec{f}(\vec{r}, \omega) = \frac{2}{\omega} \left[ \delta(\omega - \omega) + \delta(\omega + \omega) \right] \int \frac{d^3 r}{\rho} \cos(\omega \rho) \cos(\omega \rho)
$$

$$
\omega \vec{F}_d(\omega) = \vec{F}_d(\omega) \left( \delta(\omega - \omega) + \delta(\omega + \omega) \right)
$$

It follows that the radiation field is $\vec{E}(kr) \frac{e^{i k r}}{r} \left( \delta(\omega - \omega) + \delta(\omega + \omega) \right)$

Now the expression in terms of Fourier transform is not useful for the periodic case.

We need to apply a Fourier series, with coefficients $\tilde{E}_n(r)$.

Since $\mathcal{I} \propto \cos(\omega t)$, we only have the fundamental component, and $\vec{F}_d = \vec{F}_d^0 = \int d^3 \vec{r} \vec{f}(\vec{r}) = \int d^3 \vec{r} \, 2 \Im(8\pi r) \cos(\omega \rho) \frac{1}{\rho} \frac{1}{\rho} \delta(\omega - \omega)$

It follows that

$$
\frac{d^2 \tilde{E}_n}{d\omega^2} = \frac{c^2}{2\pi} \frac{|\tilde{f}|^2}{\rho^2} \left( \cos(\omega \rho) \right) \left( \sin(\omega \rho) \right)
$$

This is the spectrum angular distribution. If we want to write this as a differential distribution, clearly it will be a sum of delta functions:

$$
\tilde{E} \propto \cos(\omega t) \cdot \mathcal{I} \propto \cos(\omega t).
$$

We want to see how to express this as $dP/d\omega$. Note we are not along clearly $d^2\tilde{I}/d\omega^2$ since this changes for periodic sources. Now $\tilde{E} \propto \cos(\omega t) = \frac{1}{2} \left( e^{i \omega t} + e^{-i \omega t} \right)$

$$
\frac{d^2 \tilde{E}}{d\omega^2} = \frac{1}{2} \left( \delta(\omega - \omega) + \delta(\omega + \omega) + 2 \delta(\omega) \right) = \frac{1}{2} \left( \delta(\omega - \omega) + \delta(\omega + \omega) \right)
$$

where the last step is from restricting $\omega > 0$ and using reality of $\tilde{E}$.
Then \( \frac{d^2 L}{dp^2} = C \left( \delta(\omega-2\omega) + \delta(\omega) \right) \) and we fix the quantity \( C \) by requiring \( \int_0^\infty \frac{dL}{d\omega} \frac{d\omega}{d\omega} = \frac{1}{\pi} \int_0^\infty \frac{dL}{d\omega} d\omega = \frac{dL}{d\omega} \).

Then, using \( \int_0^\infty \left( \delta(\omega-2\omega) + \delta(\omega) \right) = 1 + \frac{1}{2} = \frac{3}{2} \) we have

\[
\frac{3}{2} C = \frac{c^2}{\pi^2} \left( \frac{2\pi}{\hbar} \right)^3 \sin^2 \theta = \frac{d^2 L}{d\Omega d\omega} = \frac{\omega^2 a^4 T^4}{3 \pi c^3} \left( \delta(\omega-2\omega) + \delta(\omega) \right)
\]

(giving either \( \frac{dL}{d\omega} \) or \( \frac{d^2 L}{d\omega^2} \) gives you credit)

The spectrum of radiation can be stated as

\[
P = P_0 + P_1 + \ldots \quad \text{all in keV/m^2}
\]

with

\[
P_i = \frac{\omega^2 a^4 T^4}{2 \pi^2 c^3} \int d\omega \sin^2 \theta = \frac{1}{2} \frac{\omega^2 a^4 T^4}{\pi^2 c^3}
\]

or

\[
\frac{dL}{d\omega} = \frac{\omega a^4 T^4}{2 \pi^2 c^3} \left( \delta(\omega-2\omega) + \delta(\omega) \right)
\]

\( P_{\text{rad}} \) is defined by

\[
P = \frac{1}{2} \int P_{\text{rad}} \, \text{d}t \quad \text{where} \quad I_0 \quad \text{is the peak current}
\]

at \( t = 0 \) in our case \( I_0 = I_m \). So need

\[
P_{\text{rad}} = \frac{8}{3} \frac{\omega a^4}{\pi^2 c^3}
\]
\((\omega)\) No dipole approx:

\[
\mathcal{N}_\text{m} = \frac{\mathcal{F}_1 (\mathcal{E})}{\mathcal{F}_1 (\mathcal{E})} = \left[ \delta r^1 e^{i \hat{k} \cdot \mathcal{F}_1 (\mathcal{E})} \right]
\]

\[
= \int d^3r e^{i \hat{k} \cdot r} \left[ \frac{1}{2} \mathcal{I}_m \left( \frac{\delta \mathcal{E} (\mathcal{E})}{\mathcal{E}} \right) \cos \left( \frac{2 \pi r_a}{a} \right) \right]
\]

\[
= \frac{1}{i} \mathcal{I}_m \left( \frac{\pi}{a} \right) \int_{-l/2}^{l/2} d^3r e^{i \hat{k} \cdot r} \cos \left( \frac{2 \pi r_a}{a} \right)
\]

\[
= \left( \frac{2 \pi}{a} \right) \mathcal{I}_m \left( \frac{\pi}{a} \right) \left[ \frac{1}{k_x^2 + \frac{1}{a^2}} \cos \left( \frac{2 \pi k_y}{a} \right) \right]
\]

\[
= \frac{\pi}{2} \cos \left( \frac{\pi k_x}{a} \right) \left[ \frac{2 \pi}{k_x^2 - \frac{1}{a^2}} \right]
\]

\[
= \frac{\pi}{2} \cos \left( \frac{\pi k_x}{a} \right) \left[ \frac{2 \pi}{k_x^2 - \frac{1}{a^2}} \right] \quad (\text{for } k_x = 0)
\]

Now the next is straightforward. Recall \( \mathcal{E} = \frac{\omega}{c} \hat{n} \) so \( k_x = \frac{\omega}{c} \cos \theta \)

So take result for power above and multiply by \( \left[ \frac{\cos \left( \frac{\pi k_x a}{2} \right)}{1 - (\omega k/e)^2} \right]^2 \) (\( k_x = \frac{\omega}{c} \cos \theta \))

\[
\frac{\partial P}{\partial r} = \frac{\omega}{2 \pi c} \left( \frac{\alpha \ln \gamma}{\pi} \right)^2 \sin \theta \left[ \frac{\cos \left( \frac{\alpha \ln \gamma}{\pi} \cos \theta \right)}{1 - (\omega k/e)^2} \right]^2
\]

(\( \text{and going from this, } d^3P/d\mathcal{E} d\mathcal{E} \text{ is identical to above).} \)

All factors of \( \frac{\omega \alpha}{c} \) can be written in terms of \( \lambda \): \( \frac{\omega \alpha}{c} = \frac{2 \pi}{\lambda} \Rightarrow \frac{2 \pi a}{\lambda} = 2 \pi (\lambda / a) \).

Now

\[
P_i = \frac{1}{c} \left( \frac{\omega \alpha a}{\pi c} \right)^2 \int_1^\infty ds \left( \frac{2 \pi a}{\lambda} \right)^2 \cos \left( \frac{\pi a}{\lambda} s \right) \left[ 1 - \left( \frac{2 \pi a}{\lambda} s \right)^2 \right]^{-1/2}
\]

and

\[
P_{\text{nl}} = \frac{2}{c} \left( \frac{2 \pi a}{\lambda} \right)^2 \int_1^\infty ds \left( \frac{2 \pi a}{\lambda} s \right) \cos \left( \frac{\pi a}{\lambda} s \right) \left[ 1 - \left( \frac{2 \pi a}{\lambda} s \right)^2 \right]^{-1/2}
\]
4. If the charge distribution of the sphere is

\[ \rho = \sum_a q_a \delta^3(r - \vec{r}_a(t)) \]

Then, \( \vec{J} = \vec{\omega} \times \sum_a q_a \vec{r}_a(t) \delta^3(r - \vec{r}_a(t)) \)

\[ = \vec{\omega} \times \vec{r} \rho(r,t) \]

And \( \frac{\partial \vec{J}}{\partial t} = \vec{\omega} \times \frac{\partial \vec{r}}{\partial t} = \vec{\omega} \times \vec{r} \cdot \frac{\partial \vec{\omega}}{\partial t} \)

Now \( \vec{\nabla} \cdot (\vec{\omega} \times \vec{r}) = \varepsilon_{ijk} \frac{\partial}{\partial t} \varepsilon_{klm} \omega_i \omega_k \delta_{lm} = 0 \)

\[ \frac{\partial \vec{J}}{\partial t} = -\vec{\omega} \times \left[ (\vec{\omega} \times \vec{r}) \cdot \vec{\nabla} \rho \right] \]

We need \( \int d^3r \frac{\partial \vec{J}^i(r,t)}{\partial t} = -\varepsilon_{mjk} \omega_k \rho \sum_a \int d^3r' \delta^3(r-r') \delta^3(r-r') \frac{\partial \rho}{\partial t} \)

(integrate by parts)

\[ = -\varepsilon_{mjk} \omega_k \rho \sum_a \int d^3r' \delta^3(r-r') \frac{\partial \rho}{\partial t} \]

\[ = (\omega_m \omega_k - \omega^2 \delta_{mk}) \delta^3(r) \]

\[ = \left[ \vec{\omega} \cdot (\vec{\omega} \times \vec{r}) \right] \]

So we need to compute the electric dipole \( \vec{d} \) (or the component \( \parallel \vec{\omega} \))

and from this the component \( \perp \vec{\omega} \) and we are done.

Now \( \vec{d} = \int d^3r \sigma \vec{r} \) is along \( \vec{\omega} \), by symmetry, \( \sigma = \frac{2}{3} \sigma \)

\( \vec{d} = \int d^2r \sigma \omega \sin \Theta \cos \Theta = 2\pi R^2 \sigma \int ds \cos^2 \Theta = \frac{4\pi R^2 \sigma}{3} \)

So \( \frac{\partial \vec{d}}{\partial t} = -\omega^2 \left( \frac{4\pi R^2 \sigma}{3} \right) \)

But as a function of time \( \vec{d} \) rotates:

\( \vec{d}(t) = \left( \sigma, -\sin \omega t \sigma, \omega(t) \right) \sigma \)
Now, from \( \vec{E}(\vec{r}, t) = -\frac{1}{c^2} \mathcal{L} \frac{\partial \vec{D}(\vec{r}, t)}{\partial t} \)

(I have used \( r^2 = \) rather than \( R = \) in the text to avoid confusion with the radius of the sphere). Then

\[
\vec{E}(\vec{r}, t) = -\frac{1}{c^2} (\omega^2)(\vec{d}(\vec{k}) - \hat{\vec{R}} \cdot \vec{d}(\vec{R}))
\]

\[
\frac{dP}{d\Omega} = r^2 \frac{\omega^2}{4\pi c^3} E^2 = \frac{\omega^2}{4\pi c^3} \left( \frac{\vec{d}}{r} - \hat{\vec{R}} \cdot \vec{d} \right)^2
\]

\[
P = \int d\Omega \frac{dP}{d\Omega} = \frac{\omega^2}{4\pi c^3} \frac{4\pi \cdot (1 - \frac{1}{3})}{2} = \frac{2}{3} \frac{\omega^2}{c^3}
\]

\[
= \frac{2}{3} \frac{\omega^2}{c^3} \left( \frac{4\pi}{3} R_0^3 \right)^2
\]

\[
= \frac{32\pi^2}{9} \frac{\omega^2 R_0^6}{c^3}
\]
5. From lecture we had

\[ E_{\text{ref}} \propto \overline{D}_D \text{ where } \overline{D}_D = \hat{R}_x D_y \]

and

\[ \overline{D}_D^T = (\overline{D} - \hat{R}_x \overline{D} \hat{R}_x) = \hat{R}_x D_y - \hat{R}_x \hat{R}_y \hat{R}_x D_y = \hat{R}_x. \]

Now, from (19.25) (or from scratch, following the construction of \( \gamma_{en} \) presented in lectures):

\[ q_{20} = D_{2e} \]

(This is \( 3 \hat{e}_2 \cdot r^2 = 2 \hat{e}_2 \cdot (\hat{e}_2 \cdot r^2) = 2 \hat{e}_2 \cdot (x + y) = q_{20} \))

But also \( D_{2e} = 3 \hat{e}_2 r - \delta_{2e} r^2 \).

\[ q_{21} = \frac{1}{\sqrt{6}} \left( D_{xx} - i D_{xy} \right) \]

\[ q_{22} = \frac{1}{\sqrt{6}} \left( D_{xx} + D_{yy} + 2 i D_{xy} \right) \]

\[ q_{20}: \text{ If } q_{21} = 0 \Rightarrow D_{xx} = D_{yy} \text{ and } D_{xy} = 0. \text{ So } D_{2e} = -(D_{xx} + D_{yy}) = -2 D_{xx} \]

Then

\[ \overline{D}^T \cdot \overline{D} = \hat{R}_x \hat{R}_y \hat{R}_x \hat{R}_y - (\hat{R}_x \hat{R}_y \hat{R}_x \hat{R}_y)^T \]

\[ = \hat{R}_x \hat{R}_y + \hat{R}_y \hat{R}_x \hat{R}_x \hat{R}_y \hat{R}_x \hat{R}_y - (\hat{R}_x \hat{R}_y \hat{R}_x + \hat{R}_y \hat{R}_x \hat{R}_y \hat{R}_x)^2 \]

\[ = D_{xx} \left[ \hat{e}_x^T + \frac{1}{2} (\hat{e}_y^T + \hat{e}_x^T) \right] = \left( \hat{e}_x^T - \frac{1}{2} (\hat{e}_y^T + \hat{e}_x^T) \right) \]

\[ = D_{xx} \left[ \hat{e}_x^T + \frac{1}{4} (1 - \hat{e}_x^T) \right] = \left( \hat{e}_x^T - \frac{1}{2} (1 - \hat{e}_x^T) \right) \]

\[ = D_{xx} \left[ \frac{1}{2} \cos \Theta + \frac{1}{4} - \left( \frac{1}{2} \cos \Theta \cos \Theta \right) \right] \]

\[ = D_{xx} \left[ \frac{1}{4} \cos 2 \Theta \right] \]

\[ = D_{xx} \left[ \frac{1}{2} \sin 2 \Theta \right] \]

\[ = 0_{22} \frac{1}{2} \sin (2 \Theta) \]

\[ = 0_{22} \frac{1}{2} \sin (2 \Theta) \]

So the pattern is \( \frac{d \hat{P}}{d \hat{R}} \propto \sin^2 (2 \Theta) \) or \( \sin^2 (2 \Theta) \cos (2 \Theta) \).
\[ \begin{align*}
q_{11} & : \quad -\frac{1}{\bar{r}^2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \\
\text{If} \quad q_{11} = 0 & \Rightarrow D_{xx} + i D_{xy} = 0 \quad \text{and} \quad q_{12} = -\frac{2}{\bar{r}_0} D_{xx} = \frac{2i}{\bar{r}_0^2} D_{xy} \\
\text{Then} \quad \bar{D}^\perp \cdot \bar{D}^\perp & = \left( \hat{\rho}_i \hat{\rho}_j \right) D_{ij} - \left[ \hat{\rho}_i \hat{\rho}_j O_{ij} \right]^2 \\
& = \hat{\rho}_z \left( |D_{xx}|^2 + |D_{xy}|^2 + \hat{\rho}_y \left( D_{xx}^2 + \hat{\rho}_y |D_{xy}|^2 \right) \right) \\
& \quad - \left( 2 \hat{\rho}_z \hat{\rho}_x D_{xx} + 2 \hat{\rho}_z \hat{\rho}_y D_{xy} \right) \\
& = \left| D_{xx} \right|^2 \left[ 2 \hat{\rho}_z^2 + \hat{\rho}_x^2 - \hat{\rho}_y \left( \hat{\rho}_x^2 - \hat{\rho}_y \right) \right] \\
& \quad - \left( 2 \hat{\rho}_z \hat{\rho}_x D_{xx} + 2 \hat{\rho}_z \hat{\rho}_y D_{xy} \right) \\
& = \left| D_{xx} \right|^2 \left[ 4 \hat{\rho}_z^2 - 3 \hat{\rho}_x^2 + 4 \hat{\rho}_y^2 \right] \\
& \quad - \left( 2 \hat{\rho}_z \hat{\rho}_x D_{xx} + 2 \hat{\rho}_z \hat{\rho}_y D_{xy} \right)
\end{align*} \]

\[ \begin{align*}
\frac{d}{d\bar{r}_0} \rho & \propto \left( 1 - 3 \cos^2 \theta + 4 \sin^4 \theta \right) \\
q_{22} & : \quad \frac{1}{\bar{r}^2} \left( D_{xx} - D_{yy} - 2i D_{xy} \right) \\
\text{If} \quad q_{22} = 0 & \Rightarrow D_{xx} - D_{yy} + 2i D_{xy} = 0 \quad \Rightarrow \quad 2i D_{xy} = -D_{xx} + D_{yy} \\
\text{and} \quad q_{12} = 0 & \Rightarrow D_{xx} + D_{yy} = 0 \quad \Rightarrow \quad D_{xx} = -D_{yy} \\
S_0 & \Rightarrow \quad 2i D_{xy} = 2 \hat{\rho}_y \quad \Rightarrow \quad D_{xy} = i\hat{\rho}_y \quad D_{yy} = -D_{xx} \\
\hat{\rho}_x \text{and} \quad \bar{D}^\perp \cdot \bar{D}^\perp & = \left( \hat{\rho}_i \hat{\rho}_j \right) D_{ij} - \left[ \hat{\rho}_i \hat{\rho}_j O_{ij} \right]^2 \\
& = \hat{\rho}_x \left( \hat{\rho}_y \right) D_{xy} - \left( \hat{\rho}_y \right)^2 \left( D_{xx}^2 + \hat{\rho}_y |D_{xy}|^2 \right) \\
& \quad + \hat{\rho}_x \left( |D_{xx}|^2 + \hat{\rho}_y |D_{xy}|^2 \right) \\
& \quad - \left( \hat{\rho}_x \hat{\rho}_y O_{xx} + \hat{\rho}_y \hat{\rho}_x O_{xy} \right) \\
& = \left| D_{xx} \right|^2 \left[ 2 \left( \hat{\rho}_x^2 + \hat{\rho}_y^2 \right) - \left( \hat{\rho}_x^2 - \hat{\rho}_y^2 \right) \right] \\
& \quad - \left| D_{xy} \right|^2 \left[ 2 \left( \hat{\rho}_x^2 \hat{\rho}_y^2 \right) - \left( \hat{\rho}_x^2 + \hat{\rho}_y^2 \right)^2 \right]
\end{align*} \]
\[
\frac{d^2}{dx^2} \left[ 2(1-r_2) - (1-r_2) \right]
\]
\[
= \frac{d}{dx} \left( 2(1-r_2) - (1-r_2) \right)
\]
\[
= \frac{d}{dx} \left( 1 - \cos^2 \theta \right)
\]

\[\frac{d^2}{dx^2} \left( 1 - \cos^2 \theta \right)\]

Summary: The patterns are

\[q_{10} = \sin \theta \cos \theta\]

\[q_{11} = 1 - 3 \cos^2 \theta + 4 \cos^4 \theta\]

\[q_{22} = 1 - \cos^4 \theta\]

(b) \(h\)  The only \(q_{xy} = q_{yx} \neq 0\) when \(q_{10}\) and \(q_{11}\) vanish and we have \(q_{22} = -q_{22}\) both of which give the same \(1 - \cos^4 \theta\) pattern.

(c) \(h\)  \(q_{xy} + q_{yx} = q_{22} = 0\). Now \(q_{10}\) and \(q_{11}\) are in \(q_{20}\) but \(q_{20} = 0\) when \(q_{10} = 0\). So we add again to obtain (d)

(c) \(h\)  \(q_{xy} = q_{yx}\) and \(q_{22}\) is pure \(q_{20}\).