Units and Dimensions

We have postponed discussion of units. Most of our studies have been formal. But at some point we would like to plug in numbers, compare with measurements and so on.

Two most common systems:

(A) Gaussian: CGS

(B) SI - MKS

CGS = cm, gram, second
MKS = m, kg, sec
denote the units used for mechanical quantities.

Gaussian is more natural:

1. \( \vec{E} \) and \( \vec{B} \) have same dimensions

2. All units are derived from CGS.

SI is more common (volts, amperes, coulombs)

1. \( \vec{E} \) and \( \vec{B} \) have different dimensions (and units)

2. Introduces one new basic unit: Ampere for current

Will not discuss how units are defined. See

https://www.nist.gov/si-redefinition/definitions-si-base-units

That \( \vec{E} \) and \( \vec{B} \) do not have same dimensions in CGS vs SI implies that formulae containing them change from one system to the other.

More detail:
For mechanics, formulae have the same form in MKS, CGS, etc.

\[ F = \frac{d}{dt} \mathbf{p}, \quad E = \nabla \phi, \quad \mathbf{p} = m \mathbf{v}, \text{ etc.} \]

For EM, formulae depend on system of units:

- **Gaussian**
  \[ F = \frac{9}{4\pi} \frac{S I}{\lambda} \]
  \[ \mathbf{E} = \frac{1}{\varepsilon_0} \frac{9}{4\pi} \frac{S I}{\lambda} \]
  \[ U = \frac{1}{8\pi} \left( \mathbf{E}^2 + \mathbf{B}^2 \right) \]
  \[ U = \frac{1}{2} \left( \varepsilon_0 \mathbf{E}^2 + \frac{1}{\mu_0} \mathbf{B}^2 \right) \]
  \[ \mathbf{E} = q (\mathbf{E} + \frac{\mathbf{V}}{c} \times \mathbf{B}) \]
  \[ \mathbf{B} = q (\mathbf{E} + \mathbf{V} \times \mathbf{B}) \]

\[ \Rightarrow \text{Translating between systems requires changes in formulas.} \]

Denote by [\text{J}] dimension (as usual) with \( [\text{J}] = \text{LT} \), \( [\text{m}] = \text{M} \), \( [\text{t}] = \text{T} \)
(with units that are measured in cm- g - s in CGS or m- kg - s in MKS).

We can see what the dimensions are for each quantity in each system. In particular the new (non-mechanical) quantities \( q \) (charge), \( \mathbf{E} \), \( \mathbf{B} \).

- **Gaussian**
  
  From \( F = q \frac{q}{\mathbf{E}} \), \( [q] = \left( [F] [\text{m}]^2 \right)^{1/2} = [M]^{1/2} [L]^{3/2} [T]^{-1} \) is statcoulomb
  
  \[ U = \frac{1}{8\pi} \left( \mathbf{E}^2 + \mathbf{B}^2 \right) \]
  \[ [U] = [\text{J}] = \left( [\text{J}] [\text{T}]^2 \right)^{1/2} = [M]^{1/2} [L]^{3/2} [T]^{-1} \text{ statvolt cm}^{-1} \]

Sanity check: \( F = q \mathbf{E} = [\text{M}] [\text{L}] [\text{T}]^{-2} \left( [\text{M}]^{1/2} [\text{L}]^{3/2} [\text{T}]^{-1} \right) \)

Other quantities trivially follow, e.g. \( \mathbf{E} = -\nabla \phi \) (\( \phi = \text{A} \))

\[ \Rightarrow [\phi] = [\text{J}] [\text{L}] = [\text{M}]^{1/2} [\text{L}]^{3/2} [\text{T}]^{-1} \left( [\text{Fm}][\text{m}] \right) = \text{statvolt} \]
\[ S \]

\[ [I] \text{ is a new basic dimension} \rightarrow \text{ampm} \]

\[ I = \frac{dq}{dt} = [Q] = [I][T] \rightarrow [Q] = [I][T] \]

\[ F = \frac{1}{\mu_0} \frac{q_1}{r^2} \rightarrow [M][L][T]^{-2} = \left[\varepsilon_0\right]^{-1} [I][T][I][T]^{-2} \rightarrow [\varepsilon_0] = [I]^{-1} [I][M][L]^{-2}[M]^{-1} \]

\[ U = \frac{1}{\mu_0} \left( e_1 \vec{B} - J \vec{B}' \right) \text{ -- two relations} \]

\[ \left[ \vec{E} \right] = \left[ M \right][L][T]^{-1} \left( \left[ J \right][T][L][T]^{-2} \right)^{1/2} \]

\[ \left[ \vec{B} \right] \left[ \mu_0 \right]^{-1} = \left[ M \right][L][T]^{-3} \left[ I \right]^{-1} \]

\[ \left[ \vec{E} \right] = \left[ M \right][L][T]^{-2} \left[ I \right][T]^{-1} = \text{same as above} \quad \checkmark \]

\[ \text{From } U: \quad \left[ \mu_0 \right] = \left[ \vec{B} \right] \left[ M \right][L][T]^{-1} = \left[ I \right][L][T]^{-2}[I] \]

\[ \text{Check: } \left[ \varepsilon_0 \right] \left[ \mu_0 \right] = \left[ I \right][T]^{-2} \left[ T \right]^{-1} \text{ consistent with } \varepsilon_0 = \frac{1}{\mu_0} \frac{1}{\varepsilon_0} \]

**Exercise:** Check that Maxwell’s Equations in SI are dimensionally consistent.

Translating between systems

We can go from an expression (say any of Maxwell’s) in one system to another if we develop a dictionary.

For example

\[ F = \frac{q^2}{x^2} \left( \text{Gauss} \right) \quad \Rightarrow \quad \frac{1}{\varepsilon_0} \frac{q^2}{x^2} \left( \text{SI} \right) \]

Since \( F \) and \( \vec{x} \) are mechanical quantities, they are the same in both systems. We

\[ \text{in SI} \quad q_0^2 = \frac{q^2}{\varepsilon_0} \quad \text{or} \quad q_0 = \frac{q}{\sqrt{\varepsilon_0}} \quad \text{in CGS} \]
And from \( u = \frac{1}{3} (E_0^2 + B_0^2) = \frac{1}{2} (E_0 \cdot E_0 + \mu_0 B_0) \),
we get \( \vec{E}_0 = \sqrt{\frac{2}{\epsilon_0 \mu_0}} \vec{E}_0 \) and \( \vec{B}_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \vec{B}_0 \).

As above, \( \vec{E}, \vec{B} \) are sufficient to obtain the rest of the dictionary.

The Lorentz force gives nothing new — except if we do not know a priori that \( c^2 = \frac{1}{\epsilon_0 \mu_0} \).

\( \vec{F} = q_0 (\vec{E}_0 + \vec{v} \times \vec{B}_0) = q_0 \vec{E}_0 + \vec{v} \times \vec{B}_0 \)

\( \Rightarrow q_0 \vec{E}_0 = q_0 \vec{E}_0 \) consistent with the above \( q_0 \vec{E}_0 = \left( \frac{q_0 \vec{E}_0}{\sqrt{\frac{2}{\epsilon_0 \mu_0}}} \right) \left( \frac{\sqrt{2}}{\epsilon_0 \mu_0} \vec{E}_0 \right) \)

and \( q_0 \vec{B}_0 = q_0 \vec{B}_0 \) \( \Rightarrow \frac{1}{c} \left( \frac{q_0 \vec{E}_0}{\sqrt{\frac{2}{\epsilon_0 \mu_0}}} \right) \left( \frac{\sqrt{2}}{\epsilon_0 \mu_0} \vec{B}_0 \right) = q_0 \vec{B}_0 \) \( \Rightarrow \frac{1}{c} \frac{q_0}{\epsilon_0 \mu_0} = 1 \) \( \Rightarrow c^2 = \frac{1}{\epsilon_0 \mu_0} \).

We can correct any formula Gaussian \( \Rightarrow SJ \). For example, we derived Larmor's formula in Gaussian, so

\( \rho = \frac{2}{3} \frac{q^3 a^2}{c} \), \( \rho \) a are mechanical \( \Rightarrow P = \frac{2}{3} \frac{q^3 a^2}{\sqrt{\frac{2}{\epsilon_0 \mu_0}}} \frac{q_0 a^2}{\epsilon_0 \mu_0} = \frac{1}{67 \sqrt{\epsilon_0}} \frac{q_0 a^2}{\epsilon_0 \mu_0} \)

Or Maxwell's equations, \( \vec{E} \),

\( \vec{\nabla} \times \vec{E}_0 + \frac{1}{c^2} \frac{\partial \vec{B}_0}{\partial t} = 0 \Rightarrow \vec{\nabla} (\mu_0 \epsilon_0 \vec{E}_0) + \frac{\epsilon_0}{\epsilon_0 \mu_0} \frac{\partial \vec{B}_0}{\partial t} = 0 \)

\( \Rightarrow \vec{\nabla} \times \vec{E}_0 + \frac{1}{c^2} \frac{\partial \vec{B}_0}{\partial t} = 0 \).

Student can derive the rest in SJ from Gaussian.

Exercise: Derive the point vector in SJ by translating from Gaussian.

Numerics. How do convert quantities from one system of units to the other? And what about \( \mu_0 \epsilon_0 \)? As physicist I like Gaussian, but the "meter" does not give standards, nor statesmum, etc.
1. Since the only new dimension is \([I]\) is SI, we should be able to translate any amount of anything between systems once we know how to translate current \(\rightarrow\) or, equivalently, charge.

Of course we also need \(1\text{m} = 10^2\text{cm} \quad 1\text{kg} = 10^3\text{g} \quad 1\text{s} = 1\text{s}\).

Consider Coulomb's law. In CGS

\[
F = \frac{q_1 q_2}{4\pi\varepsilon_0} \quad \text{means two charges of}\ q_1 \text{q_2 (statcoulomb)} \quad 1\text{cm apart experience a force of 1 dyn}
\]

\(\text{esu}\) is derived very much like dyne or Erg.

(\(\text{esu}\) and \(\text{statcoulomb}\) are used inter changeably; franklin \(\text{fr}\) is also sometimes used - less common)

The same two charges (\(\text{esu}\)) at the same distance (\(1\text{cm}\)) experience the same force (\(\frac{1}{4}\)) in other systems. In SI the force is \(1\text{dyn} = 10^{-5}\text{newton}\), distance \(1\text{cm} = 10^{-2}\text{m}\) and charge is \(1\text{esu} = x C\) (\(C\) = coulomb). So we have

\[
10^{-5} = \frac{1}{4\pi\varepsilon_0} \frac{x^2}{(10^{-2})^2} \quad \text{or} \quad x = \sqrt[4]{4\pi\varepsilon_0 10^{-9}}
\]

With \(\varepsilon_0 = 8.854 \times 10^{-12}\ \text{F/m}\)

\[x = \sqrt[4]{4\pi\times 8.854 \times 10^{-12}} = 3.39 \times 10^{-10} = \frac{1}{(2.998 \times 10^8)}\]

Usually written \(1\text{esu} = 10^3\ C\) or \(1\text{C} = 3 \times 10^8\ \text{esu}\)

but "3" is 2.998, suspiciously the same digits don\text{t} appear in \(C\) = speed of light
2. \( E_0 \) vs \( C \\

Because \([J]\) is a new dimension in SI, formulae like

\[
F \propto \frac{q^2}{r^2} \quad \text{and} \quad B \propto \frac{1}{r} \quad \text{or} \quad F/r^2 \propto \frac{1}{r^2}
\]

require introduction of dimensionful constants. For example, we saw

\[
F = \frac{1}{4\pi \epsilon_0} \frac{q^2}{r^2}, \quad \text{with} \quad [\epsilon_0] = [J^2][T]^{-1}[L]^{-2}[M]^{-1}
\]

And, say, \( B = \frac{\mu_0 E}{2\pi r} \). But also \([\mu_0][E_0] = [J^2][L]^{-2} \Rightarrow \] there is no need for two different conversion factors. This is clear just from dimensional analysis, has nothing to do with speed of light. (Had we introduced \( E_0 \) with different coefficients in Coulomb and Ampere’s laws, we would still get \( \mu_0 E_0 \propto \frac{1}{r^2} \) but with some proportionality constant.) Since only one is needed, one may define one and measure the other, or define one, measure \( C \), and infer the other from \( \mu_0 E_0 = \frac{1}{C^2} \).

You probably know \( \mu_0 = 4\pi \times 10^{-7} \text{ H/m} \) by definition. This fixes

\[
\epsilon_0 = \frac{1}{\mu_0 C^2} = 8.854 \times 10^{-12} \text{ F/m}. \quad \text{But there is a better way to write this}
\]

\[
y \epsilon_0 = \left(\frac{4\pi}{\mu_0}\right)^{-\frac{1}{2}} \quad \text{so} \quad 4\pi \epsilon_0 = 10^7 \text{ C}^2 \quad \text{(in F/m if C in m/s).}
\]

As we saw above \( 1 \text{ esu} = x \text{ C} \) with \( x = \sqrt{4\pi \epsilon_0} \times 10^9 \)

\[
\Rightarrow x = \sqrt{\frac{10^9}{10^7}} = 100 = \left(\frac{1}{2.998 \times 10^8}\right) \quad \text{as before.}
\]

Exercise: Derive the conversion factors for 5 other quantities in the conversion tables 1.1.1.2 of Garg or Table 4 of Appendix of Jackson.