Refs: L&L: 66 and 73
Jackson: 14.1-2
Lechner: Chap 7

Chapter 4: Fields of Moving Charges

Liénard-Wiechert potential & field

To determine the field due to a point charge in arbitrary motion given by a specified trajectory $y^\mu(\lambda)$ (we use y^μ rather than x^μ, so as to not get confused with the argument of $A_\mu(x)$) we use the retarded Green's function $G_{ret}(x,x')$. Recall

$$G_{ret}(x,x') = G_{ret}(x-x')$$

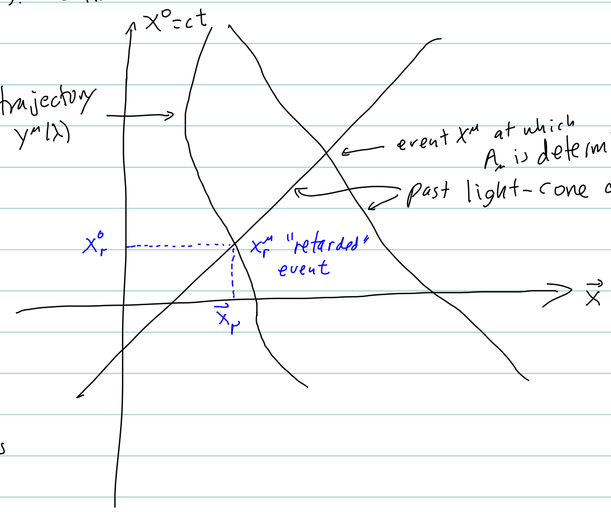
with $\partial^2_x G_{ret}(x) = \delta^{(4)}(x)$

so that $G_{ret}(x) = \frac{1}{4\pi|\vec{x}|}\delta(x^0-|\vec{x}|)$

and

$$A_\mu(x) = A^{(in)}_\mu(x) + \int d^4_x{}' \, G_{ret}(x,x')\left(\frac{4\pi}{c}J_\mu(x')\right)$$

(recall $\partial^2 A_\mu = \frac{4\pi}{c}J_\mu$ in Lorentz gauge)



Notice that the δ-function in G_{ret} means that the field at $(x^0,\vec{x})$ is determined by the charge at the retarded time $x^0_r = ct_r$, given by $x^0 - x^0_r - |\vec{x}-\vec{x}_r| = 0 \quad \Rightarrow \quad x^0_r = x^0 - |\vec{x}-\vec{x}_r|$. That is $t_r = t - \frac{1}{c}\vec{R}$ where $|\vec{R}|$ is the distance from the charge to $\vec{x}$ at time t_r.

$A^{in}_\mu(x)$ has $\partial^2 A^{in}_\mu = 0$ and has a simple interpretation: if the charge is infinitely far away as $t\to-\infty$, then the only contribution to $A_\mu(x)$ is from $A^{in}_\mu(x) \Rightarrow$ it's the "initial" value of $A_\mu(x)$, specified at $t=-\infty$. We set it to zero (can add it back at no cost).

The 4-current of the point charge q is $J^\mu(x) = cq\frac{dy^\mu}{dy^0}\delta^{(3)}(\vec{x}-\vec{y}(y^0))$

Trick: multiply by $\int d\lambda\, \delta(x^0-y^0(\lambda))\frac{dy^0}{d\lambda} = 1$ to obtain a covariant expression and, more usefully, a $\delta^4(x)$:

$$\frac{1}{c}J^\mu(x) = \int d\lambda\, q u^\mu \delta^{(4)}(x^\mu - y^\mu(\lambda)) \quad \text{where } u^\mu = \frac{dy^\mu}{d\lambda}$$

So we have

$$A_\mu(x) = \int d^4x'\, G_{ret}(x,x')\left[\frac{4\pi}{c}\int d\lambda\, cqu^\mu \delta^4(x'-y^\mu(\lambda))\right] = 4\pi q\int d\lambda\, u^\mu G_{ret}(x-y(\lambda))$$

There are two ways to do the integral. The elegant way involves expressing $G_{ret}(x)$ in a Lorentz invariant way: since $x^0 = |\vec{x}|$ is on the future light cone, consider

$$\delta(x^2)\theta(x^0) = \delta(x^{0\,2}-\vec{x}^2)\,\theta(x^0) = \frac{\delta(x^0-|\vec{x}|)}{2|\vec{x}|} \quad \text{so} \quad G_{ret} = \frac{1}{2\pi}\delta((x-y)^2)\,\theta(x^0-y^0)$$

$$\Rightarrow A_\mu(x) = 2q\int d\lambda\, u^\mu \delta((x-y(\lambda))^2)\,\theta(x^0-y^0(\lambda)) = 2qu^\mu \frac{1}{\left|\frac{d}{d\lambda}(x-y(\lambda))^2\right|}\Bigg|_{\substack{y(\lambda_0)=x \\ y^0<x^0}} = q\frac{u^\mu}{|(x-y)\cdot u|}\Bigg|_{\lambda_0}$$

$\boxed{\text{Liénard-Wiechert potentials}}$

Here, evaluated at λ_0 means, as anticipated, at retarded time: λ_0 is the solution to

$$(x-y(\lambda_0))^2=0 \qquad \text{with } x^0>y^0(\lambda_0)$$

which, of course, is just $\quad y^0(\lambda_0)=x^0-|\vec{x}-\vec{y}(\lambda_0)|$

The alternative way is to use $G_{ret}(x)=\frac{1}{4\pi|x|}\delta(x^0-|\vec{x}|)$ directly:

$$A_\mu(x)=q\int d\lambda\, v^\mu\,\frac{1}{|\vec{x}-\vec{y}|}\,\delta(x^0-y^0(\lambda)-|\vec{x}-\vec{y}(\lambda)|)$$

$$=q\,\frac{v^\mu}{|\vec{x}-\vec{y}(\lambda)|}\Big|_{\lambda_0}\frac{1}{\left|\frac{d}{d\lambda}(x^0-y^0-|\vec{x}-\vec{y}(\lambda)|)\right|}$$

where $\quad \frac{d}{d\lambda}\left(y^0+|\vec{x}-\vec{y}|\right)=\frac{dy^0}{d\lambda}+\frac{(\vec{x}-\vec{y})\cdot\left(-\frac{d\vec{y}}{d\lambda}\right)}{|\vec{x}-\vec{y}|}$

This equals the previous expression, since equals $\frac{1}{|\vec{x}-\vec{y}|}\left[(x^0-y^0)\frac{dy^0}{d\lambda}-(\vec{x}-\vec{y})\cdot\frac{d\vec{y}}{d\lambda}\right]=\frac{1}{|\vec{x}-\vec{y}|}(x-y)\cdot v$

so $\quad A_\mu=\frac{q\,v_\mu}{|x-y|\cdot\frac{1}{|x-y|}[(x-y)\cdot v]}=\frac{q v_\mu}{(x-y)\cdot v}$

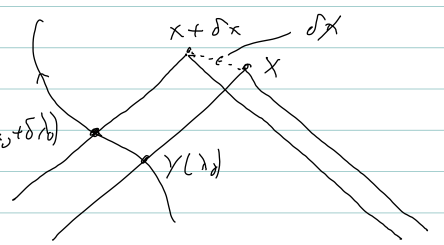
Let's write the potentials in terms of the velocity $\vec{\beta}$ of the charge and the distance $\vec{R}$ from retarded charge to $\vec{x}$. Using $\lambda=y^0$ $\frac{dy^\mu}{dy^0}=(1,\vec{\beta})$. As above,

$$A^\mu(x)=q\,\frac{v^\mu}{|\vec{x}-\vec{y}(\lambda)|}\Big|_{\lambda_0}\frac{1}{\frac{dy^0}{d\lambda}-\frac{(\vec{x}-\vec{y})}{|\vec{x}-\vec{y}|}\cdot\frac{d\vec{y}}{d\lambda}}=q\,\frac{(1,\vec{\beta})}{R}\cdot\frac{1}{1-\hat{R}\cdot\vec{\beta}}$$

where $\vec{\beta}$ is at t_r (retarded).

We can also compute $\vec{E}$ & $\vec{B}$. The complication is that in taking $\frac{\partial}{\partial x^\mu}$ we are changing not just x^μ but also λ_0 (think of $\lambda_0=\lambda_0(x)$ determined by $(x-y(\lambda_0))^2=0$).

$$(x-y(\lambda_0))^2=0 \quad\text{and}\quad (x+\delta x-y(\lambda_0+\delta\lambda_0))^2=0 \quad\text{or}\quad (x+\delta x-[y(\lambda_0)+v\,\delta\lambda_0])^2=0$$



$$(x-y)\cdot(\delta x-v\,\delta\lambda_0) \quad 0$$

$$\Rightarrow \partial_\mu\lambda_0=\frac{\partial\lambda_0}{\partial x^\mu}=\frac{(x-y)_\mu}{(x-y)\cdot v}$$

For $F_{\mu\nu}$ we need $\partial_\mu A_\nu$, and for this $\partial_\mu v_\nu=\frac{\partial\lambda_0}{\partial x^\mu}\cdot\frac{dv_\nu}{d\lambda}\Big|_{\lambda_0}=\frac{(x-y)_\mu}{(x-y)\cdot v}a_\nu$

where $a^\nu=\frac{dv^\nu}{d\lambda}=\frac{d^2y^\nu}{d\lambda^2}=\ddot{y}^\nu$. Similarly $\partial_\mu y_\nu=\frac{(x-y)_\mu v_\nu}{(x-y)\cdot v}$

Then $$F_{\mu\nu} = \partial_\mu\left(\frac{q\,u_\nu}{(x-y)\cdot u}\right) - \mu\leftrightarrow\nu$$

$$= \frac{q}{[(x-y)\cdot u]^2}\left\{(x-y)_\mu a_\nu - u_\nu\left[\left(\eta^{\mu\lambda} - \frac{(x-y)^\mu u^\lambda}{(x-y)\cdot u}\right)u_\lambda + (x-y)_\lambda \frac{(x-y)_\mu a^\lambda}{(x-y)\cdot u}\right]\right\} - \mu\leftrightarrow\nu$$

This gives $\vec{E}$ and $\vec{B}$ in terms of retarded $y^\mu(\lambda_0)$, $u^\mu(\lambda_0)$ and $a^\mu(\lambda_0)$. Note that it is reparametrization invariant. It is useful to write this more explicitly in terms of $\vec{\beta}$, $\vec{R}$ and t_r. So take $\lambda = y^0$.

Let's separate this into the a-dependent part $F^{acc}_{\mu\nu}$ and the rest, $F^{vel}_{\mu\nu}$

$$F^{acc}_{\mu\nu} = \frac{q}{[(x-y)\cdot u]^2}(x-y)_\mu\left(a_\nu - \frac{u_\nu\,(x-y)\cdot a}{(x-y)\cdot u}\right) - \mu\leftrightarrow\nu$$

$$F^{vel}_{\mu\nu} = \frac{q\,u^2}{[(x-y)\cdot u]^3}\left[(x-y)_\mu u_\nu - (x-y)_\nu u_\mu\right]$$

Then (recall $u^\mu = (1,\vec{\beta})$, $(x-y)\cdot u = |\vec{x}-\vec{y}(\lambda_0)| - (\vec{x}-\vec{y}(\lambda_0))\cdot\vec{\beta} = R(1-\hat{R}\cdot\vec{\beta})$)

$$F^{vel}_{\mu\nu} = \frac{q}{\gamma^2 R^3}\frac{(x-y)_\mu u_\nu - \mu\leftrightarrow\nu}{(1-\hat{R}\cdot\vec{\beta})^3} \Rightarrow \vec{E}^{vel} = -F^{0i} = -\frac{q}{\gamma^2 R^3}\frac{1}{(1-\hat{R}\cdot\vec{\beta})^3}\left[R\beta^i - \vec{R}\right] = \frac{q}{R^2}\frac{\hat{R}-\vec{\beta}}{\gamma^2(1-\hat{R}\cdot\vec{\beta})^3}$$

and $$\vec{B}^{vel} = -\tfrac{1}{2}\epsilon^{ijk}F^{jk} = -\frac{q}{\gamma^2 R^3}\frac{\epsilon^{ijk}R^j\beta^k}{(1-\hat{R}\cdot\vec{\beta})^3} = -\frac{q}{\gamma^2 R^2}\frac{\hat{R}\times\vec{\beta}}{(1-\hat{R}\cdot\vec{\beta})^3}$$

which we recognize as the $\vec{E}$ & $\vec{B}$ fields of a moving charge with $\vec{\beta}$ = constant. But for $\dot{\vec{\beta}}\neq 0$ we have additional terms. We need

$$a^\mu = \frac{d^2y^\mu}{dy^{0\,2}} = \frac{d}{dy^0}(1,\vec{\beta}) = (0,\vec{\alpha}) \quad \text{with } \vec{\alpha} = \frac{d\vec{\beta}}{dy^0} = \frac{1}{c^2}\vec{a}$$

$$\vec{E}^{acc} = -\frac{q}{R^2(1-\hat{R}\cdot\vec{\beta})^2}\left[R\left(\vec{\alpha} + \frac{\vec{\beta}\,\hat{R}\cdot\vec{\alpha}}{1-\hat{R}\cdot\vec{\beta}}\right) - \vec{R}\left(0 + \frac{\hat{R}\cdot\vec{\alpha}}{1-\hat{R}\cdot\vec{\beta}}\right)\right] = -\frac{q}{R}\frac{1}{(1-\hat{R}\cdot\vec{\beta})^2}\left[\vec{\alpha} - \hat{R}\cdot\vec{\alpha}\,\frac{\hat{R}-\vec{\beta}}{1-\hat{R}\cdot\vec{\beta}}\right]$$

and
$$\vec{B}^{acc} = -\frac{q}{R(1-\hat{R}\cdot\vec{\beta})^2}\hat{R}\times\left(\vec{\alpha} + \frac{\vec{\beta}\,\hat{R}\cdot\vec{\alpha}}{1-\hat{R}\cdot\vec{\beta}}\right)$$

Note that $\vec{B}^{acc} = \hat{R}\times\vec{E}^{acc}$ and $|\vec{E}^{acc}| \sim \frac{1}{R}$. So $E \sim \underbrace{\frac{1}{R}}_{\text{"radiation field"}} + \frac{1}{R^2}$

Note $\hat{R}\times[(\hat{R}-\vec{\beta})\times\vec{\alpha}] = (\hat{R}-\vec{\beta})\,\hat{R}\cdot\vec{\alpha} - \vec{\alpha}\,\hat{R}\cdot(\hat{R}-\vec{\beta}) = -(1-\hat{R}\cdot\vec{\beta})\left(\vec{\alpha} - \hat{R}\cdot\vec{\alpha}\,\frac{\hat{R}-\vec{\beta}}{1-\hat{R}\cdot\vec{\beta}}\right)$ so one may write

$$\vec{E}^{acc} = \frac{q}{R}\frac{1}{(1-\hat{R}\cdot\vec{\beta})^3}\,\hat{R}\times[(\hat{R}-\vec{\beta})\times\vec{\alpha}]$$

At long distances only the radiation field is significant. It has $\vec{E}$ & $\vec{B}$ perpendicular to $\vec{R}$ — the retarded position vector, and are perpendicular to each other.

Non-relativistic limit: $\vec{E}^{acc} = \frac{q}{R}\hat{R}\times(\hat{R}\times\vec{\alpha}) = \frac{q}{c^2R}\hat{R}\times(\hat{R}\times\vec{a})$

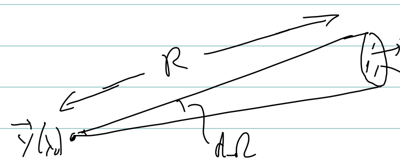
Rectilinear motion, $\vec{\beta} = \beta\hat{\alpha} \Rightarrow \vec{E}^{acc} = \frac{q}{c^2R}\frac{1}{(1-\beta\hat{a}\cdot\hat{R})^3}\left[\hat{R}\times(\hat{R}\times\vec{a})\right]$

Power Radiated

Poynting vector: energy flux

$$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B} = \frac{c}{4\pi}|E|^2\hat{R} \quad \left(= cu\hat{R} \text{ with } u = \tfrac{1}{8\pi}(E^2+B^2)\right).$$

Consider at t a sphere centered at the retarded charge position $\vec{y}(\lambda_0)$. The energy



$d^2x = (\text{area}) = R^2\,d\Omega = R^2\sin\theta\,d\theta\,d\phi$

dP = power = $\frac{\text{energy}}{\text{time}}$ through area $= (\vec{S}\cdot\hat{R})(R^2d\Omega)$

$$\Rightarrow \frac{dP'}{d\Omega} = \vec{S}\cdot\hat{R}R^2 = \frac{c}{4\pi}(R|E|)^2 = \frac{c}{4\pi}\left|\frac{q}{(1-\hat{R}\cdot\vec{\beta})^3}\hat{R}\times\left[(\hat{R}-\vec{\beta})\times\vec{\alpha}\right]\right|^2$$

Note that as $R\to\infty$ the contribution of $\vec{E}^{vel}$ vanishes, hence we have neglected it here.

This expression has the energy per unit time in the inertial frame, that is, measured by a far-away observer. Sometimes we are interested in a time interval measured at the particle, $dP = dP'\frac{\partial x^0}{\partial y^0}$. Now, recall

$$\frac{\partial y^\mu}{\partial x^\nu} = \frac{(x-y)_\nu U^\mu}{(x-y)\cdot U} \quad \text{so that} \quad \frac{\partial y^0}{\partial x^0} = \frac{1}{1-\hat{R}\cdot\vec{\beta}}$$

and we have then

$$\frac{dP}{d\Omega} = \frac{cq^2}{4\pi}\frac{1}{(1-\hat{R}\cdot\vec{\beta})^5}\left|\hat{R}\times\left((\hat{R}-\vec{\beta})\times\vec{\alpha}\right)\right|^2$$

Note added: see discussion (below) on synchrotron radiation. The width of a pulse of radiation changes by this factor, it is the same issue, but there it is conceptually clearer.

(The reason for using time as seen by particle is (i) want total power radiated between particle at y_1^0 and y_2^0, and (ii) dP is then a Lorentz invariant: see below).

The relativistic expression presents some challenges with the integration:

$$\frac{dP}{d\Omega} = \frac{cq^2}{4\pi}\frac{1}{(1-\hat{R}\cdot\vec{\beta})^5}\left|\vec{\alpha}(1-\hat{R}\cdot\vec{\beta}) + (\vec{\beta}-\hat{R})\hat{R}\cdot\vec{\alpha}\right|^2$$

$$= \frac{cq^2}{4\pi}\frac{1}{(1-\hat{R}\cdot\vec{\beta})^5}\Big[\alpha^2(1-\hat{R}\cdot\vec{\beta})^2 + 2\left[\vec{\alpha}\cdot\vec{\beta}-\hat{R}\cdot\vec{\alpha}\right]\hat{R}\cdot\vec{\alpha}(1-\hat{R}\cdot\vec{\beta})$$
$$+ (\hat{R}\cdot\vec{\alpha})^2(1+\beta^2-2\vec{\beta}\cdot\hat{R})\Big]$$

$$= \frac{cq^2}{4\pi}\frac{1}{(1-\hat{R}\cdot\vec{\beta})^5}\left[\alpha^2(1-\hat{R}\cdot\vec{\beta})^2 - (\hat{R}\cdot\vec{\alpha})^2(1-\beta^2) + 2\vec{\alpha}\cdot\vec{\beta}\,\hat{R}\cdot\vec{\alpha}\,(1-\hat{R}\cdot\vec{\beta})\right]$$

To write this in terms of angles we pick a frame ($\vec{\beta} \propto \hat{z}$, $\vec{\alpha}$ in xz-plane)

$$\vec{\alpha} = \alpha(\sin\xi, 0, \cos\xi)$$
$$\vec{\beta} = \beta(0,0,1)$$
$$\hat{R} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$\Rightarrow \vec{\beta}\cdot\hat{R} = \beta\cos\theta, \quad \vec{\alpha}\cdot\hat{R} = \alpha(\cos\theta\cos\xi + \sin\theta\sin\xi\cos\phi)$$

(and $\vec{\alpha}\cdot\vec{\beta} = \alpha\beta\cos\xi$ but these are independent of $d\Omega$).

Now $\int_0^{2\pi} d\phi$ is trivial, so, with $x = \cos\theta$

$$\underline{P} = \frac{cq^2}{2}\int_{-1}^{1} dx\,\frac{1}{(1-\beta x)^5}\left[\alpha^2(1-\beta x)^2 - (1-\beta^2)\alpha^2\left(\cos^2\xi\, x^2 + \tfrac{1}{2}\sin^2\xi(1-x^2)\right) + 2\vec{\alpha}\cdot\vec{\beta}\,\alpha\cos\xi\, x(1-x\beta)\right]$$

(we used $\int_0^{2\pi} d\phi\,(1, \cos\phi, \cos^2\phi) = 2\pi(1, 0, \tfrac{1}{2})$.). The remaining integral is straightforward but tedious (and we have Mathematica):

$$\underline{P} = \frac{cq^2}{2}\left[\frac{4}{3}\alpha^2\frac{1-s^2\beta^2}{(1-\beta^2)^3}\right] \qquad \text{where } s^2 = \sin^2\xi = |\hat{\alpha}\times\hat{\beta}|^2$$

or $$\boxed{\underline{P} = \frac{2}{3}\frac{q^2}{c}\gamma^6\left[\vec{a}^2 - (\vec{\beta}\times\vec{a})^2\right]}$$ Lienard (1898)

In the non-relativistic limit, $\beta \ll 1$, we obtain "Larmor's formula"

$$\boxed{\underline{P} = \frac{2}{3}\frac{q^2}{c}a^2}$$ (NR limit, "Larmor's formula).

Comment: It is easy to obtain Larmor's formula from the non-relativistic limit of

$$\frac{dP'}{d\Omega} = \frac{cq^2}{4\pi}\frac{1}{(1-\hat{R}\cdot\vec{\beta})^6}\left|\hat{R}\times\left[(\hat{R}-\vec{\beta})\times\vec{\alpha}\right]\right|^2 \simeq \frac{cq^2}{4\pi}|\hat{R}\times\vec{\alpha}|^2 = \frac{cq^2\alpha^2}{4\pi}\sin^2\theta$$

Then $P'\simeq P$ and integrating over $d\Omega$ (recall $\vec{a}=\frac{1}{c^2}\vec{\alpha}$)

$$P \simeq \frac{2}{3}\frac{q^2a^2}{c^3}$$

Many textbooks then obtain Lienard's formula as follows: argue that P is frame invariant, then find a Lorentz scalar that reduces to Larmor's formula in the NR limit.

The 1st part of the argument is this: with $P=\frac{\Delta E}{\Delta y^0}$, this is energy/time as measured by comoving observer.

2nd part: substitute $\vec{a}=\frac{1}{m}\frac{d\vec{p}}{dt}$ in Larmor's:

$$a^2 = \frac{1}{m^2}\frac{d\vec{p}}{dt}\cdot\frac{d\vec{p}}{dt} = \frac{1}{m^2}\left[-c^2\frac{dp^\mu}{ds}\frac{dp_\mu}{ds}\right]$$

where the last step is valid in the NR limit (we'll verify below).

Now

$$-\frac{c^2}{m^2}\frac{dp^\mu}{ds}\frac{dp_\mu}{ds} = -\frac{c^2}{m^2}\left(\frac{dt}{ds}\right)^2\left[\left(\frac{d}{dt}(mc\gamma)\right)^2 - \left(\frac{d}{dt}(mc\gamma\vec{\beta})\right)^2\right]$$

$$= -c^2\gamma^2\left[\left(\gamma^3\vec{\beta}\cdot\dot{\vec{\beta}}\right)^2 - \left(\gamma^3\vec{\beta}\cdot\dot{\vec{\beta}}\,\vec{\beta} + \gamma\dot{\vec{\beta}}\right)^2\right]$$

$$= -c^2\gamma^2\left[\gamma^6(\vec{\beta}\cdot\dot{\vec{\beta}})^2(1-\beta^2) - 2\gamma^4(\vec{\beta}\cdot\dot{\vec{\beta}})^2 - \gamma^2(\dot{\vec{\beta}})^2\right]$$

$$= c^2\gamma^6\left[\dot{\vec{\beta}}^2(1-\beta^2) + (\vec{\beta}\cdot\dot{\vec{\beta}})^2\right]$$

$$= c^2\gamma^6\left[\dot{\vec{\beta}}^2 - \left(\beta^2\dot{\vec{\beta}}^2 - (\vec{\beta}\cdot\dot{\vec{\beta}})^2\right)\right] = c^2\gamma^6\left[\dot{\vec{\beta}}^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2\right]$$

so

$$P = \frac{2}{3}\frac{q^2\gamma^6}{c^3}\left[\vec{a}^2 - (\vec{\beta}\times\vec{a})^2\right] \quad \text{as before.}$$

Ref: Jackson 14.3
L.L. 73
Lechner Chap 10

Angular Distribution

We already have the basic equation for this in $\frac{dP}{d\Omega}$ given in terms of θ & φ angles in p.5 above.

We look at special cases:

1. Linear motion: $\vec{\beta} \parallel \vec{\alpha}$.

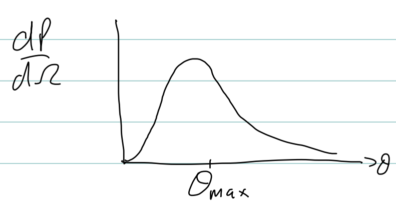
$\frac{dP}{d\Omega}$ is independent of ϕ (symmetric under rotations about axis defined by $\vec{\beta}$).

Explicitly
$$\frac{dP}{d\Omega} = \frac{cq^2}{4\pi} \frac{1}{(1-\hat{R}\cdot\vec{\beta})^5} \left| \hat{R} \times \{ (\hat{R}-\vec{\beta}) \times \vec{\alpha} \} \right|^2$$
$$= \frac{cq^2}{4\pi} \frac{\alpha^2 \sin^2\theta}{(1-\beta\cos\theta)^5}$$

The direction of maximum radiation can be found analytically, but can also be quickly approximated by noting that for $\beta \sim 1$ it is at small θ and the denominator is small for

$$1-\beta\cos\theta \simeq 1-\beta(1-\tfrac{1}{2}\theta^2) \simeq 1-\beta+\tfrac{1}{2}\theta^2 = \text{small}, \text{ or with } \frac{1}{\gamma^2} = (1-\beta)(1+\beta) \simeq 2(1-\beta)$$

$\Rightarrow \theta^2 \sim \frac{1}{\gamma^2} \Rightarrow \frac{dP}{d\Omega}$ is peaked at $\theta_{max} \sim \frac{1}{\gamma} \ll 1$, but vanishes at $\theta = 0$



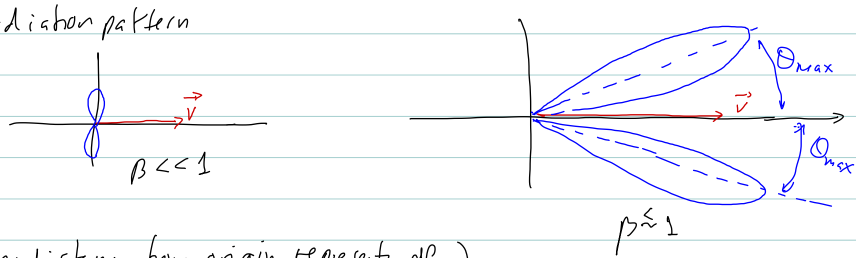
writing $1-\beta\cos\theta \simeq \frac{1}{2\gamma^2}(1+\gamma^2\theta^2)$

$$\frac{dP}{d\Omega} \simeq \frac{cq^2\alpha^2}{4\pi}(2\gamma^2)^5 \frac{\theta^2}{(1+(\gamma\theta)^2)^5}$$

or
$$\frac{dP}{d\Omega} = \frac{8q^2}{\pi} c\alpha^2 \gamma^8 \frac{(\gamma\theta)^2}{(1+(\gamma\theta)^2)^5}$$

In the opposite limit, $\beta \ll 1$ $\quad \frac{dP}{d\Omega} \simeq \frac{cq^2}{4\pi}\alpha^2\sin^2\theta$ (Larmor), so $\theta_{max} \simeq \frac{\pi}{2}$

Radiation pattern



Note: Figures of revolution (ie, the latter is a forward cone).

(The distance from origin represents $\frac{dP}{d\Omega}$)

Another case of interest is when $\vec{\beta} \perp \vec{\alpha}$, as in circular motion. Then, from previous lecture

$$\frac{dP}{d\Omega} = \frac{cq^2}{4\pi}\frac{1}{(1-\hat{R}\cdot\vec{\beta})^5}\left[\alpha^2(1-\hat{R}\cdot\vec{\beta})^2 - (\hat{R}\cdot\vec{\alpha})^2(1-\beta^2) + 2\vec{\alpha}\cdot\vec{\beta}\,\hat{R}\cdot\vec{\alpha}\,(1-\hat{R}\cdot\vec{\beta})\right]$$

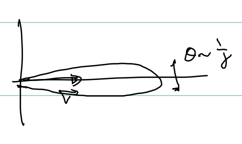
$$= \frac{cq^2}{4\pi}\frac{\alpha^2}{(1-\hat{R}\cdot\vec{\beta})^5}\left[(1-\hat{R}\cdot\vec{\beta})^2 - (\hat{R}\cdot\hat{\alpha})^2(1-\beta^2)\right]$$

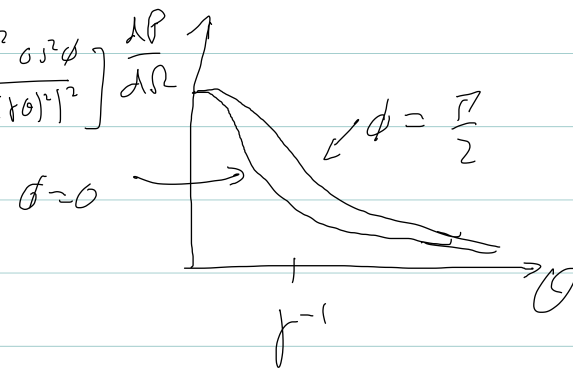
In the coordinate system we used earlier ($\hat{\beta}=\hat{z}$, $\vec{\alpha}$ in xz plane) we have now $\hat{\alpha}=\hat{x}$ and $\hat{R}\cdot\hat{\beta}=\cos\theta$, $\hat{R}\cdot\hat{\alpha}=\sin\theta\cos\phi$ (that is $\hat{R}\cdot\hat{\beta}=\hat{R}_z$ and $\hat{R}\cdot\hat{\alpha}=\hat{R}_x$)

$$\frac{dP}{d\Omega} = \frac{cq^2\alpha^2}{4\pi}\left[\frac{1}{(1-\beta\cos\theta)^3} - (1-\beta^2)\frac{\sin^2\theta\cos^2\phi}{(1-\beta\cos\theta)^5}\right]$$

Using $1-\beta\cos\theta \simeq \frac{1}{2\gamma^2}(1+(\gamma\theta)^2)$ for $\beta\simeq 1$ we have

$$\frac{dP}{d\Omega} \simeq \frac{2cq^2\alpha^2}{\pi}\gamma^6\frac{1}{(1+(\gamma\theta)^2)^3}\left[1 - \frac{4(\gamma\theta)^2\cos^2\phi}{\gamma^2(1+(\gamma\theta)^2)^2}\right]$$





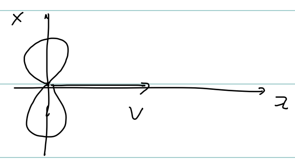
The radiation is emitted preferentially in the direction of $\vec{\beta}$, to within a cone of angular size $\theta\sim\frac{1}{\gamma}$. There is slightly more power radiated off the $\vec{\alpha}$-$\vec{\beta}$ plane (ie $\phi=\frac{\pi}{2}$) than on plane ($\phi=0$), but the difference is order $1/\gamma^2$.

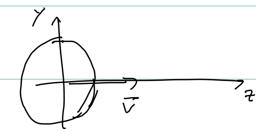
At small β, retaining lowest order in β:

$$\frac{dP}{d\Omega} = \frac{cq^2\alpha^2}{4\pi}\left[1-\sin^2\theta\cos^2\phi + \beta\cos\theta\left(3-5\sin^2\theta\cos^2\phi\right)\right]$$

$$= \frac{cq^2\alpha^2}{4\pi} \times \begin{cases} \cos^2\theta - \beta\cos\theta\left(2-5\cos^2\theta\right) & \text{for } \phi = 0 \\ 1+3\beta\cos\theta & \phi = \frac{\pi}{2} \end{cases}$$

Patterns:





How about polarization? Recall we 1st derived (Lienord-Weichert)

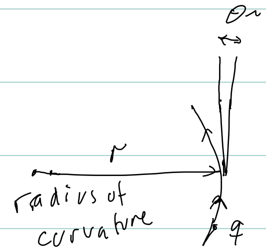
$$\vec{E}_{rad} = -\frac{q}{R}\frac{1}{(1-\hat{R}\cdot\vec{\beta})^2}\left[\vec{\alpha} - \hat{R}\cdot\vec{\alpha}\,\frac{\hat{R}-\vec{\beta}}{1-\hat{R}\cdot\vec{\beta}}\right]$$

For $\gamma \gg 1$ the radiation is mostly along $\hat{R} = \hat{\beta}$, so focusing on that direction and noting that $\vec{\alpha}\cdot\vec{\beta} = 0$ for circular motion, $\vec{E}_{rad} \simeq -\frac{q}{R}\frac{1}{(1-\beta)^2}\vec{\alpha}$

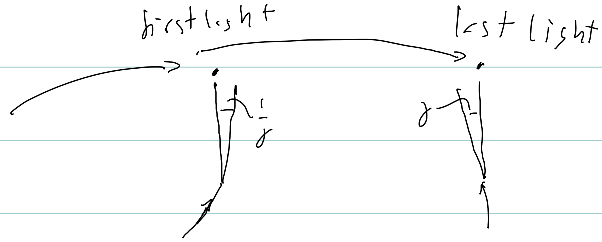
That is, fully in the plane of the circular motion to good approximation.

Heuristic discussion of radiation from Arbitrary motion with $\gamma >> 1$ and intro to spectral decomposition

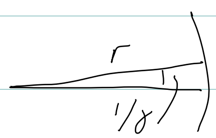
Ref: Jackson 14.4



Arbitrary relativistic motion $(\gamma >> 1)$:
Radiation in forward cone $\theta \sim \frac{1}{\gamma}$

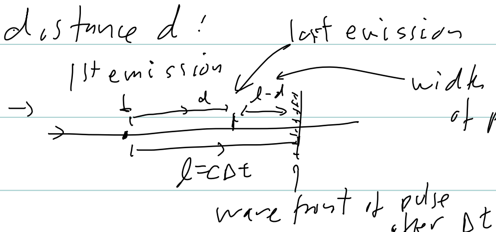


For a fixed observer as the particle transits a small section of the curved path there is radiation within the cone $\theta \sim 1/\gamma$ at observer.



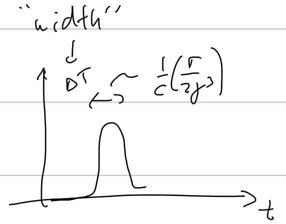
so burst of emission over time $\Delta t = \frac{d}{v} = \frac{r}{c\gamma\beta}$

Front edge of radiation moves a distance $l = c\Delta t = \frac{r}{\gamma\beta}$ by time the back edge of radiation "pulse" is emitted from particle that moved distance d:

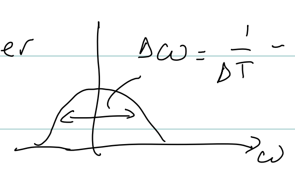


width of pulse $= l - d = \frac{r}{\gamma\beta} - \frac{r}{\gamma} = \frac{r}{\gamma\beta}(1-\beta) \simeq \frac{r}{2\gamma^3}$

Observed intensity



$\Longleftrightarrow$ Fourier



For circular motion $\frac{c}{r} = \omega_q$ angular frequency and the observed frequencies $\omega \lesssim \Delta\omega \sim \gamma^3 \omega_q$

The amplification factor γ^3 is important.

For example, for $\omega_q \sim$ MHz, once can produce 10 keV X-rays ($\omega \sim 10^{19}\,s^{-1}$) with $\gamma \sim (10^{19}/10^6)^{1/3} \sim 10^4$. For electrons with $mc^2 \simeq 1$ MeV this requires $E \sim 10$ GeV → see energies of synchrotrons used as X-ray sources!

Ref: Lechner 10

Jackson 14.5

Spectral Analysis

Clearly of interest to have a more quantitative analysis.

We would like to have an expression for

$$\frac{dI}{d\Omega\, d\omega} = \frac{\text{observed intensity of radiation}}{\text{solid angle} \cdot \text{frequency}}$$

Now $\frac{dI}{d\Omega}$ is just $\frac{dP'}{d\Omega}$ integrated over time. We use P' which refers to per time of lab frame, ie, observer's time as is appropriate for this question. The expression is in these notes (p.4).

$$\frac{dI}{d\Omega} = \int_{-\infty}^{\infty} dt\, \frac{dP}{d\Omega}$$

Parseval's theorem

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{f}(k) \iff \int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} \frac{dk}{2\pi} |\tilde{f}(k)|^2$$

Physicist's proof:

$$\int_{-\infty}^{\infty} dx\, |f(x)|^2 = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{dk'}{2\pi}\, e^{ikx} \tilde{f}(k)\, e^{-ik'x} \tilde{f}^*(k')$$

$$= \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} \tilde{f}(k) \tilde{f}^*(k') \underbrace{\int_{-\infty}^{\infty} dx\, e^{i(k-k')x}}_{= 2\pi\, \delta(k-k')}$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi} |\tilde{f}(k)|^2$$

Since we had

$$\frac{dP'}{d\Omega} = \vec{S}\cdot\hat{R}R^2 = \frac{c}{4\pi} (R|E|)^2 = \frac{c}{4\pi} \left| \frac{q}{(1-\hat{R}\cdot\vec{\beta})^3} \hat{R}\times[(\hat{R}-\vec{\beta})\times\vec{\alpha}] \right|^2$$

we write this as $\frac{dP'}{d\Omega} = |\vec{P}(t)|^2$ $\quad \vec{P}(t) = \sqrt{\frac{c}{4\pi}} \frac{q}{(1-\hat{R}\cdot\vec{\beta})^3} \hat{R}\times(\hat{R}-\vec{\beta})\times\vec{\alpha}$

Then $$\frac{dI}{d\Omega} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |\vec{\tilde{P}}(\omega)|^2$$ (where $\vec{\tilde{P}}(\omega)$ is the Fourier transform of $\vec{P}(t)$)

$$= \int_{0}^{\infty} \frac{d\omega}{2\pi} \left[|\vec{\tilde{P}}(\omega)|^2 + |\vec{\tilde{P}}(-\omega)|^2 \right]$$

$$\Rightarrow \frac{dI}{d\Omega\, d\omega} = \frac{1}{2\pi} \left[|\vec{\tilde{P}}(\omega)|^2 + |\vec{\tilde{P}}(-\omega)|^2 \right] = \frac{1}{\pi} |\vec{\tilde{P}}(\omega)|^2$$

the last step if $\vec{P}(t)$ is real, then $\vec{\tilde{P}}(-\omega) = \vec{\tilde{P}}^*(\omega)$.

Useful approximations:

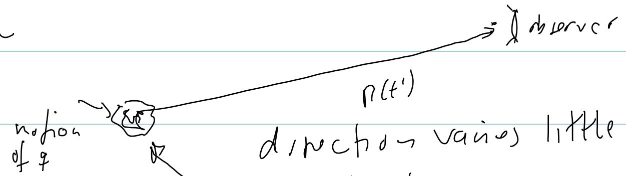
in $\displaystyle\int_{-\infty}^{\infty} dt\, e^{i\omega t}\, (\text{func})_{\text{retarded}}$

$\uparrow$ evaluated at $t - t' = R/c$

Change integration variable $dt \to dt'$. Then func = func(t')

$$\Rightarrow \int_{-\infty}^{\infty} dt'\,(1 - \hat{n}\cdot\vec{\beta})\, e^{i\omega(t' + R(t')/c)} \quad (\text{func}(t'))$$

Now



direction varies little so

(i) choose origin somewhere in blob (ie, particle trajectory)

(ii) If observer is at $\vec{r}$

$$R(t') = |\vec{r} - \vec{y}(t')| \simeq r - \hat{r}\cdot\vec{y}(t')$$

$\uparrow$ recall, trajectory of q.

So (drop primes and $e^{i\omega r}$ = constant)

$$\vec{p}(\omega) = \sqrt{\frac{c}{4\pi}}\, q \int_{-\infty}^{\infty} dt\, e^{i\omega\left(t - \frac{\hat{r}\cdot\vec{y}(t)}{c}\right)} \frac{\hat{r}\times[(\hat{r}-\vec{\beta})\times\vec{\alpha}]}{(1-\hat{r}\cdot\vec{\beta})^2}$$

and

$$\frac{d^2 I}{d\Omega\, d\omega} = \frac{c q^2}{4\pi^2} \left| \int_{-\infty}^{\infty} dt\, e^{i\omega\left(t - \frac{\hat{r}\cdot\vec{y}(t)}{c}\right)} \frac{\hat{r}\times\{(\hat{r}-\vec{\beta})\times\vec{\alpha}\}}{(1-\hat{r}\cdot\vec{\beta})^2} \right|^2$$

This integral is hard. There is a trick that simplifies significantly. Note that

$$\frac{\hat{r}\times[(\hat{r}-\vec{\beta})\times\vec{\alpha}]}{(1-\hat{r}\cdot\vec{\beta})^2} = \frac{1}{c}\frac{d}{dt}\left[\frac{\hat{r}\times(\hat{r}\times\vec{\beta})}{1-\hat{r}\cdot\vec{\beta}}\right]$$

Proof:

$$\frac{1}{c}\frac{d}{dt}\left[\frac{\hat{r}\times(\hat{r}\times\vec{\beta})}{1-\hat{r}\cdot\vec{\beta}}\right] = \frac{\hat{r}\cdot\vec{\alpha}}{(1-\hat{r}\cdot\vec{\beta})^2}\hat{r}\times(\hat{r}\times\vec{\beta}) + \frac{\hat{r}\times(\hat{r}\times\vec{\alpha})}{1-\hat{r}\cdot\vec{\beta}}$$

$$= \frac{1}{(1-\hat{r}\cdot\vec{\beta})^2}\left[\hat{r}\cdot\vec{\alpha}\,(\hat{r}\,\hat{r}\cdot\vec{\beta}-\vec{\beta}) + \hat{r}\times(\hat{r}\times\hat{\alpha}) - \hat{r}\cdot\hat{\beta}\,(\hat{r}\,\hat{r}\cdot\vec{\alpha}-\vec{\alpha})\right]$$

$$= \frac{1}{(1-\hat{r}\cdot\vec{\beta})^2}\left[(\hat{r}\cdot\vec{\beta})\vec{\alpha}-(\hat{r}\cdot\vec{\alpha})\vec{\beta} + \hat{r}\times(\hat{r}\times\vec{\alpha})\right]$$

$$= \frac{1}{(1-\hat{r}\cdot\vec{\beta})^2}\left[-\hat{r}\times(\vec{\beta}\times\vec{\alpha}) + \hat{r}\times(\hat{r}\times\vec{\alpha})\right]$$

Now, we willy-willy integrate by parts

$$\vec{\mathcal{P}}(\omega) = \sqrt{\frac{c}{4\pi}}\,q\int_{-\infty}^{\infty}dt\; e^{i\omega(t-\hat{r}\cdot\frac{\vec{y}(t)}{c})}\,\frac{1}{c}\frac{d}{dt}\left[\frac{\hat{r}\times(\hat{r}\times\vec{\beta})}{1-\hat{r}\cdot\vec{\beta}}\right]$$

$$\text{"="} -\sqrt{\frac{c}{4\pi}}\,\frac{q}{c}\int_{-\infty}^{\infty}dt\,\frac{d}{dt}e^{i\omega(t-\hat{r}\cdot\vec{y}/c)}\,\frac{\hat{r}\times(\hat{r}\times\vec{\beta})}{1-\hat{r}\cdot\vec{\beta}}$$

and noting that $\frac{d}{dt}(t-\hat{r}\cdot\vec{y}/c) = 1-\hat{r}\cdot\vec{\beta}$ we have

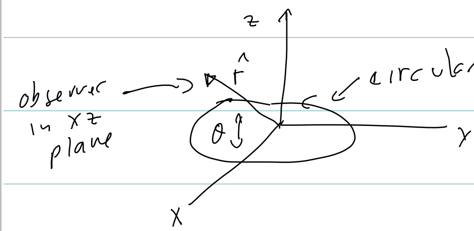
$$\vec{\mathcal{P}}(\omega) = -i\omega\sqrt{\frac{c}{4\pi}}\,\frac{q}{c}\int_{-\infty}^{\infty}dt\; e^{i\omega(t-\hat{r}\cdot\vec{y}/c)}\;\hat{r}\times(\hat{r}\times\vec{\beta})$$

This is much simpler! Now

$$\frac{d^2I}{d\Omega\,d\omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty}dt\; e^{i\omega(t-\hat{r}\cdot\vec{y}/c)}\;\hat{r}\times(\hat{r}\times\vec{\beta})\right|^2$$

Spectrum of circular motion
(Synchrotron radiation)

Refs: Lechner 11
Jackson 14.6



Note: I use θ for angle with x-axis (not z!).

$$\vec{y}(t) = \rho\left(\sin\left(\frac{vt}{\rho}\right), -\cos\left(\frac{vt}{\rho}\right), 0\right)$$

(putting $\vec{y}(0) = -\rho\,(0,1,0)$)
and $\vec{\beta}(0) = \beta(1,0,0)$)

and the argument of exp in $\tilde{\vec{\rho}}$ is

$$\omega\left(t - \frac{\hat{r}\cdot\vec{y}(t)}{c}\right) = \omega\left(t - \frac{\rho}{c}\sin\left(\frac{vt}{\rho}\right)\cos\theta\right)$$

$$\simeq \frac{\omega}{2}\left(\left(\frac{1}{\gamma^2}+\theta^2\right)t + \frac{c^2}{3\rho^2}t^3\right)$$

where we used our previous heuristic discussion that for $\gamma \gg 1$ the radiation is in a cone $\theta \sim 1/\gamma$ around $\hat{\beta}$, and since the observer is in the xz plane the burst of radiation that reaches him/her is from when $\hat{\beta} = \pm\hat{x}$, ie, from a small time interval around $\vec{y}(t) \simeq \rho(\pm 1, 0, 0) \Rightarrow$ expand about $\sin\left(\frac{vt}{\rho}\right) = 0$; and of course, expand about $\theta \simeq 0$.

The higher order terms are supressed by either $\theta^2 \sim 1/\gamma^2$ or by $\left(\frac{vt}{\rho}\right)^2 \sim \left(\frac{v\Delta t}{\rho}\right)^2$ with $c\Delta t \sim \frac{\rho}{\gamma\beta}$ from our previous arguments

So $\left(\frac{vt}{\rho}\right)^2 \sim \frac{1}{\gamma^2}$.

Note that this makes sense if we only integrate over one cycle of the circular motion. If we really integrate over all

times, we get infinitely many equal contributions $\to \infty$
The reason is clear: it is periodic motion, with fixed frequency = angular frequency $= \omega_q = \frac{v}{\rho}$

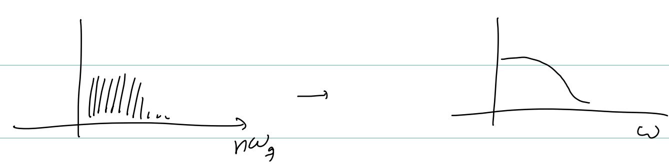
The proper mathematical analysis is then to do a Fourier series rather than an integral:

$$\vec{p}(t) = \sum_{n=-\infty}^{\infty} e^{in\omega_q t} \vec{\tilde{p}}_n$$

with

$$\vec{\tilde{p}}_n = \frac{1}{2\pi} \int_0^{\frac{2\pi}{\omega_q}} dt\, e^{-in\omega_q t} \vec{p}(t)$$

Physically we expect $n\omega_q \sim \gamma^3 \omega_q$, the number of modes that contribute is huge; the approximation above of just retaining one pulse replaces the large (but exact) number of discrete modes by a continuum:



and $\frac{1}{2\pi} \sum_n |\tilde{p}_n|^2 \to \frac{1}{2\pi} \int d\omega\, |\tilde{p}(\omega)|^2$.

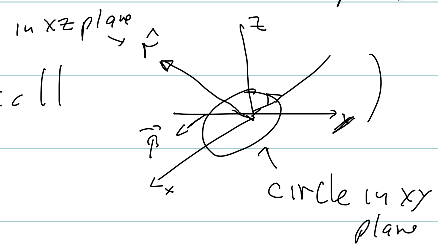
The approximation to $\vec{p}(t) \propto \hat{r}\times[(\hat{r}-\vec{\beta})\times\vec{\alpha}]$

is $\perp$ to $\hat{r}$: physically the EM wave at the observer has $\vec{E}$ (and $\vec{B}$) $\perp$ to line of sight. We can decompose $\vec{E}$ into 2 polarizations in the plane $\perp$ to $\hat{r}$:

$\vec{\epsilon}_{\parallel}$: in the xy plane

$\vec{\epsilon}_{\perp}$: $\perp$ direction, $\vec{\epsilon}_{\perp} = \hat{r}\times\vec{\epsilon}_{\parallel}$

Since radiation is from $\hat{\beta} = \pm\hat{x}$ $\quad$ ($+\hat{x}$ if $\hat{r}\cdot\hat{x} > 0$)

(recall 

)

$\Rightarrow \quad \vec{\epsilon}_{\parallel} = (0,1,0)$

(and $\vec{\beta} = \beta(\cos(\frac{vt}{\rho}), \sin(\frac{vt}{\rho}), 0)$).

Then $\vec{\epsilon}_{\perp} = \hat{r}\times\vec{\epsilon}_{\parallel} = (\cos\theta, 0, \sin\theta)\times\vec{\epsilon}_{\parallel} = (-\sin\theta, 0, \cos\theta)$

Decompose

$$\hat{r}\times(\hat{r}\times\vec{\beta}) = P_{\parallel}\,\vec{\epsilon}_{\parallel} + P_{\perp}\,\vec{\epsilon}_{\perp}$$

$$P_{\parallel} = \vec{\epsilon}_{\parallel}\cdot[\hat{r}\times(\hat{r}\times\vec{\beta})] = (\vec{\epsilon}_{\parallel}\times\hat{r})\cdot(\hat{r}\times\vec{\beta}) = -\vec{\epsilon}_{\perp}\cdot(\hat{r}\times\vec{\beta})$$

$$= -(\vec{\epsilon}_{\perp}\times\hat{r})\cdot\vec{\beta} = -\vec{\epsilon}_{\parallel}\cdot\vec{\beta} = -\sin\left(\frac{vt}{\rho}\right) \simeq -\frac{c\,t}{\rho}$$

$$P_{\perp} = \vec{\epsilon}_{\perp}\cdot[\hat{r}\times(\hat{r}\times\vec{\beta})] = (\vec{\epsilon}_{\perp}\times\hat{r})\cdot(\hat{r}\times\vec{\beta}) = \vec{\epsilon}_{\parallel}\cdot(\hat{r}\times\vec{\beta})$$

$$= (\vec{\epsilon}_{\parallel}\times\hat{r})\cdot\vec{\beta} = -\vec{\epsilon}_{\perp}\cdot\vec{\beta} = \beta\cos\left(\frac{vt}{\rho}\right)\sin\theta \simeq \theta$$

So

$$\frac{d^2I}{d\Omega\, d\omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} dt\; e^{i\frac{\omega}{2}\left[\left(\frac{1}{\gamma^2}+\theta^2\right)t + \frac{1}{3}\frac{c^2}{\rho^2}t^3\right]}\left(-\frac{c\,t}{\rho}\vec{\epsilon}_{\parallel} + \theta\,\vec{\epsilon}_{\perp}\right)\right|^2$$

The integral can be expressed in terms of the Airy function

$$Ai(x) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{i(xt+\frac{1}{3}t^3)} = \frac{1}{\pi}\int_0^{\infty} dt \cos\left(xt+\frac{1}{3}t^3\right)$$

and it's derivative

$$Ai'(x) = \frac{i}{2\pi}\int_{-\infty}^{\infty} dt\, t\, e^{i(xt+\frac{1}{3}t^3)}$$

Our integral is of the form $\int_{-\infty}^{\infty} dt\, e^{i(at+\frac{1}{3}bt^3)}$

So rescale $t \to \frac{1}{b^{1/3}}t$, $\frac{1}{b^{1/3}}\int_{-\infty}^{\infty} dt\, e^{i\left(\frac{a}{b^{1/3}}t+\frac{1}{3}t^3\right)} = \frac{2\pi}{b^{1/3}} Ai\left(\frac{a}{b^{1/3}}\right)$

and $\int_{-\infty}^{\infty} dt\, t\, e^{i(at+\frac{1}{3}bt^3)} = \frac{2\pi}{b^{2/3}} Ai'\left(\frac{a}{b^{1/3}}\right)$

In our case $a = \frac{\omega}{2}\left(\frac{1}{\gamma^2}+\theta^2\right)$ and $b = \frac{\omega}{2}\frac{c^2}{\rho^2}$. Let $z = \frac{\frac{\omega}{2}\left(\frac{1}{\gamma^2}+\theta^2\right)}{\left[\frac{\omega}{2}\cdot c^2/\rho^2\right]^{1/3}}$

or $z = \left(\frac{\omega\rho}{2c}\right)^{2/3}\left(\frac{1}{\gamma^2}+\theta^2\right)$

With this

$$\frac{d^2 I}{d\Omega d\omega} = \frac{q^2\omega^2}{4\pi^2 c}\left| -\frac{c}{\rho}2\pi \frac{1}{\left(\frac{\omega}{2}\frac{c^2}{\rho^2}\right)^{2/3}} Ai'(z)\,\vec{\epsilon}_{\parallel} + \theta\frac{2\pi}{\left(\frac{\omega}{2}\frac{c^2}{\rho^2}\right)^{1/3}} Ai(z)\,\vec{\epsilon}_{\perp}\right|^2$$

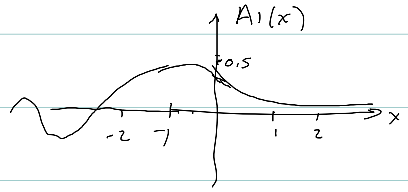
$$= \frac{q^2}{c}\left[\left(\frac{4\omega\rho}{c}\right)^{2/3}[Ai'(z)]^2 + \theta^2\left(\sqrt{2}\,\frac{\omega\rho}{c}\right)^{4/3}[Ai(z)]^2\right]$$

Note that the angular frequency of the circular motion is $\omega_0 = \frac{c}{\rho}$

so the result is given in terms of the ratio $\frac{\omega\rho}{c} = \frac{\omega}{\omega_0}$.

Garg: App. E

Now

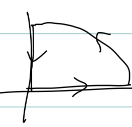


We can compute

$$Ai(0) = \frac{1}{\pi} \mathrm{Re} \int_0^\infty e^{i\frac{1}{3}t^3} dt$$. Change variables $u = \frac{1}{3}t^3$

$$\left(du = t^2 dt \Rightarrow dt = \frac{du}{(3u)^{2/3}} \right)$$

Then $$\int_0^\infty e^{i\frac{1}{3}t^3} dt = \int_0^\infty e^{iu} \frac{du}{3^{2/3} u^{2/3}}$$

Then change $u = iv$ (formally consider $\oint_C dz\, e^{iz} = 0$ ).

$$= \frac{1}{3^{2/3}} \int_0^\infty e^{-v} \frac{i\, dv}{i^{2/3} v^{2/3}} = \frac{i^{1/3}}{3^{2/3}} \int_0^\infty e^{-v} v^{\frac{1}{3}-1} dv = \frac{e^{i\pi/6}}{3^{2/3}} \Gamma(1/3)$$

So $Ai(0) = \frac{1}{\pi} \frac{\cos(\pi/6)}{3^{2/3}} \Gamma(1/3)$, or using $\Gamma(\frac{1}{3})\Gamma(\frac{2}{3}) = \frac{\pi}{\cos(\pi/6)}$

$$Ai(0) = \frac{1}{3^{2/3}\Gamma(\frac{2}{3})} \simeq 0.355$$

Similarly one finds $Ai'(0) = -\frac{1}{3^{1/3}\Gamma(1/3)} \simeq -0.259$

The large x behavior can be obtained by stationary phase:

$$\frac{d}{dt}\left(xt + \frac{1}{3}t^3\right) = 0 \Rightarrow t^2 = -x \; ; \; \frac{d^2}{dt^2}\left(xt + \frac{1}{3}t^3\right) = 2t$$

So $$xt + \frac{1}{3}t^3 = x\sqrt{-x} + \frac{1}{3}(-x)^{3/2} + \frac{1}{2}\, 2\sqrt{-x}\,(t - \sqrt{-x})^2 + \cdots$$

so that $$\int_{-\infty}^{\infty} dt\, e^{i(tx + \frac{1}{3}t^3)} \simeq \int_{-\infty}^{\infty} dt\, e^{i\left(\sqrt{-x}\,\frac{2}{3}x + \sqrt{-x}\,(t-\sqrt{-x})^2\right)}$$

Use $\sqrt{-x} = ix$ (the other solution blows up (steepest "ascent").

So

$$= e^{-\frac{2}{3}x^{3/2}} \int_{-\infty}^{\infty} dt\, e^{-\sqrt{x}(t-i\sqrt{x})^2}$$

$$= e^{-\frac{2}{3}x^{3/2}} \int_{-\infty}^{\infty} dv\, e^{-\sqrt{x}\,v^2} = \frac{e^{-\frac{2}{3}x^{3/2}}}{x^{1/4}}\, 2\int_0^{\infty} du\, e^{-u^2}$$

$\xi = u^2$

$du = \frac{d\xi}{2\sqrt{\xi}}$

$$= \frac{2e^{-\frac{2}{3}x^{3/2}}}{x^{1/4}} \int_0^{\infty} \frac{d\xi}{2\sqrt{\xi}} e^{-\xi} = \frac{e^{-\frac{2}{3}x^{3/2}}}{x^{1/4}} \Gamma(1/2) = \sqrt{\pi}\, \frac{e^{-\frac{2}{3}x^{3/2}}}{x^{1/4}}$$

And $Ai(x) \simeq \dfrac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}\, x^{1/4}}$ as $x \to +\infty$

For $Ai'(x)$, differentiate this

$$Ai'(x) \simeq -\frac{x^{1/4}}{2\sqrt{\pi}} e^{-\frac{2}{3}x^{3/2}}$$

For $x \to -\infty$ and relation to K_ν see Garg.

We can now analyze the behavior of $\frac{d^2 I}{d\Omega\, d\omega}$.

Recall

$$\frac{d^2 I}{d\Omega\, d\omega} = \frac{q^2}{c}\left[\left(\frac{4\omega}{\omega_0}\right)^{2/3}\left[A_i'(z)\right]^2 + \theta^2\left(\frac{\sqrt{2}\,\omega}{\omega_0}\right)^{4/3}\left[A_i(z)\right]^2\right]$$

where $z = \left(\frac{\omega}{2\omega_0}\right)^{2/3}\left(\frac{1}{\gamma^2}+\theta^2\right)$

For fixed θ^2, we have at small ω

$$\frac{d^2 I}{d\Omega\, d\omega} \simeq \frac{q^2}{c}\left[\left(\frac{4\omega}{\omega_0}\right)^{2/3}\left[A_i'(0)\right]^2 + \theta^2\left(\frac{\sqrt{2}\,\omega}{\omega_0}\right)^{4/3}\left[A_i(0)\right]^2\right] \simeq \frac{q^2}{c}\left[A_i'(0)\right]^2\left(\frac{4\omega}{\omega_0}\right)^{2/3}$$

and for large ω

$$\frac{d^2 I}{d\Omega\, d\omega} \simeq \frac{q^2}{c}\frac{1}{4\pi}\, e^{-\frac{2}{3}\left(\frac{1}{\gamma^2}+\theta^2\right)^{3/2}\frac{\omega}{\omega_0}}\left[\left(\frac{4\omega}{\omega_0}\right)^{2/3}\left(\frac{1}{\gamma^2}+\theta^2\right)^{1/2}\left(\frac{\sqrt{2}\,\omega}{\omega_0}\right)^{1/3} + \theta^2\left(\frac{\sqrt{2}\,\omega}{\omega_0}\right)^{\frac{4}{3}}\left(\frac{\sqrt{2}\,\omega}{\omega_0}\right)^{-1/3}\left(\frac{1}{\gamma^2}+\theta^2\right)^{-1/2}\right]$$

$$= \frac{\sqrt{2}\, q^2}{4\pi c}\, e^{-\frac{2}{3}\left(\frac{1}{\gamma^2}+\theta^2\right)^{3/2}\frac{\omega}{\omega_0}}\left(\frac{\omega}{\omega_0}\right)\left[\left(\frac{4}{\sqrt{2}}\right)^{2/3}\left(\frac{1}{\gamma^2}+\theta^2\right)^{1/2} + \frac{\theta^2}{\left(\frac{1}{\gamma^2}+\theta^2\right)^{1/2}}\right]$$

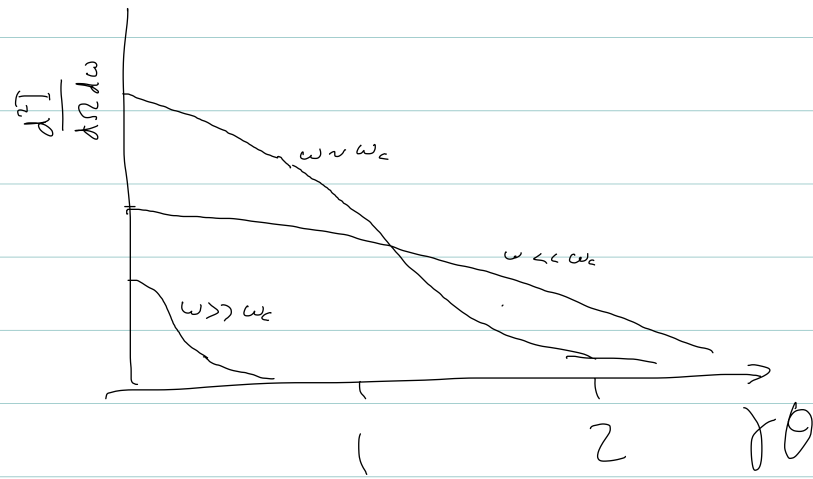
$$= \frac{3\sqrt{2}\, q^2}{2\pi c}\, e^{-2\left(1+\theta^2\gamma^2\right)^{3/2}\frac{\omega}{\omega_c}}\gamma^2\left(1+\gamma^2\theta^2\right)^{1/2}\frac{\omega}{\omega_c}\left[1+\frac{1}{2}\frac{\gamma^2\theta^2}{1+\gamma^2\theta^2}\right]$$

The exponential gives a rapid falloff $e^{-2\left(1+\theta^2\gamma^2\right)^{3/2}\frac{\omega}{\omega_c}}$

where the critical frequency $\omega_c = 3\gamma^3\omega_0$ characterizes the frequency beyond which $(\omega \gtrsim \omega_c)$ radiation is negligible even for $\theta = 0$.

Note also that for small ω the polarization is largely $\parallel$. Likewise, at $\theta=0$ only $\epsilon_{\parallel}$ contributes.

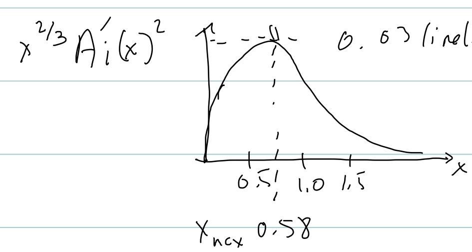
Jackson has a nice plot:



where $\omega_c \equiv 3\gamma^3\omega_0$ is defined by $z=1$ at $\theta=0$, the critical frequency beyond which there is negligible radiation for any angle.

Added: At $\theta=0$ we have $z = \left(\frac{\omega}{2\omega_0}\right)^{2/3}\frac{1}{\gamma^2} = \left(\frac{\omega}{2\gamma^3\omega_0}\right)^{2/3} = \left(\frac{3}{2}\frac{\omega}{\omega_c}\right)^{2/3}$

and
$$\left.\frac{d^2I}{d\Omega d\omega}\right|_{\theta=0} = 4\frac{q^2}{c}\gamma^2 \left(\frac{3\omega}{2\omega_c}\right)^{2/3}\left[Ai'\left(\left(\frac{3\omega}{2\omega_c}\right)^{2/3}\right)\right]^2$$



This explains the $\theta \approx 0$ behavior of curve above.

Garg - Sec. 4
Jackson - Appendix

Units and Dimensions

We have postponed discussion of units. Most of our studies have been formal. But at some point we would like to plug in numbers, compare with measurements and so on.

Two most common systems:

(i) Gaussian - CGS.

(ii) SI - MKS

CGS = cm, gram, second MKS = m, kg, sec

denote the units used for mechanical quantities.

Gaussian is more natural:

1. $\vec{E}$ and $\vec{B}$ have same dimensions
2. All units are derived from CGS.

SI is more common (volts, amperes, coulombs)

1. $\vec{E}$ and $\vec{B}$ have different dimensions (and units)
2. Introduces one new basic unit: Ampere for current

Will not discuss how units are defined. See

https://www.nist.gov/si-redefinition/definitions-si-base-units

That $\vec{E}$ & $\vec{B}$ do/don't have same dimensions in CGS vs SI implies that formulae containing them change from one system to the other.

More detail:

For mechanics, formulae have the same form in MKS & CGS, eg

$$\vec{F} = \frac{d\vec{p}}{dt} \qquad \vec{L} = \vec{r} \times \vec{p}, \qquad \vec{p} = m\vec{v}, \text{ etc.}$$

For EM, formulae depend on system of units:

Gaussian	SI
$\vec{F} = \frac{q_1 q_2}{\lvert\vec{x}\rvert^2}\hat{x}$	$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{\lvert\vec{x}\rvert^2}\hat{x}$
$u = \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2)$	$u = \frac{1}{2}(\epsilon_0 \vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2)$
$\vec{F} = q(\vec{E} + \frac{\vec{v}}{c} \times \vec{B})$	$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

$\Rightarrow$ Translating between systems requires changes in Formulas.

Denote by [ ] dimensions (as usual) with $[\vec{x}] = L$, $[m] = M$, $[t] = T$ (with units that are measured in cm-g-s in CGS or m-kg-s in MKS).

We can see what the dimensions are of each quantity in each system. In particular the new (non-mechanical) quantities q (charge), $\vec{E}$ & $\vec{B}$.

Gaussian:

From $F = \frac{q^2}{\lvert\vec{x}\rvert^2}$, $[q] = \left([F=ma][x^2]\right)^{1/2} = [M]^{1/2}[L]^{3/2}[T]^{-1} = \text{statcoulomb}$

$u = \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2) \quad [\vec{E}] = [\vec{B}] = \left[\frac{mv^2}{vol}\right]^{1/2} = [M]^{1/2}[L]^{-1/2}[T]^{-1} = \text{statvolt}\cdot\text{cm}^{-1}$

Sanity check: $F = qE \Rightarrow [M][L][T]^{-2} \stackrel{?}{=} \left([M]^{1/2}[L]^{3/2}[T]^{-1}\right)\cdot\left([M]^{1/2}[L]^{-1/2}[T]^{-1}\right)$ ✓

Other quantities trivially follow, eg $\vec{E} = -\vec{\nabla}\phi$ $(\phi = A^0)$

$\Rightarrow [\phi] = [\vec{E}][L] = [M]^{1/2}[L]^{1/2}[T]^{-1} \left(= [\text{Force}]^{1/2}\right) = \text{statvolt}$

SI

$[I]$ is a new basic dimension $\to$ Ampere

$I = \frac{dq}{dt} \Rightarrow [Q] = [I][T] \quad \to$ Coulomb

$F = \frac{1}{4\pi\epsilon_0}\frac{q^2}{|\vec{x}|^2} \to [M][L][T]^{-2} = [\epsilon_0]^{-1}[I]^2[T]^2[L]^{-2} \to [\epsilon_0] = [I]^2[T]^4[L]^{-3}[M]^{-1}$

$u = \frac{1}{8\pi}\left(\epsilon_0 \vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2\right) \to$ two relations

$$[\vec{E}] = \left([M][L]^{-1}[T]^{-2}\right)^{1/2}\left([I]^2[T]^4[L]^{-3}[M]^{-1}\right)^{-1/2} = [M][L][T]^{-3}[I]^{-1}$$

$$[\vec{B}]^2[\mu_0]^{-1} = [M][L]^{-1}[T]^{-2}$$

$F = q(\vec{E} + \vec{v}\times\vec{B}) \quad \to$ two relations

$$[\vec{B}] = [\vec{E}][\vec{v}]^{-1} = [M][T]^{-2}[I]^{-1}$$

$$[\vec{E}] = [M][L][T]^{-2}[I]^{-1}[T]^{-1} = \text{same as above} \quad ✓✓$$

From u: $[\mu_0] = [B]^2[M]^{-1}[L][T]^2 = [M][L][T]^{-2}[I]^{-2}$

Check: $[\epsilon_0][\mu_0] = [L]^{-2}[T]^2$ consistent with $c^2 = \frac{1}{\epsilon_0\mu_0}$

Exercise: Check that Maxwell's Equations in SI are dimensionally consistent.

Translating between systems

We can go from an expression (say any of Maxwell's) in one system to another if we develop a dictionary.

For example

$F = \frac{q^2}{x^2}$ (Gauss) $\quad F = \frac{1}{4\pi\epsilon_0}\frac{q^2}{x^2}$ (SI)

Since F & $\vec{x}$ are mechanical quantities, they are the same in both systems. We infer $q_G^2 = \frac{q_{SI}^2}{4\pi\epsilon_0}$ or $q_G = \frac{q_{SI}}{\sqrt{4\pi\epsilon_0}}$

And from $u = \frac{1}{8\pi}(E_G^2 + B_G^2) = \frac{1}{2}\left(\epsilon_0 \vec{E}_{SI}^2 + \frac{1}{\mu_0}\vec{B}_{SI}^2\right)$

we get $\vec{E}_G = \sqrt{4\pi\epsilon_0}\,\vec{E}_{SI}$ and $\vec{B}_G = \sqrt{\frac{4\pi}{\mu_0}}\,\vec{B}_{SI}$

As above, $q, \vec{E} \,\&\, \vec{B}$ are sufficient to obtain the rest of the dictionary

The Lorentz force gives nothing new – except if we do not know a priori that $c^2 = \frac{1}{\epsilon_0\mu_0}$. $\vec{F} = q_G\left(\vec{E}_G + \frac{\vec{v}}{c}\times\vec{B}_G\right) = q_{SI}\left(\vec{E}_{SI} + \vec{v}\times\vec{B}_{SI}\right)$

$\Rightarrow q_G\vec{E}_G = q_{SI}\vec{E}_{SI}$ consistent with the above $q_G\vec{E}_G = \left(\frac{q_{SI}}{\sqrt{4\pi\epsilon_0}}\right)\left(\sqrt{4\pi\epsilon_0}\,\vec{E}_{SI}\right)$

and $\Rightarrow \frac{1}{c}q_G\vec{B}_G = q_{SI}\vec{B}_{SI} \Rightarrow \frac{1}{c}\left(\frac{q_{SI}}{\sqrt{4\pi\epsilon_0}}\right)\left(\sqrt{\frac{4\pi}{\mu_0}}\,\vec{B}_{SI}\right) = q_{SI}\vec{B}_{SI} \Rightarrow \frac{1}{c}\frac{1}{\sqrt{\mu_0\epsilon_0}} = 1 \Rightarrow c^2 = \frac{1}{\epsilon_0\mu_0}$

We can convert any formula Gaussian $\longleftrightarrow$ SI. For example, we derived Larmor's formula in Gaussian, so

$$P = \frac{2}{3}\frac{q_G^2}{c}a^2 \quad , \; P \,\&\, a \text{ are mechanical} \rightarrow P = \frac{2}{3}\frac{q_{SI}^2}{4\pi\epsilon_0\left(\sqrt{\epsilon_0\mu_0}\right)^{-1}}a^2 = \frac{1}{6\pi}\sqrt{\frac{\mu_0}{\epsilon_0}}\,q_{SI}^2\,a^2$$

Or Maxwell's equations, eg

$$\vec{\nabla}\times\vec{E}_G + \frac{1}{c}\frac{\partial\vec{B}_G}{\partial t} = 0 \Rightarrow \vec{\nabla}\times\left(\sqrt{4\pi\epsilon_0}\,\vec{E}_{SI}\right) + \sqrt{\epsilon_0\mu_0}\,\frac{\partial}{\partial t}\sqrt{\frac{4\pi}{\mu_0}}\,\vec{B}_{SI} = 0$$

$$\Rightarrow \vec{\nabla}\times\vec{E}_{SI} + \frac{\partial}{\partial t}\vec{B}_{SI} = 0$$

Student can derive the rest in SI from Gaussian.

Exercise: Derive the Poynting vector in SI by translating from Gaussian.

<u>Numerics</u>. How to convert quantities from one system of units to the other? And what about μ_0 & ϵ_0? As physicist I like Gaussian, but the "volt"meter does not give statvolts, nor statamperes, etc.

1. Since the only new dimension is [I] is SI, we should be able to translate any amount of anything between systems once we know how to translate current → or, equivalently, charge.

Of course we also need $1m = 10^2 cm$ $1kg = 10^3 g$, $1s = 1s$.

Consider Coulomb's law. In CGS

$F = \frac{q^2}{1 \times r^2}$ means two charges of 1 esu (= 1 statcoulomb) 1 cm apart experience a force of 1 dyn

esu is <u>derived</u>, very much like dyne or Erg.

(esu and statcoulomb are used interchangeably; franklin (fr) is also sometimes used - less common)

The same two charges (1 esu) at the same distance (1cm) experience the same force (1!) in other systems. In SI the force is $1 dyn = 10^{-5}$ Newton, distance $1 cm = 10^{-2} m$ and charge is $1 esu = x C$ (C = coulomb). So we have

$$10^{-5} = \frac{1}{4\pi\epsilon_0} \frac{x^2}{(10^{-2})^2} \quad \text{or} \quad x = \sqrt{4\pi\epsilon_0 10^{-9}}$$

With $\epsilon_0 = 8.854 \times 10^{-12}$ F/m

$$x = \sqrt{4\pi \times 8.854 \times 10^{-21}} = 3.336 \times 10^{-10} = 1/(2.998 \times 10^9)$$

Usually written $1 esu = 10^{-9}/3$ C or $1C = 3 \times 10^9$ esu

but "3" is 2.998, suspiciously the same digits that appear in c = speed of light

2. ϵ_0 & μ_0 vs c

Because $[I]$ is a new dimension in SI, formulae like

$$F \propto \frac{q^2}{r^2} \quad \text{and} \quad B \propto \frac{I}{r} \quad \text{or} \quad F/\ell \propto \frac{I^2}{r}$$

require introduction of dimensionful constants. For example, we saw

$$F = \frac{1}{4\pi\epsilon_0}\frac{q^2}{r^2} \quad \text{with } [\epsilon_0] = [I]^2[T]^4[L]^{-3}[M]^{-1}$$

And, say, $B = \frac{\mu_0 I}{2\pi r}$. But also $[\mu_0][\epsilon_0] = [T]^2[L]^{-2}$ $\Rightarrow$ there is no need for two different conversion factors. This is clear just from dimensional analysis, has nothing to do with speed of light. (Had we introduced ϵ_0 & μ_0 with different coefficients in Coulomb and Ampere's laws, we would still get $\mu_0\epsilon_0 \propto 1/c^2$, but with some proportionality constant $\neq 1$).

Since only one is needed, one may define one and measure the other, or define one, measure c, and infer the other from $\mu_0\epsilon_0 = 1/c^2$.

You probably know $\mu_0 = 4\pi \times 10^{-7}$ H/m by definition. This fixes $\epsilon_0 = 1/\mu_0 c^2 = 8.854 \times 10^{-12}$ F/m. But there is a better way to write this

$$4\pi\epsilon_0 = \left(\frac{4\pi}{\mu_0}\right)\cdot\frac{1}{c^2} \quad \text{so} \quad 4\pi\epsilon_0 = 10^7/c^2 \quad \text{(in F/m if } c \text{ in m/s)}.$$

As we saw above $1\,\text{esu} = x\,\text{C}$ with $x = \sqrt{4\pi\epsilon_0 \cdot 10^{-9}}$

$$\Rightarrow x = \sqrt{\frac{10^7}{c^2}10^{-9}} = 1/10c = 1/(2.998 \times 10^9) \quad \text{as before.}$$

Exercise: Derive the conversion factors for 5 other quantities in the conversion tables 1.1 & 1.2 of Garg or table 4 of Appendix of Jackson.

Jackson 2nd Ed: Chap 17
Garg: secs 18, 39, 67

RADIATION REACTION

A charge is the source of EM field. A field produces a force on a charge. What are the effects of a field produced by a charge on the charge itself?

Start with a simple question: what is the energy in the field of a point charge? (Say the electron, which, as best we know, is a point particle).

If q is at $\vec{x}'$, then $\vec{E}(\vec{x}) = \frac{q}{|\vec{x}-\vec{x}'|^3}(\vec{x}-\vec{x}')$

(For continuous distribution ρ, $\vec{E}(\vec{x}) = \int d^3x' \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|^3}(\vec{x}-\vec{x}')$)

Then $u = \frac{1}{8\pi}\vec{E}^2$ ($+\vec{B}^2$, but $\vec{B}=0$ for stationary charge, assumed here for simplicity). Then

$$\mathcal{E}\,(\text{energy}) = \int d^3x \frac{1}{8\pi}\vec{E}^2 = \frac{q^2}{8\pi}\int d^3x \frac{1}{|\vec{x}-\vec{x}'|^4} = \infty$$

the divergence arising from the region $\vec{x} = \vec{x}'$.

Is this a problem? An additive but constant (infinite) energy does not affect dynamics — which depends on energy differences. BUT an <u>accelerated</u> charge looses (radiates) energy, so it does matter.

If instead we replace point charge by smeared distribution of charge, we get a finite self-energy:

For a continuum $\mathcal{E} = \frac{1}{8\pi}\int d^3x \int d^3x' d^3x'' \frac{\rho(x')\rho(x'')}{|\vec{x}-\vec{x}'|^3|\vec{x}-\vec{x}''|^3}(\vec{x}-\vec{x}')\cdot(\vec{x}-\vec{x}'')$

$$= \frac{1}{2}\int d^3x' \int d^3x'' \frac{\rho(x')\rho(x'')}{|\vec{x}'-\vec{x}''|}$$ is finite (except for $\rho(\vec{x}) \sim \delta^{(3)}(\vec{x})$).

Mathematical digression: (not for class)

We have used our knowledge from electrostatics that

$$\mathcal{E} = \frac{1}{2}\int d^3x' d^3x'' \frac{\rho(x')\rho(x'')}{|\vec{x}' - \vec{x}''|}$$

which is usually derived from $\mathcal{E} = \frac{q_1 q_2}{r}$ for two charges, so

$$\mathcal{E} = \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \frac{q_i q_j}{r_{ij}} = \frac{1}{2}\sum_{i,j=1}^{N} \frac{q_i q_j}{r_{ij}} \quad \text{(introducing infinite self energies)}$$

then replacing $\sum_i q_i \to \int d^3x' \rho(x')$.

Question: can we obtain this directly from the integral of $\frac{1}{8\pi}\int \vec{E}^2$?

Answer: yes! We need to show

$$I \equiv \int d^3x \frac{(\vec{x}-\vec{x}')\cdot(\vec{x}-\vec{x}'')}{|\vec{x}-\vec{x}'|^3\,|\vec{x}-\vec{x}''|^3} = 4\pi \cdot \frac{1}{|\vec{x}'-\vec{x}''|}$$

Writing $\vec{x}' = \vec{R} + \vec{r}$, $\vec{x}'' = \vec{R} - \vec{r}$ so $\vec{r} = \frac{1}{2}(\vec{x}-\vec{x}')$, we have (after shifting $\vec{x} \to \vec{x} - \vec{R}$)

$$I \equiv \int d^3x \frac{x^2 - r^2}{\left((x^2+r^2)^2 - 4x^2r^2\cos^2\theta\right)^{3/2}} = 2\pi\int_0^\infty x^2 dx \int_{-1}^{1} d\xi \frac{x^2 - r^2}{\left[(x^2+r^2)^2 - 4x^2r^2\xi^2\right]^{3/2}}$$

Now $\frac{d}{d\xi}\frac{1}{a}\frac{\xi}{\sqrt{a-b\xi^2}} = \frac{1}{(a-b\xi^2)^{3/2}}$ so

$$I = 4\pi\int_0^\infty dx \frac{x^2(x^2-r^2)}{(x^2+r^2)^2}\cdot\frac{1}{|x^2-r^2|} = \frac{4\pi}{r}\left[\int_1^\infty dx \frac{x^2}{(x^2+1)^2} - \int_0^1 dx \frac{x^2}{(x^2+1)^2}\right] = \frac{2\pi}{r} = \frac{4\pi}{|\vec{x}'-\vec{x}''|}$$

End digression.

Generally a difficult integral. But we can gain insight by dimensional analysis. If ρ is for total charge e ($q=e$ for electron) and is for a spherical distribution with radius r_e, then we expect $E \sim \frac{e^2}{r_e}$ and this should be no larger than the rest energy $m_e c^2$. The lower bound on r_e is the classical electron radius (I use the same symbol)

$$\frac{e^2}{r_e} = m_e c^2 \quad \Rightarrow \quad r_e = \frac{e^2}{m_e c^2} \simeq 2.8 \times 10^{-13}\,\text{cm}$$

The time it takes light to traverse this is $\tau_e = \frac{r_e}{c} = \frac{e^2}{m_e c^3} \simeq 10^{-23}\,\text{s}$.

Therefore: we should not trust classical E&M at length scales shorter than r_e – or time scales shorter than τ_e. Note that this is just from internal consistency of Maxwell's theory, regardless of quantum limitations: we should not trust it either for distance scales shorter than the Compton wave-length $\lambda_e = \frac{\hbar}{m_e c}$. Note that $\frac{\lambda_e}{r_e} = \frac{\hbar c}{e^2} = 137$, the breakdown of classical E&M occurs well before the self-inconsistency at $|\vec{x}| \sim r_e$ kicks in!

If no point charges are allowed in classical E&M, what is the electron? We need a mathematical model of a charge distribution that consistently accounts for it. (See digression next page)

Alternatively redefine "bare" mass to also be infinite and precisely cancel the infinity from self-energy. This can be done consistently – the result is independent of the precise manner in which the calculation is regulated (meaning, modified to make intermediate steps finite → discussed later).

Historical digression:

Abraham & Lorentz proposed the electron's structure is purely electromagnetic. In particular that its mass/energy and momentum are completely due to the EM field. There is a charge density $\rho(\vec{x})$ localized to $\sim r_e$, and we know from above this should give an energy $m_e c^2$ for the electron.

To explore the relation between $\mathcal{E}$ & $\vec{p}$, consider electron in motion, with vel $\vec{\beta}$. In the rest frame fields are $\vec{E}_0$ and $\vec{B}_0 = 0$. Boosting to frame with vel $-\vec{\beta}$, we have $\vec{E}$ & $\vec{B}$ related by $\vec{B} = \vec{\beta} \times \vec{E}$

Exercise: check this!

$\left(\vec{B} = \gamma \vec{\beta} \times \vec{E}_0 \quad \vec{E} = \gamma \left(\vec{E}_0 - \frac{\gamma}{\gamma+1} \vec{\beta}\cdot\vec{E}_0\, \vec{\beta}\right), \gamma \vec{E}_0 = \vec{E} + \frac{\gamma^2}{\gamma+1} \vec{\beta}\cdot\vec{E}_0 \vec{\beta} \Rightarrow \vec{B} = \vec{\beta} \times (\gamma \vec{E}_0) = \vec{\beta} \times \vec{E} \right)$

Now $\mathcal{E} = \frac{1}{8\pi} \int d^3x \left(E^2 + B^2\right) = \frac{1}{8\pi} \int d^3x \left[E^2 + \left(\beta^2 E^2 - (\vec{\beta}\cdot\vec{E})^2\right)\right]$

The momentum is $\vec{p} = \int d^3x\, \vec{g} = \frac{1}{4\pi c} \int d^3x\, \vec{E} \times \vec{B} = \frac{1}{4\pi c} \int d^3x\, \vec{E} \times (\vec{\beta} \times \vec{E}) = \frac{1}{4\pi c} \int d^3x \left[\vec{\beta} E^2 - (\vec{\beta}\cdot\vec{E})\vec{E}\right]$

In the NR limit $(\beta \ll 1)$ $\vec{E} = \vec{E}_0 + \mathcal{O}(\beta^2)$ is spherically symmetric so

$$\int d^3x\, E^i E^j = \frac{1}{3} \delta^{ij} \int d^3x\, E^2.$$

Hence $\vec{p} = \frac{\vec{\beta}}{4\pi c} \int d^3x\, E^2 \left(1 - \frac{1}{3}\right) = \frac{2}{3} \frac{\vec{\beta}}{4\pi c} \cdot (8\pi \mathcal{E}) = \frac{4}{3} \vec{\beta} \left(\mathcal{E}/c\right)$

Oops! Is the $4/3$ (which should be 1) an algebra mistake?

No! It is an inherent failure of the model.

Poincare proposed a solution, with two important ingredients:

(i) There must be additional non-EM forces holding the charge together.

Then $T^{\mu\nu} = T^{\mu\nu}_{EM} + T^{\mu\nu}_{X}$ where "X" = other mysterious force field.

(ii) Proper treatment of Lorentz covariance:

$$p^\mu = \int d^3x\, T^{0\mu}$$

is a 4-vector $\underline{\underline{\text{if}}}$ $\partial_\mu T^{\mu\nu} = 0$

He showed $T^{\mu\nu}_X$ contributes $-\frac{1}{3}$ to $\mathcal{E}/c$ so that $\vec{p} = \vec{\beta}\,\mathcal{E}$, and gives fully covariant results (this came later: Fermi, Kwal, Rohrlich, Wilson, ...).

None of this addresses successfully other issues with radiation-reaction that we turn to next.

Digression:

If $\partial_\mu T^{\mu\nu}=0$ then $p^\mu = \int d^3x\, T^{0\mu}$ is a 4-vector.

Proof: $T^{\mu\nu}(x)$ is a tensor: $T'^{\mu\nu}(x) = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta\, T^{\alpha\beta}(\Lambda^{-1}x)$.

We aim at showing that $p'^\mu = \Lambda^\mu{}_\nu p^\nu$ where p'^μ is constructed as above by an observer in a frame K', and p^μ in frame K.

$$p'^\mu = \int d^3x'\, T'^{0\mu}(x') \qquad p^\mu = \int d^3x\, T^{0\mu}(x)$$

Note that $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ does not imply $d^3x' = d^3x$

We do have $d^3x' = d^4x'\,\delta(x'^0) = d^4x\,\delta(\Lambda^0{}_\mu x^\mu)$

So $p'^\mu = \int d^4x\,\delta(\Lambda^0{}_\mu x^\mu)\,\Lambda^0{}_\alpha \Lambda^\mu{}_\beta\, T^{\alpha\beta}(x)$

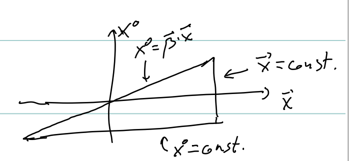
We have two issues to contend with: (i) the integral is not over d^3x at $x^0=0$ (or any constant), and (ii) we have $T^{\alpha\beta}$ rather than $T^{0\beta}$.

Now $\Lambda^0{}_\mu x^\mu = \gamma(x^0 - \vec\beta\cdot\vec x)$. So we have

$$p'^\mu = \Lambda^\mu{}_\nu \int d^3x\, \frac{1}{\gamma}\cdot\gamma\left[T^{0\nu} - \beta^i T^{i\nu}\right] \quad \text{with } T^{\mu\nu}(x) = T^{\mu\nu}(x^0 = \vec\beta\cdot\vec x, \vec x)$$

We'd like to show that this does not depend on the choice of spacelike hypersurface. Consider

$$0 = \int_V d^4x\, \partial_\mu T^{\mu\nu} = \int_{\partial V} d^3x\, n_\mu T^{\mu\nu}$$ where n_μ is the normal to the surface ∂V at element d^3x. Now, we want V to be



We assume $T\to 0$ as $|\vec x|\to\infty$ so the $|\vec x| = \text{const}\to\infty$ surface does not contribute.

The other two (spacelike) surfaces have $n_\mu = (1,0,0,0)$ and $n_\mu = \gamma(1,-\beta^i)$

so $$\int d^3x\,\frac{1}{\gamma}\,\gamma\left(T^{0\nu} - \beta^i T^{i\nu}\right)\Big|_{x^0=\vec x\cdot\vec\beta} = \int d^3x\, T^{0\nu}\Big|_{x^0=0}$$

End Digression

Going back to question on trusting classical E&M, we may ask more pointedly: under what conditions can we neglect the reactive effects of radiation?

Qualitatively: simple criterion $\mathcal{E}_{radiated} \ll \mathcal{E}$ (typical energy relevant to problem)

By the same token $\mathcal{E}_{rad} \sim \mathcal{E}$ should give us an estimate of when radiation reaction cannot be neglected.

Now $P = \frac{2}{3}\frac{q^2}{c^3}a^2$ So $\mathcal{E}_{rad} \sim \frac{q^2}{c^3}a^2\Delta t$

where Δt is time over which particle is accelerated.

For $\mathcal{E}$ we have to look case by case, and be judicious (after all $\mathcal{E}$ is defined up to additive constant; eg, we can swamp all energies in the NR case if we write $\mathcal{E} = mc^2 +$ kinetic).

Two typical cases:

(i) Accelerate particle from rest $\rightarrow$ $\mathcal{E} = \frac{1}{2}mv_{final}^2 \sim ma^2(\Delta t)^2$

RR non-neglible: $\frac{q^2}{c^3}a^2\Delta t \sim ma^2(\Delta t)^2 \Rightarrow \Delta t \sim \frac{q^2}{mc^3}$

For $q=e$ and $m=m_e$ this is $\Delta t \sim \tau_e$. And RR can be neglected for $\Delta t \gg \tau_e$

A bit of a surprise: $\tau_e = c r_e$ was introduced above as the minimum size of a charge distribution so that mass does not exceed $m_e c^2$. This was only from static fields. If we get around that somehow (by subtraction?) the scales τ_e, r_e reappear!

(ii) Circular motion:

acceleration: $\omega_0^2 \rho$ energy: $\frac{1}{2} m v^2 = \frac{1}{2} m (\omega_0 \rho)^2$

Criterial $\frac{q^2}{c^3} (\omega_0^2 \rho)^2 \Delta t \sim m(\omega_0 \rho)^2$ $\left[\frac{q^2}{mc^3}\right] \omega_0^2 \Delta t \sim 1$

For electron $\Delta t \, \tau_e \omega_0^2 \sim 1$

We cannot use such long Δt that ρ changes appreciably (then no longer circular and formulae are incorrect). So we need this to work for at least one period, or $\Delta t \sim \frac{1}{\omega_0}$. This condition is then $\tau_e \omega_0 \sim 1$ For angular frequencies $\omega_0 \gtrsim \frac{1}{\tau_e} \sim 10^{23}$ Hz RR cannot be neglected.

Note that this argument applies equally to any (quasi)-periodic motion with angular frequency ω_0 (and, irrelevant, amplitude ρ).

In both cases we see τ_e sets the relevant scale. And it is so small ($\frac{1}{\tau_e}$ so large) that for most (but not all!) practical purposes RR can be safely neglected (justifying the success of all you've learned in EM that neglects this).

Quantifying RR: baby version.

If q looses energy by radiation it must decelerate but $F=ma$ says there must be a force acting on it. Of course in order to radiate it has to accelerate so there already was an external force ($\vec{F}_{ext}$) acting on it.

$$m\ddot{\vec{x}} = \vec{F}_{ext} + \vec{F}_{RR} \qquad \text{RR = "radiative reaction"}$$

How to determine $\vec{F}_{RR}$? We'll see below a derivation, fully covariant, using retarded Green's function. Abraham and Lorentz's method is similar but (i) NR and (ii) r_e (small but non-zero) used as cut-off.

For now, cheat a little. Set

$$\begin{pmatrix}\text{energy radiated}\\ \text{in } \Delta t\end{pmatrix} = -\begin{pmatrix}\text{work done by}\\ \vec{F}_{RR} \text{ in } \Delta t\end{pmatrix}$$

$$\int_{\Delta t} P\,dt = \frac{2}{3}\frac{q^2}{c^3}\int_{\Delta t}|\ddot{\vec{x}}|^2 dt \qquad\qquad -\int_{\Delta t}\vec{F}_{RR}\cdot\dot{\vec{x}}\,dt$$

Now the LHS has $\displaystyle\int \ddot{\vec{x}}\cdot\ddot{\vec{x}}\,dt = \underbrace{\dot{\vec{x}}\cdot\ddot{\vec{x}}\Big|_{t_i}^{t_f}}_{\text{ignore}} - \int_{\Delta t}\dddot{\vec{x}}\cdot\dot{\vec{x}}\,dt$

(because $\ddot{\vec{x}}=0$ at t_i & t_f or periodic).

Then we identify $\boxed{\vec{F}_{RR} = \frac{2}{3}\frac{q^2}{c^3}\dddot{\vec{x}}}$

Since it is T-odd ($t\to -t \Rightarrow \dddot{x}\to-\dddot{x}$) we expect dissipation (much like for $\vec{F}=\gamma\dot{\vec{x}}$, air drag). Of course dissipation is precisely what we expect.

We can now write

$$m(\ddot{\vec{x}} - \tau \dddot{\vec{x}}) = \vec{F}_{ext}$$

where $\tau = \frac{2}{3}\frac{q^2}{mc^3}$ is just as before, but now including the factor of $\frac{2}{3}$ so that we do not have to carry it around.

WEIRD!

For $\vec{F}_{ext} = 0 \Rightarrow \frac{d}{dt}\ddot{x} = \frac{1}{\tau}\ddot{x} \Rightarrow \ddot{\vec{x}} = \vec{a}_0 e^{t/\tau}$

Clearly only $\vec{a}_0 = 0$ is physical. But useful lesson: when we turn on $\vec{F}_{ext}$ there will also be un-physical, as well as physical, solutions.

In preparation for that consider the case of a general time dependent $\vec{F}_{ext}(t)$. We use the method of Green functions:

$$\left(\frac{d^2}{dt^2} - \tau\frac{d^3}{dt^3}\right) G(t) = \delta(t) \Rightarrow \vec{x}(t) = \int_{-\infty}^{\infty} dt'\, G(t-t')\,\vec{F}_{ext}(t')$$

To find G consider its Fourier transform $\tilde{G}(\omega)$:

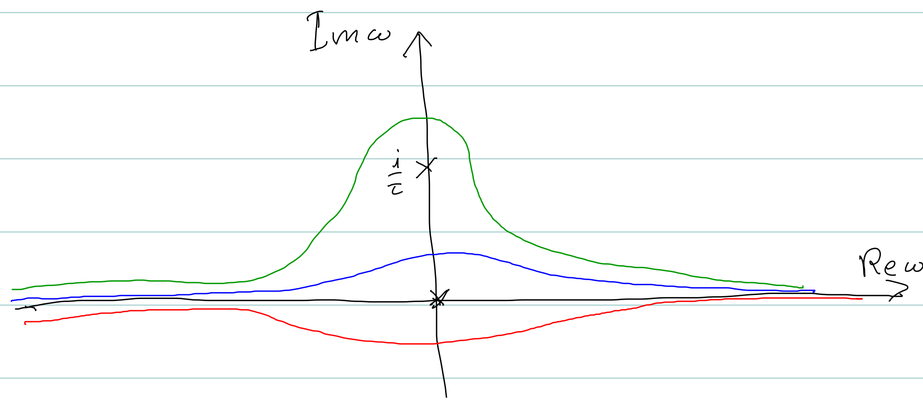
$$G(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\,\tilde{G}(\omega)$$

$$\Rightarrow \left(\frac{d^2}{dt^2} - \tau\frac{d^3}{dt^3}\right) G = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\left(-\omega^2 - i\omega^3\tau\right)\tilde{G}(\omega)$$

and set this equal to $\delta(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}$

$$\Rightarrow \tilde{G}(\omega) = -\frac{1}{\omega^2(1+i\omega\tau)} \quad \text{and} \quad G(t) = -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\,\frac{e^{-i\omega t}}{\omega^2(1+i\omega\tau)}$$

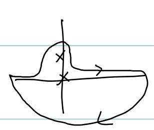
As with other Green functions we encounter poles and we need to choose a contour of integration about them. We should expect our choice will determine behavior appropriate for various boundary conditions, to which we now turn.



Poles of $\tilde{G}(\omega)$ in complex ω-plane (and possible contours of integration)

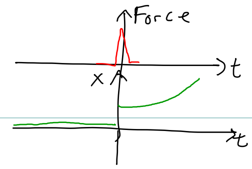
Consider each possibility:

- Top (green): Can close contour on upper half and get $G(t)=0$ provided the argument of $\exp(-i\omega t)$ has negative real part $\Leftrightarrow$ $\mathrm{Re}\left[-i\left(\mathrm{Re}(\omega)+i\,\mathrm{Im}(\omega)\right)t\right] = \mathrm{Im}(\omega)\,t$ is negative for $t<0$. This is the analogue of a retarded Green function: it is causal. For $t>0$ we get contribution from $\omega = i\tau$ and $\omega=0$.



$$\oint \frac{d\omega\, e^{-i\omega t}}{\omega^2(1+i\omega\tau)} = -2\pi i\left[i\tau e^{t/\tau} - i(t+\tau)\right] \Rightarrow G(t) = -\tau e^{t/\tau} + (t+\tau)$$

This is unphysical: an impulsive force at time $t=0$ sets the charge in accelerated motion for all $t>0$ times

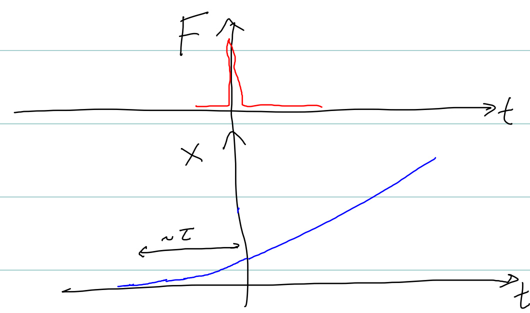


· Bottom (red): we can close the contour to get $G(t)=0$ for $t>0$. Analogous to advanced Green function for radiation. Useful only if we know that $\vec{x}=\vec{x}_{free}$ for $t>0$.

· Middle (blue): neither advanced nor retarded: for $t<0$ close contour above, and for $t>0$ close it below: in both cases pick up poles: neither advanced nor retarded:

$$G = \begin{cases} \tau e^{t/\tau} & t<0 \\ t+\tau & t>0 \end{cases}$$

In this case



The charge moves at constant velocity after hit by a hammer. But it starts moving $\tau \simeq \frac{2}{3}\frac{q^2}{mc^3}$ before contact. This is acausal behavior. This is not a concern because on those time (and distance) scales, quantum mechanical effects take over. As best we know QED (quantum electrodynamics) is fully causal - in the quantum mechanical sense.

Line Breadth and shift of oscillator.

Consider a harmonic oscillator $\vec{F}_{ext} = -m\omega_0^2 \vec{x}$

$$\Rightarrow \quad \ddot{\vec{x}} - \tau \dddot{\vec{x}} + \omega_0^2 \vec{x} = 0$$

For simplicity, do 1-dim only:

Look for solutions $A e^{-i\alpha t}$: $-\alpha^2 - i\alpha^3\tau + \omega_0^2 = 0$

There are 3 solutions. Two of them survive even if we take $\tau \to 0$ and correspond to the solutions in the absence of RR. The 3rd solution gives the unphysical exponential growth and we must discard it.

Since τ is small, solve perturbatively (in powers of $\omega_0\tau$).

$\alpha_i = \alpha_i^{(0)} + \alpha_i^{(1)} + \alpha_i^{(2)} + \cdots$ where $\alpha_i^{(n)} \sim (\omega_0\tau)^n$ and $i = +/-$ labels the sols.

Clearly $\alpha^{(0)2} = \omega_0^2 \Rightarrow \alpha_+^{(0)} = \omega_0 \quad \alpha_-^{(0)} = -\omega_0$

Then $-(\pm\omega_0 + \alpha_\pm^{(1)})^2 - i(\pm\omega_0)^3\tau + \omega_0^2 = 0 \Rightarrow \mp 2\alpha_\pm^{(1)} \mp i\omega_0^2\tau = 0 \Rightarrow \alpha_\pm^{(1)} = -\frac{i}{2}\omega_0^2\tau$

and we can write the solution to this order

$$x = A_+ e^{-i[\omega_0 - \frac{i}{2}\omega_0^2\tau]t} + A_- e^{-i[-\omega_0 - \frac{i}{2}\omega_0^2\tau]t}$$

This is a damped oscillator, with $x \propto e^{-\frac{1}{2}\Gamma t}$ $\quad \Gamma = \omega_0^2\tau$

exactly as expected from energetic considerations early on.

This charge is radiating. The field oscillates with the same frequency.

More precisely the spectral analysis of the radiation is

$$\frac{dI}{d\omega} \propto \left| \text{Fourier transform of } \vec{x}(t) \right|^2$$

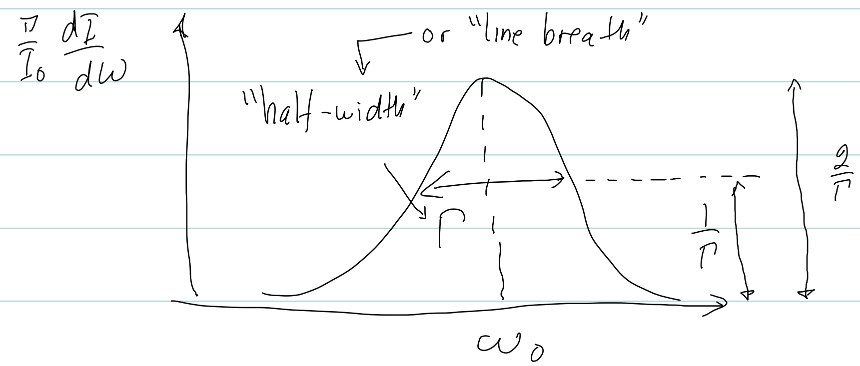
Now $$\int_0^\infty dt\, e^{i\omega t}\left[e^{-i(\omega_0-\frac{i}{2}\Gamma)t}\right] = i\,\frac{1}{\omega-\omega_0+\frac{i}{2}\Gamma}$$

where we have assumed the oscillator is excited from $t=0$ onward and neglect the process of turning on. (One can clearly do a better, cleaner job).

Then $$\frac{dI}{d\omega} \propto \left|\frac{1}{\omega-\omega_0+\frac{i}{2}\Gamma}\right|^2$$
$$= \frac{1}{(\omega-\omega_0)^2+(\Gamma/2)^2}$$

$$\left(\int_0^\infty d\omega\,\frac{1}{(\omega-\omega_0)^2+(\Gamma/2)^2} \simeq \int_{-\infty}^\infty d\omega\,\frac{1}{\omega^2+(\Gamma/2)^2} = 2\pi i\cdot\frac{1}{i\Gamma} = \frac{2\pi}{\Gamma} \right)$$ done by contour integration

or $$\frac{dI}{d\omega} = \frac{1}{\pi}I_0\left[\frac{\Gamma/2}{(\omega-\omega_0)^2+(\Gamma/2)^2}\right] \quad \text{with} \int\frac{dI}{d\omega}d\omega = I_0$$



If the spectrum is given as function of wavelength, $\omega\lambda = 2\pi c$ then a small interval in λ is $\delta\lambda \simeq \frac{2\pi c}{\omega^2}\delta\omega$. Since the half-width is small, $\Gamma/\omega_0 = \omega_0\tau \ll 1$ (by assumption), the half-width in λ-space is $\Delta\lambda = c\frac{2\pi}{\omega_0^2}\Gamma = 2\pi c\tau$; for the lectron $c\tau_e = r_e$ the classical electron radius (up to $\times\frac{2}{3}$), independent of λ.

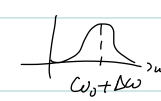
Final comments:

- Quantum mechanics: for atomic transitions photons are not monochromatic. Same phenomenon. QM line widths depend on strength of transition, and are related to partial lifetimes of levels $T_{1/2}^{-1} = \sum_{\substack{i \\ \text{levels}}} \Gamma_i$

- Had we retained $\mathcal{O}(\omega_0\tau)^2$ in our solution for α (ie $\alpha^{(2)}$) we would've obtained

$$\alpha = (\omega_0 + \Delta\omega_0) - \frac{i}{2}\Gamma$$

with Γ as before and $\Delta\omega_0 = -\frac{5}{8}\omega_0^3\tau^2$

This is the line-shift. It moves the center of the curve 

But very little $\frac{\Delta\omega}{\Gamma} \approx \mathcal{O}(\omega_0\tau) << 1$. QM effects have $\Delta\omega \sim \Gamma$.
In atoms this is called a Lamb shift (Lamb 1st observed).

$\frac{\Delta\omega_0}{\omega_0} \sim (\omega_0\tau)\ln\frac{m_e c^2}{\hbar\omega_0}$ vs classical $\frac{\Delta\omega_0}{\omega_0} \sim (\omega_0\tau)^2$

Coleman, Acausality
Barut, Chaps V, VI

Self-Field of Electron & the Force of Radiation Reaction

I believe this amounts to a modern (and relativistic) version of Abraham-Lorentz.

Rather than infering the self-force on the electron (due to its radiation field) by matching its rate of kinetic energy loss to the power radiated, which only gives us a time average so the inference of $\vec{F}_{RR}$ ("RR" = radiation reaction) is not completely justified, can we obtain $\vec{F}_{RR}$ directly?

The program should be clear:

(i) Compute $A_\mu \to F_{\mu\nu}$ due to electron

(ii) Compute $\vec{F}_{RR}$ due to $F_{\mu\nu}$ $\to$ give motion of electrons

Of course, there is no ordering here (which is first, the chicken of the egg? Ans: both/neither). These are simultaneous equations. As often done with simultaneous equations, solve for one variable in terms of the other, then plug into second equation.

To keep the aim clear, let's list expectations:

* The static component of self-force should give an infinite self-energy (ie, mass). We should regulate this (ie, cut-off the integral near $\vec{x} = \vec{x}_{electron}$), then subtract it using a "bare" mass (i.e, a contribution to the energy which is not of electromagnetic origin). Call this m_0.
* The radiation field should give rise to a force responsible for energy loss: it should be T-odd (dissipation! think air drag $\vec{F} \propto \vec{v}$) and we expect $\vec{F}_{RR} \propto \dddot{\vec{x}}$.

The two equations are

- Field due to point charge (electron):

$$A_\mu(x) = 4\pi q \int d\lambda\, u_\mu\, G_{ret}(x - y(\lambda))$$

(Notes from PHYS 203A, p.1 of chapter 4 "Fields of Moving charges")

and

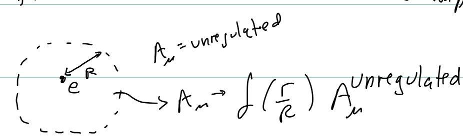
- Equation of motion

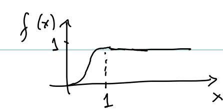
$$\frac{dp_\alpha}{d\lambda} = \frac{q}{c} F_{\alpha\beta} U^\beta$$

(PHYS 203A, chap 2., p.6)

And we take $p^\alpha = m_0 U^\alpha$ with m_0 as explained above.

The integral giving A_μ will diverge at the electron. To get around this problem we introduce a cut-off. How it's introduced should not matter provided (i) It only affects $|\vec{x} - \vec{x}_{el}| < r_e$ and (ii) I has a paramter that removes the cutting-off in some limit. For example





Remove cut-off by $R \to 0$.

Our choice of cut-off is in wave-number space: recall

$$G(x) = -\int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot x}}{k^2}$$

(PHYS203A chap. 2, p.13 of revised notes).

The $x \to 0$ region corresponds to $k \to \infty$. So we take

$$G_\Lambda(x) = -\int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \left[\frac{1}{k^2} - \frac{1}{k^2 - \Lambda^2} \right]$$

(cut-off removed by $\Lambda \to \infty$).

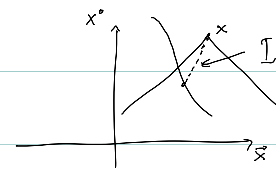
It is $k^2 - \Lambda^2$ rather than $k^2 + \Lambda^2$ so that poles are at $k^0 = \pm\sqrt{\vec{k}^2 + \Lambda^2}$, real.

We are ready to compute. We need $F_{\mu\nu}$ so take $\partial_\mu A_\nu$ above:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 4\pi q \int d\lambda\, U_\nu \partial_\mu G_\Lambda^{ret}(x-y(\lambda)) - (\mu \leftrightarrow \nu)$$

Here $U^\nu = \frac{dy^\nu}{d\lambda}$. The integral runs over $-\infty < \lambda < \lambda_0$, where λ_0 solves the retarded condition $x - y(\lambda_0) = 0$. We choose as parameter

$$\lambda = z^2 = (x-y)^2$$



This is useful because $G_\Lambda^{ret}(x)$ is a scalar function with dimensions of L^{-2}, ie, of wave-vector, so it depends on x and Λ only through the combination $\Lambda^2 x^2$ which is dimensionless, and to get dimensions right we write

$$G_\Lambda^{ret}(x-y) = \Lambda^2 f(\Lambda z)$$

for some function f. This function can be computed by performing the integral explicitly; this is difficult and not illuminating, and not necessary.

Use $\partial_\mu^{\,} G_\Lambda^{ret}(x-y) = \Lambda^2 \partial_\mu z \frac{df}{dz}$ and $2z\,\partial_\mu z = 2(x-y)^\lambda \partial_\mu (x-y)_\lambda$

Recall

$$\partial_\mu y_\lambda = \frac{(x-y)_\mu U_\lambda}{(x-y)\cdot U} \qquad \text{(PHYS703A, chap 4, p.2)}$$

so $z\,\partial_\mu z = (x-y)^\lambda \left[\frac{(x-y)_\mu U_\lambda}{(x-y)\cdot U}\right] = (x-y)_\mu$ so we have

$$\partial_\mu G_\Lambda^{ret} = \Lambda^2 \frac{(x-y)_\mu}{z} \quad \text{and} \quad F_{\mu\nu} = 4\pi q \Lambda^2 \int_0^\infty dz\, \frac{dy_\nu}{dz} \frac{(x-y)_\mu}{z} \frac{df}{dz} - (\mu \leftrightarrow \nu)$$

Integrate by parts

$$F_{\mu\nu}(x) = -4\pi q \Lambda^2 \int_0^\infty dz\, f(\Lambda z) \frac{d}{dz}\left[\frac{(x-y)_\mu}{z}\frac{dy_\nu}{dz}\right] - (\mu \leftrightarrow \nu)$$

$$= 4\pi q \Lambda^2 \int_0^\infty dz\, f(\Lambda z) \left[\frac{(x-y)_\mu}{z}\frac{d^2 y_\nu}{dz^2} - \frac{(x-y)_\mu}{z^2}\frac{dy_\nu}{dz}\right] - (\mu \leftrightarrow \nu)$$

Now, we are interested in $F_{\mu\nu}(x)$ for x = location of charge q. So at some time x^0, we want $X = y(\lambda_*)$ with λ_* determined by $x^0 = y^0(\lambda_*)$. In terms of z, λ_* is $z=0$. So $x-y = y(0)-y(z)$.

Since the divergences are associated with the field at $x = x_{\text{electron}} = y(0)$, we expand the integrand in powers of z.

Note that $\displaystyle\int_0^\infty dz\, f(\Lambda z)\, z^n = \frac{1}{\Lambda^{n+1}} \underbrace{\int_0^\infty d\xi\, f(\xi)\, \xi^n}_{\text{some pure number}} \equiv c_n \frac{1}{\Lambda^{n+1}}$

So only a finite number of terms need be retained: beyond some power the expansion terms vanish as $\Lambda \to \infty$. This will leave us with some divergent terms (expected, like self-energy), and some Λ-independent terms, the big pay-off of this long computation.

In fact, since there is a Λ^2 in front we need include only $n=1$ above.

So we have

$$F_{\mu\nu}(x) = 4\pi q \Lambda^2 \int_0^\infty dz\, f(\Lambda z) \left[\frac{(x-y)_\mu}{z} \frac{d^2 y_\nu}{dz^2} - \frac{(x-y)_\mu}{z^2} \frac{dy_\nu}{dz} \right] - (\mu \leftrightarrow \nu)$$

Use $y^\mu(z) = y^\mu(0) + z \left.\frac{dy^\mu}{dz}\right|_0 + \frac{z^2}{2} \left.\frac{d^2 y^\mu}{dz^2}\right|_0$ and $\frac{d^2 y_\nu}{dz^2} = \left.\frac{d^2 y_\nu}{dz^2}\right|_0 + z \left.\frac{d^3 y_\nu}{dz^3}\right|_0$

and let dots denote derivatives at current time, ie, $\dot{y}^\mu = \left.\frac{dy^\mu}{dz}\right|_0$, etc.

$$F_{\mu\nu}(y(0)) = -4\pi q \Lambda^2 \int_0^\infty dz f(z\Lambda) \left[(\dot{y}_\mu + \tfrac{1}{2} z \ddot{y}_\mu)(\ddot{y}_\nu + z \dddot{y}_\nu) - \mu \leftrightarrow \nu \right] + \mathcal{O}(1/\Lambda)$$

↳ ignore henceforth

$$= -4\pi q \left[c_0 \Lambda (\dot{y}_\mu \ddot{y}_\nu - \dot{y}_\nu \ddot{y}_\mu) + c_1 (\dot{y}_\mu \dddot{y}_\nu - \dot{y}_\nu \dddot{y}_\mu) \right]$$

Postpone determination of c_0 & c_1. Instead, we are ready to compute $\vec{F}_{RR}$

$\frac{d}{dz} m_0 U_\alpha = \frac{q}{c} F_{\alpha\beta} U^\beta$ or

$$m_0 \ddot{y}_\mu = \frac{q}{c} F_{\mu\nu} \dot{y}^\nu = -\frac{4\pi q^2}{c} \left[c_0 \Lambda (\dot{y}_\mu \ddot{y} \cdot \dot{y} - \dot{y}^2 \ddot{y}_\mu) + c_1 (\dot{y}_\mu \dddot{y} \cdot \dot{y} - \dot{y}^2 \dddot{y}_\mu) \right]$$

Now, as $z \to 0$, $\frac{dz}{ds} \to 1$, so we can interpret the derivatives as w.r.t s

so $\dot{y}^2 = 1$ and $\dot{y} \cdot \ddot{y} = 0$ (and $\dot{y} \cdot \dddot{y} + \ddot{y}^2 = 0$). So

$$\left(m_0 - \frac{4\pi q^2}{c} c_0 \Lambda\right) \ddot{y}_\mu = \frac{4\pi q^2}{c} c_1 \left(\dddot{y}_\mu + \dot{y}_\mu \ddot{y}^2\right)$$

The divergent self-energy can be combined with a divergent bare mass $m_0(\Lambda)$ to leave a finite mass, the physical electron mass $m_e = m_0 - \frac{4\pi q^2}{c} c_0 \Lambda$ (so we don't much care what c_0 is). So we have

$$m_e \ddot{y}^\mu = \frac{4\pi q^2}{c} c_1 \left(\dddot{y}^\mu + \dot{y}^\mu \ddot{y}^2 \right)$$

In the nonrelativistic limit, $\dot{\vec{y}} \ll 1$ and we recognize the NR version of $\vec{F}_{RR}$, proportional to $\dddot{\vec{y}}$. Comparing with our simplistic energy conservation-on-average argument we can read off the constant c_1:

$$\vec{F}_{RR} = \frac{2}{3}\frac{q^2}{c^3}\frac{d^3\vec{y}}{dt^3} \Rightarrow 4\pi c_1 = \frac{2}{3} \quad \left(c_1 = \frac{1}{6\pi}\right).$$ So finally

$$\boxed{m\ddot{y}^\mu = \frac{2}{3}\frac{q^2}{c}\left(\dddot{y}^\mu + \dot{y}^\mu \ddot{y}^2\right)}$$

Coleman, Acausality
Barut, Chaps V, VI

Self-Field of Electron & the Force of Radiation Reaction

I believe this amounts to a modern (and relativistic) version of Abraham-Lorentz.

Rather than infering the self-force on the electron (due to its radiation field) by matching its rate of kinetic energy loss to the power radiated, which only gives us a time average so the inference of $\vec{F}_{RR}$ ("RR" = radiation reaction) is not completely justified, can we obtain $\vec{F}_{RR}$ directly?

The program should be clear:

(i) Compute $A_\mu \rightarrow F_{\mu\nu}$ due to electron

(ii) Compute $\vec{F}_{RR}$ due to $F_{\mu\nu}$ → give motion of electrons

Of course, there is no ordering here (which is first, the chicken of the egg? Ans: both/neither). These are simultaneous equations. As often done with simultaneous equations, solve for one variable in terms of the other, then plug into second equation.

To keep the aim clear, let's list expectations:

* The static component of self-force should give an infinite self-energy (ie, mass). We should regulate this (ie, cut-off the integral near $\vec{x} = \vec{x}_{electron}$), then subtract it using a "bare" mass (i.e, a contribution to the energy which is not of electromagnetic origin). Call this m_0.
* The radiation field should give rise to a force responsible for energy loss: it should be T-odd (dissipation! think air drag $\vec{F} \propto \vec{v}$) and we expect $\vec{F}_{RR} \propto \dddot{\vec{x}}$.

The two equations are

- Field due to point charge (electron):

$$A_\mu(x) = 4\pi q \int d\lambda\, u_\mu\, G_{ret}(x-y(\lambda))$$

(Notes from PHYS 203A, p.1 of chapter 4 "Fields of Moving charges")

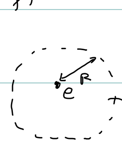
and

- Equation of motion

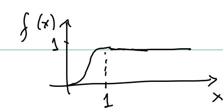
$$\frac{dp_\alpha}{d\lambda} = \frac{q}{c} F_{\alpha\beta} U^\beta \qquad \text{(PHYS 203A, chap 2., p.6)}$$

And we take $p^\alpha = m_0 U^\alpha$ with m_0 as explained above.

The integral giving A_μ will diverge at the electron. To get around this problem we introduce a cut-off. How it's introduced should not matter provided (i) It only affects $|\vec{x}-\vec{x}_{el}| < r_e$ and (ii) I has a paramter that removes the cutting-off in some limit. For example



$\to A_\mu \to f\left(\frac{r}{R}\right) A_\mu^{unregulated}$



Remove cut-off by $R \to 0$.

Our choice of cut-off is in wave-number space: recall

$$G(x) = -\int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot x}}{k^2}$$

(PHYS203A chap. 2, p.13 of revised notes).

The $x \to 0$ region corresponds to $k \to \infty$. So we take

$$G_\Lambda(x) = -\int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \left[\frac{1}{k^2} - \frac{1}{k^2-\Lambda^2} \right]$$

(cut-off removed by $\Lambda \to \infty$).

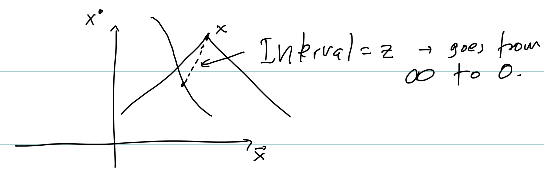
It is $k^2-\Lambda^2$ rather than $k^2+\Lambda^2$ so that poles are at $k^0 = \pm\sqrt{\vec{k}^2+\Lambda^2}$, real.

We are ready to compute. We need $F_{\mu\nu}$ so take $\partial_\mu A_\nu$ above:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 4\pi q \int d\lambda \, U_\nu \, \partial_\mu G^{ret}_\Lambda (x - y(\lambda)) \quad - (\mu \leftrightarrow \nu)$$

Here $U^\nu = \frac{dy^\nu}{d\lambda}$. The integral runs over $-\infty < \lambda < \lambda_0$, where λ_0 solves the retarded condition $x - y(\lambda_0) = 0$. We choose as parameter

$$\lambda = z^2 = (x-y)^2$$



This is useful because $G^{ret}_\Lambda(x)$ is a scalar function with dimensions of L^{-2}, ie, of wave-vector, so it depends on x and Λ only through the combination $\Lambda^2 x^2$ which is dimensionless, and to get dimensions right we write

$$G^{ret}_\Lambda (x-y) = \Lambda^2 f(\Lambda z)$$

for some function f. This function can be computed by performing the integral explicitly; this is difficult and not illuminating, and not necessary.

Use $\partial^\lambda_\mu G^{ret}_\Lambda (x-y) = \Lambda^2 \partial^\lambda_\mu z \frac{df}{dz}$ and $2z\partial^\lambda_\mu z = 2(x-y)^\lambda \partial_\mu (x-y)_\lambda$

Recall
$$\partial^\lambda_\mu y_\lambda = \frac{(x-y)_\mu U_\lambda}{(x-y)\cdot U} \qquad \text{(PHYS703A, chap 4, p. 2)}$$

so $z \partial^\lambda_\mu z = (x-y)^\lambda \left[\frac{(x-y)_\mu U_\lambda}{(x-y)\cdot U} \right] = (x-y)_\mu$ so we have

$$\partial_\mu G^{ret}_\Lambda = \Lambda^2 \frac{(x-y)_\mu}{z} \frac{df}{dz} \quad \text{and} \quad F_{\mu\nu} = 4\pi q \Lambda^2 \int_0^\infty dz \, \frac{dy_\nu}{dz} \frac{(x-y)_\mu}{z} \frac{df}{dz} \quad - (\mu \leftrightarrow \nu)$$

Integrate by parts

$$F_{\mu\nu}(x) = -4\pi q \Lambda^2 \int_0^\infty dz\, f(\Lambda z) \frac{d}{dz}\left[\frac{(x-y)_\mu}{z}\frac{dy_\nu}{dz}\right] - (\mu \leftrightarrow \nu)$$

$$= 4\pi q \Lambda^2 \int_0^\infty dz\, f(\Lambda z) \left[\frac{(x-y)_\mu}{z}\frac{d^2 y_\nu}{dz^2} - \frac{(x-y)_\mu}{z^2}\frac{dy_\nu}{dz}\right] - (\mu \leftrightarrow \nu)$$

Now, we are interested in $F_{\mu\nu}(x)$ for x = location of charge q. So at some time x^0, we want $X = y(\lambda_*)$ with λ_* determined by $x^0 = y^0(\lambda_*)$. In terms of z, λ_* is $z=0$. So $x-y = y(0) - y(z)$.

Since the divergences are associated with the field at $x = x_{\text{electron}} = y(0)$, we expand the integrand in powers of z.

Note that $\displaystyle\int_0^\infty dz\, f(\Lambda z)\, z^n = \frac{1}{\Lambda^{n+1}} \underbrace{\int_0^\infty d\xi\, f(\xi)\, \xi^n}_{\text{some pure number}} \equiv c_n \frac{1}{\Lambda^{n+1}}$

So only a finite number of terms need be retained: beyond some power the expansion terms vanish as $\Lambda \to \infty$. This will leave us with some divergent terms (expected, like self-energy), and some Λ-independent terms, the big pay-off of this long computation.

In fact, since there is a Λ^2 in front we need include only $n=1$ above.

So we have

$$F_{\mu\nu}(x) = 4\pi q \Lambda^2 \int_0^\infty dz\, f(\Lambda z) \left[\frac{(x-y)_\mu}{z} \frac{d^2 y_\nu}{dz^2} - \frac{(x-y)_\mu}{z^2} \frac{dy_\nu}{dz} \right] - (\mu \leftrightarrow \nu)$$

Use $y^\mu(z) = y^\mu(0) + z \left.\frac{dy^\mu}{dz}\right|_0 + \frac{z^2}{2} \left.\frac{d^2 y^\mu}{dz^2}\right|_0$ and $\frac{d^2 y_\nu}{dz^2} = \left.\frac{d^2 y_\nu}{dz^2}\right|_0 + z \left.\frac{d^3 y_\nu}{dz^3}\right|_0$

and let dots denote derivatives at current time, ie, $\dot{y}^\mu = \left.\frac{dy^\mu}{dz}\right|_0$, etc.

$$F_{\mu\nu}(y(0)) = -4\pi q \Lambda^2 \int_0^\infty dz f(z\Lambda) \left[(\dot{y}_\mu + \tfrac{1}{2} z \ddot{y}_\mu)(\ddot{y}_\nu + z \dddot{y}_\nu) - \mu \leftrightarrow \nu \right] \quad + \mathcal{O}(1/\Lambda)$$

↳ ignore henceforth

$$= -4\pi q \left[c_0 \Lambda (\dot{y}_\mu \ddot{y}_\nu - \dot{y}_\nu \ddot{y}_\mu) + c_1 (\dot{y}_\mu \dddot{y}_\nu - \dot{y}_\nu \dddot{y}_\mu) \right]$$

Postpone determination of c_0 & c_1. Instead, we are ready to compute $\vec{F}_{RR}$

$$\frac{d}{dz} m_0 U_\alpha = \frac{q}{c} F_{\alpha\beta} U^\beta \quad \text{or}$$

$$m_0 \ddot{y}_\mu = \frac{q}{c} F_{\mu\nu} \dot{y}^\nu = -\frac{4\pi q^2}{c} \left[c_0 \Lambda (\dot{y}_\mu \ddot{y}\cdot\dot{y} - \dot{y}^2 \ddot{y}_\mu) + c_1 (\dot{y}_\mu \dddot{y}\cdot\dot{y} - \dot{y}^2 \dddot{y}_\mu) \right]$$

Now, as $z \to 0$, $\frac{dz}{ds} \to 1$, so we can interpret the derivatives as w.r.t s

so $\dot{y}^2 = 1$ and $\dot{y}\cdot\ddot{y} = 0$ (and $\dot{y}\cdot\dddot{y} + \ddot{y}^2 = 0$). So

$$\left(m_0 - \frac{4\pi q^2}{c} c_0 \Lambda\right) \ddot{y}_\mu = \frac{4\pi q^2}{c} c_1 \left(\dddot{y}_\mu + \dot{y}_\mu \ddot{y}^2\right)$$

The divergent self-energy can be combined with a divergent bare mass $m_0(\Lambda)$ to leave a finite mass, the physical electron mass $m_e = m_0 - \frac{4\pi q^2}{c} c_0 \Lambda$ (so we don't much care what c_0 is). So we have

$$m_e \ddot{y}^\mu = \frac{4\pi q^2}{c} c_1 \left(\dddot{y}^\mu + \dot{y}^\mu \ddot{y}^2\right)$$

In the nonrelativistic limit, $\dot{\vec{y}} \ll 1$ and we recognize the NR version of $\vec{F}_{RR}$, proportional to $\dddot{\vec{y}}$. Comparing with our simplistic energy conservation-on-average argument we can read off the constant c_1:

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$$\boxed{m\ddot{y}^\mu = \frac{2}{3}\frac{q^2}{c}\left(\dddot{y}^\mu + \dot{y}^\mu \ddot{y}^2\right)}$$

[Garg 15]

Electrostatics. Spherical Harmonics. Multipole Expansion

Electrostatics: $\frac{\partial \vec{E}}{\partial t}=0$, $\vec{B}=0$, $\vec{j}=0$ $\frac{\partial \rho}{\partial t}=0$ in Maxwell's Eqs:

$$\vec{\nabla}\times\vec{E}=0 \qquad \vec{\nabla}\cdot\vec{E}=4\pi\rho$$
$$\Downarrow$$
$$\vec{E}=-\vec{\nabla}\phi \quad (\phi=A^0,\ \vec{A}=0). \quad \Rightarrow \quad \nabla^2\phi=-4\pi\rho \quad \text{Poisson Equation}$$

We have seen the solution in terms of Green functions:

$$\phi(\vec{x})=\phi_{hom}(\vec{x})+\int G(\vec{x}-\vec{x}')\,\rho(\vec{x}')\,d^3x' \quad \text{where} \quad \nabla^2 G(\vec{x})=-4\pi\,\delta^{(3)}(\vec{x}),\quad \nabla^2\phi_{hom}=0$$

and had determined $G(\vec{x})$ by Fourier transform. We can also infer G from our knowledge of Coulombs law:

$$\phi(\vec{x})=\frac{q}{|\vec{x}|} \quad \text{is for} \quad \rho(\vec{x})=q\,\delta^{(3)}(\vec{x}) \Rightarrow \nabla^2\phi=-4\pi q\,\delta^{(3)}(\vec{x})$$

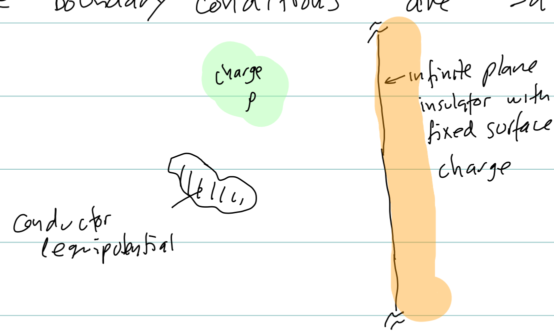
$$\Rightarrow G(\vec{x})=\frac{1}{|\vec{x}|}$$

So

$$\boxed{\phi(\vec{x})=\phi_{hom}(\vec{x})+\int\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d^3x'}$$

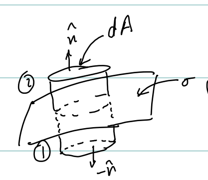
<u>Boundary value problems</u>: often concerned with region of space with boundaries on which we know something about $\vec{E}$. Then ϕ_{hom} is chosen to ensure these "boundary conditions" are satisfied.

As in



Conducting boundary: $\phi=$ constant

Surface charge density: $\Delta\vec{E}\cdot\hat{n}=4\pi\sigma$ (pillbox:



Gauss: $\int\vec{E}\cdot\hat{n}\,dA=4\pi Q$

$\Rightarrow(\vec{E}_2-\vec{E}_1)\cdot\hat{n}=4\pi\frac{Q}{dA}=4\pi\sigma$)

Regardless of the presence of charges, the central problem in electrostatics is then to solve

$$\nabla^2 \phi = 0 \quad \text{(Laplace Equation)}$$

subject to boundary conditions:

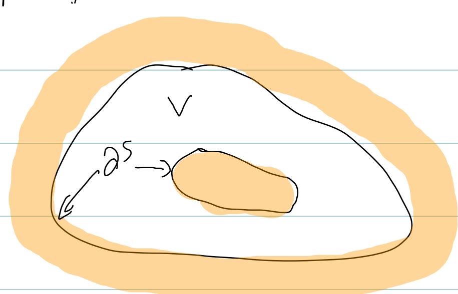
(i) ϕ specified (Dirichlet)

or (ii) $\frac{\partial \phi}{\partial n}$ specified (Neumann)

(or (iii) mixed)

Uniqueness of solution of Poisson with boundaries: [Garg 16]

If ϕ_1 & ϕ_2 are two solutions to $\nabla^2 \phi = 4\pi\rho$ with ϕ or $\frac{\partial \phi}{\partial n}$ specified on $S = \partial V$, then $\psi = \phi_2 - \phi_1$ satisfies $\nabla^2 \psi = 0$ and $\psi = 0$ or $\frac{\partial \psi}{\partial n} = 0$ on ∂V.



By Gauss's theorem

$$\int_V \vec{\nabla}\cdot(\psi \vec{\nabla}\psi)\, dV = \int_{\partial V} (\psi \vec{\nabla}\psi)\cdot \hat{n}\, dS = \int_{\partial V} \psi \frac{\partial \psi}{\partial n}\, dS$$

The RHS vanishes by assumption. The LHS

is $\int_V dV \left(\vec{\nabla}\psi \cdot \vec{\nabla}\psi + \underbrace{\psi \nabla^2 \psi}_{=0 \text{ by assumption}} \right) \Rightarrow \int_V dV \, |\vec{\nabla}\psi|^2 = 0 \Rightarrow \vec{\nabla}\psi = 0$

$\Rightarrow \psi = \text{Constant} \Rightarrow \boxed{\phi_2 = \phi_1 + \text{Constant}}$

(Note, for 2 functions ψ_1 & ψ_2

$$\int_V (\psi_1 \nabla^2 \psi_2 + \vec{\nabla}\psi_1 \cdot \vec{\nabla}\psi_2)\, dV = \int_{\partial V} \psi_1 \frac{\partial \psi_2}{\partial n}\, dS$$ is "Green's 1st identity"

subtracting $1 \leftrightarrow 2$ $\Rightarrow$ $\int (\psi_1 \nabla^2 \psi_2 - \psi_2 \nabla^2 \psi_1)\, dV = \int_{\partial V} \left(\psi_1 \frac{\partial \psi_2}{\partial n} - \psi_2 \frac{\partial \psi_1}{\partial n}\right) dS$ is "Green's 2nd identity" or Green's theorem)

Solving Laplace (PDE's): separation of variables

Cartesian: $\phi(x,y,z) = X(x)\,Y(y)\,Z(z)$

$$\frac{1}{\phi}\nabla^2\phi = \frac{1}{X}X''(x) + \frac{1}{Y}Y''(y) + \frac{1}{Z}Z''(z) = 0$$

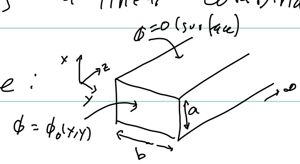
The three functions of different arguments can add up to zero only if each one is a constant:

$$\frac{1}{X}X'' = \alpha^2 \qquad \frac{1}{Y}Y'' = \beta^2 \qquad \frac{1}{Z}Z'' = \gamma^2$$

with $\alpha^2+\beta^2+\gamma^2 = 0$, $X \propto e^{\pm\alpha x}$, $Y \propto e^{\pm\beta y}$ $\quad Z = e^{\pm\gamma z}$

The boundary conditions (b.c.'s) limit values of (α,β,γ). The solution is a linear combination.

Example:



We need to specify $\phi(x,y,0) = \phi_0(x,y)$

with $\phi_0(0,y) = \phi_0(a,y) = 0 = \phi_0(x,0) = \phi_0(x,b)$

So take $\alpha^2<0$, $\beta^2<0$ above for oscillatory functions. Changing $\alpha^2\to-\alpha^2$, $\beta^2\to-\beta^2$ above

so $\gamma^2 = \alpha^2+\beta^2 \Rightarrow \phi \sim e^{\pm i\alpha x}\,e^{\pm i\beta y}\,e^{\pm\gamma z}$ $\quad \gamma = \sqrt{\alpha^2+\beta^2}$

b.c.s at $z=0$, $x=0,a$ $\Rightarrow \left(Ae^{i\alpha x}+Be^{-i\alpha x}\right)\Big|_{0,a} = 0 \Rightarrow B=-A$ and $\sin(\alpha a)=0$ $\quad \alpha = \frac{n\pi}{a}$ $\quad n=1,2,\ldots$

$$\Rightarrow \phi(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\left(A_{nm}\cosh(\gamma_{nm}z) + B_{nm}\sinh(\gamma_{nm}z)\right)$$

where $\gamma_{nm} = \pi\sqrt{\left(\frac{n}{a}\right)^2+\left(\frac{m}{b}\right)^2}$. The b.c. at $z=0$ gives

$$\phi_0(x,y) = \sum_{n,m} A_{nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)$$

With $\int_0^\pi d\xi\,\sin(n\xi)\sin(n'\xi) = \frac{\pi}{2}\delta_{nn'}$ $\quad (n>0, n'>0)$ $\Rightarrow$ $A_{nm} = \frac{4}{ab}\int_0^a dx\int_0^b dy\,\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\phi_0(x,y)$

Finally as $z\to\infty$ we do not want $|\vec{E}| \sim e^{+\gamma z} \to \infty$ choose $B_{nm} = -A_{nm}$

CLEAR THAT THE SOLUTION is MOST GENERAL, BUT IMPLEMENTING BOUNDARY CONDITIONS COMPLICATED UNLESS RECTANGULAR SYMMETRY IN PROBLEM $\longrightarrow$ consider also curvilinear coordinates

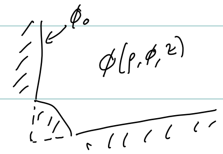
Cylindrical: $\phi(\rho,\phi,z) = R(\rho)\,\Phi(\varphi)\,Z(z)$ (use φ = angle, ϕ = potential)

$$\frac{1}{\rho R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{1}{\rho^2 \Phi}\Phi'' + \frac{1}{Z}Z'' = 0$$

$$\Rightarrow Z'' = \alpha^2 Z \quad , \quad \rho^2\left[\frac{1}{\rho R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \alpha^2\right] = -\frac{1}{\Phi}\Phi'' = \beta^2$$

$$\Rightarrow \Phi'' = -\beta^2\Phi \quad \Rightarrow \quad \left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} + \alpha^2 - \frac{\beta^2}{\rho^2}\right]R = 0$$

Φ periodic (except for "conical" configurations

eg 

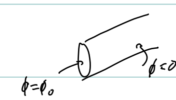
$\phi(\rho,\phi,z)$ (z out of plane).)

$$\Rightarrow \quad \beta = m \ , \ m \in \mathbb{Z} \ , \ \Phi = e^{\pm i m\varphi} .$$

$$Z = e^{\pm\alpha z}$$

R equation is Bessel's (see 203A, cavities & wave guides)

$\rightarrow R(\rho) = J_m(\alpha\rho)$. Won't review that $\rightarrow$ see last quarter notes.

(Expansion in zeroes $J_m(\xi_{mn}) = 0$, say, if $\phi(\rho=a,\varphi,0)=0$ 

$$\phi(\rho,\varphi,z) = \sum_{n,m} c_{nm} J_m\left(\xi_{mn}\frac{\rho}{a}\right) e^{im\varphi} e^{-\xi_{mn} z/a} \quad \ldots)$$

[Garg App. A]

Spherical Coordinates $\phi(r,\theta,\varphi)$

Appropriate for problems with spherical symmetry: spherical boundaries.

Recall, a scalar has $\phi'(\vec{x}') = \phi(\vec{x})$ with $\vec{x}' = R\vec{x}$ $\quad R^T R = 1$

With $R = 1 + \epsilon$, ϵ-infinitesimal $\quad R^T R = 1 \Rightarrow \epsilon^T = -\epsilon$

with $\phi'(\vec{x}) = \phi(R^{-1}\vec{x})$ we have $\delta\phi = \phi(R^{-1}\vec{x}) - \phi(\vec{x}) = \phi(\vec{x} - \epsilon\vec{x}) - \phi(\vec{x}) = -\epsilon_{ij} x_j \partial_i \phi$

$\Rightarrow$ the infinitesimal rotation is generated by $-\epsilon_{ij} x_j \partial_i = \epsilon_{ji} x_j \partial_i$

Now an antisymmetric 3x3 matrix has $\frac{3\cdot 2}{2} = 3$ independent components, so we can parametrize ϵ_{ij} as $\epsilon_{ij} = \epsilon^a \epsilon_{aij}$ $\quad \epsilon^a$, $a = 1,2,3$ the parameters (infinitesimal)

(and ϵ_{aij} = completely antisymmetric tensor with $\epsilon_{123} = +1$).

$$\Rightarrow \quad \delta\phi = \epsilon^a \epsilon_{aij} x_i \partial_j = \vec{\epsilon} \cdot (\vec{x} \times \vec{\nabla})$$

This should ring a bell! In QM $\vec{L} = \vec{r} \times \vec{p} = -i\hbar\, \vec{r} \times \vec{\nabla}$

Setting $\hbar = 1$ (because we are not doing QM) ie, $\vec{L} \equiv -i\, \vec{r} \times \vec{\nabla}$ then

$$\delta\phi = i\, \vec{\epsilon} \cdot \vec{L}\phi$$

We can use our knowledge from QM here:

$$[L_i, L_j] = i\,\epsilon_{ijk} L_k$$

(Proof: $[L_i, L_j] = (-i)^2 \epsilon_{imn} \epsilon_{jpq} [x_m \partial_n, x_p \partial_q] = (-i)^2 \epsilon_{imn} \epsilon_{jpq} (\delta_{np} x_m \partial_q - \delta_{qm} x_p \partial_n)$

$$= (-i)^2 \left((\delta_{iq}\delta_{mj} - \cancel{\delta_{ij}}\delta_{mq}) x_m \partial_q - (\delta_{ip}\delta_{nj} - \cancel{\delta_{ij}\delta_{np}}) x_p \partial_n \right)$$

$$= (-i)^2 (x_j \partial_i - x_i \partial_j)$$

$$= (-i)^2 \epsilon_{jik} \epsilon_{kmn} x_m \partial_n$$

$$= -i\, \epsilon_{jik} L_k = i\, \epsilon_{ijk} L_k \Big).$$

$L^2 \equiv \vec{L}\cdot\vec{L}$, $\quad [L^2, L_i] = 0 \qquad$ (Proof $L_j [L_j, L_i] + [L_j, L_i] L_j = \epsilon_{jik}(L_j L_k + L_k L_j) = 0$).

$L^\pm \equiv \frac{1}{\sqrt{2}}(L_1 \pm i L_2) \Rightarrow [L^+, L^-] = \frac{1}{2}\left(-i[L_1, L_2] + i[L_2, L_1]\right) = L_3$

$$[L^\pm, L_3] = \frac{1}{\sqrt{2}}\left([L_1, L_3] \pm i[L_2, L_3]\right) = \frac{1}{\sqrt{2}}(-i L_2 \mp L_1) = \mp L^\pm$$

We can simultaneously diagonalize L^2 and one of $L_{1,2,3}$, say L_3:

The eigenvectors are $Y_{\ell m}$, "spherical harmonics"

Before we find them, lets connect this to $\nabla^2\phi = 0$

Note that $L^2 = (-i)^2(\vec r \times \vec\nabla)\cdot(\vec r \times \vec\nabla) = -\epsilon_{mij}\, x_i \partial_j\, \epsilon_{mpq}\, x_i \partial_j x_p \partial_q$

$$= -(\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})\, x_i \partial_j x_p \partial_q$$

$$= -x_i \partial_j x_i \partial_j + x_i \partial_j x_j \partial_i$$

$$= -\vec x\cdot\vec\nabla - r^2\nabla^2 + 3\vec x\cdot\vec\nabla + x_i x_j \partial_j \partial_i$$

$x_j\partial_j x_i \partial_i = x_i x_j \partial_i \partial_j + \vec x\cdot\vec\nabla \Longrightarrow$

$$= (\vec x\cdot\vec\nabla)^2 + (\vec x\cdot\vec\nabla) - r^2\nabla^2$$

Now $\vec x\cdot\vec\nabla = r\frac{\partial}{\partial r}$ so $\frac{1}{r^2}L^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial r} - \nabla^2$

$$= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \nabla^2$$

$$\Rightarrow \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\vec L^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2} r - \frac{1}{r^2}L^2$$

$\Rightarrow L^2 = \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$ $\Rightarrow$ $Y_{\ell m}$'s are functions of θ & φ, $\underline{Y_{\ell m}(\theta,\varphi)}$

As we will see $L^2 Y_{\ell m} = \ell(\ell+1) Y_{\ell m}$ $\quad \ell = 1, 2, \ldots$

So $\phi = \frac{1}{r}R_\ell(r)\, Y_{\ell m}$ so $\nabla^2\phi = 0 \Rightarrow \frac{1}{R}R'' - \frac{\ell(\ell+1)}{r^2} = 0 \Rightarrow R'' - \frac{\ell(\ell+1)}{r^2}R = 0$

This is homogeneous in $r \Rightarrow R = r^\alpha$ gives $\alpha(\alpha-1) = \ell(\ell+1) \Rightarrow \alpha = \ell+1$, $\alpha = -\ell$

$$\boxed{\phi(r,\theta,\varphi) = \sum_{\ell,m} c_{\ell m}\, r^\ell\, Y_{\ell m} + d_{\ell m}\, r^{-\ell-1}\, Y_{\ell m}}$$

By the standard argument, the $Y_{\ell m}$'s form an orthonormal set. They are normalized,

$$\int d\Omega\, Y^*_{\ell' m'}(\theta,\varphi)\, Y_{\ell m}(\theta,\varphi) = \delta_{\ell\ell'}\,\delta_{mm'}$$

and form a complete basis (in the space of normalizable functions on the unit sphere), so

$$c_{\ell m} r^\ell + d_{\ell m} r^{-(\ell+1)} = \int d\Omega\, Y^*_{\ell m}(\theta,\varphi)\, \phi(r,\theta,\varphi)$$

Either determine this at two radii to solve for both $c_{\ell m}$ and $d_{\ell m}$, OR, very commonly, use some additional condition, eg, regularity at origin ($r=0 \Rightarrow d_{\ell m}=0$) or at $r=\infty$ ($\Rightarrow c_{\ell m}=0$ for $\ell>0$).

Finding eigenvalues: $L^2\psi = \lambda\psi \quad L_3\psi = \lambda'\psi$.

To streamline notation, use QM notation $L_3|\lambda'\rangle = \lambda'|\lambda'\rangle$. We really should write $|\lambda,\lambda'\rangle$ for $L^2|\lambda,\lambda'\rangle = \lambda|\lambda,\lambda'\rangle$ and $L_3|\lambda,\lambda'\rangle = \lambda'|\lambda,\lambda'\rangle$, but for now we concentrate on λ' for fixed λ, so omit λ in $|\lambda,\lambda'\rangle$.

We also need an inner product $\langle\psi|\chi\rangle = \int d\Omega\, \psi^*\chi$. Note that $\vec{L}$ is hermitian w.r.t. this inner product: the hermitian conjugate

$$\langle\chi|L^\dagger|\psi\rangle \equiv \langle\psi|L|\chi\rangle^*$$

has

$$\langle\chi|L^\dagger|\psi\rangle = \langle\chi|L|\psi\rangle$$

Note also $L_\pm^\dagger = L_\mp$.

⇛ We will show that $|\lambda'\rangle$'s come in discrete sets with $\lambda' = -l, -l+1, \ldots, l-1, l$ for some integer l.

First note: $L_3 L_\pm|\lambda'\rangle = ([L_3, L_\pm] + L_\pm L_3)|\lambda'\rangle = (\lambda' \pm 1) L_\pm|\lambda'\rangle$

$\Rightarrow$ $L_\pm|\lambda'\rangle$ is an eigenvector of L_3 with eigenvalue $(\lambda' \pm 1)$

So we have a chain $\ldots, L_-^2|\lambda'\rangle, L_-|\lambda'\rangle, |\lambda'\rangle, L_+|\lambda'\rangle, L_+^2|\lambda'\rangle, \ldots$

This will terminate if $L_+|l\rangle = 0$ for some $\lambda' = l$. Assume this.

Introduce proportionality constants into $L_+|\gamma\rangle \propto |\gamma+1\rangle$ (use "γ" because the prime in λ' is tiring).

and assume $|\gamma\rangle$ is normalized: $\langle\gamma|\gamma\rangle = 1$ (any γ)

$$L_+|\gamma\rangle = C_\gamma|\gamma+1\rangle \quad \text{and} \quad L_-|\gamma\rangle = D_\gamma|\gamma-1\rangle$$

These are not independent: $\| L_+|\gamma\rangle \|^2 = C_\gamma^2\langle\gamma+1|\gamma+1\rangle = \langle\gamma|L_-L_+|\gamma\rangle = C_\gamma D_{\gamma+1}\langle\gamma|\gamma\rangle$

$$\Rightarrow C_\gamma = D_{\gamma+1}$$

Now we use $[L_+, L_-]\,|\gamma\rangle = L_3|\gamma\rangle = \gamma|\gamma\rangle$

or $\quad C_{\gamma-1}D_\gamma - C_\gamma D_{\gamma+1} = \gamma$

$\Rightarrow \quad C_{\gamma-1}^2 - C_\gamma^2 = \gamma$

and $L_+|l\rangle = 0$ is $C_l = 0$. So we have

$$C_{l-1}^2 - 0 = l \quad \Rightarrow \quad C_{l-1}^2 = l$$

$$C_{l-2}^2 - C_{l-1}^2 = l-1 \qquad C_{l-2}^2 = 2l-1$$

$$\vdots \qquad\qquad \vdots$$

$$C_{l-(k+1)}^2 - C_{l-k}^2 = l-k \qquad C_{l-(k+1)}^2 = (k+1)l - \underbrace{(1+\dots+k)}_{\frac{1}{2}k(k+1)}$$

$$\vdots$$

$$\Rightarrow \quad C_{l-(k+1)}^2 = \tfrac{1}{2}(k+1)(2l-k)$$

This should not be negative: if we take $2l$ = integer then for $k=2l$

$$C_{-l-1} = D_{-l} = 0 \quad \Rightarrow \quad L_-|-l\rangle = 0$$

$\Rightarrow$ The set of functions is $|-l\rangle, |-l+1\rangle, \dots, |l-1\rangle, |l\rangle \Rightarrow$ call them $|m\rangle$

for l = integer or half-integer. $\quad m = -l, \dots, l$

Let's get back to $L^2 = L_1^2 + L_2^2 + L_3^2$

Note that $L_+L_- + L_-L_+ = \frac{1}{2}\left[(L_1+iL_2)(L_1-iL_2) + (L_1-iL_2)(L_1+iL_2)\right] = L_1^2 + L_2^2$

so $L^2 = L_+L_- + L_-L_+ + L_3^2$. But

$$L_+L_-|m\rangle = C_{m-1}D_m|m\rangle = C_{m-1}^2|m\rangle \qquad L_-L_+|m\rangle = C_m^2|m\rangle$$

$\Rightarrow \quad L^2|m\rangle = \left(C_{m-1}^2 + C_m^2 + m^2\right)|m\rangle$. But $C_{l-k}^2 = \frac{1}{2}k(2l+1-k) \Rightarrow C_m^2 = \frac{1}{2}(l-m)(l+1+m)$

$$C_m^2 + C_{m-1}^2 = \tfrac{1}{2}(l-m)(l+1+m) + \tfrac{1}{2}(l+1-m)(l+m) = l(l+1) - m^2 \quad \Rightarrow \quad L^2|m\rangle = l(l+1)|m\rangle$$

So all our functions have the same L^2 eigenvalue, and are fully (properly)

labeled $|l,m\rangle$, with $L^2|l,m\rangle = l(l+1)|l,m\rangle$, $L_3|l,m\rangle = m|l,m\rangle$

$l = \frac{1}{2}\mathbb{Z}_+$, $\quad m = -l, -l+1, \dots, l-1, l$

Find eigenfunctions.

$$L_\pm = \frac{1}{\sqrt{2}}(L_1 \pm i L_2) = \frac{(-i)}{\sqrt{2}}\left[(y\partial_z - z\partial_y) \pm i(z\partial_x - x\partial_z)\right]$$

$$= \frac{1}{\sqrt{2}}\left[\mp(x \pm iy)\partial_z \pm z(\partial_x \pm i\partial_y)\right]$$

Note that $(\partial_x + i\partial_y)(x+iy) = 0$, $\quad (\partial_x - i\partial_y)(x - iy) = 0$

$$\frac{1}{\sqrt{2}}(\partial_x + i\partial_y)\frac{(x-iy)}{\sqrt{2}} = 1 \qquad \frac{1}{\sqrt{2}}(\partial_x - i\partial_y)\frac{(x+iy)}{\sqrt{2}} = 1$$

So $L_+ \left(\frac{x+iy}{\sqrt{2}}\right)^{n_+}\left(\frac{x-iy}{\sqrt{2}}\right)^{n_-} z^{n_3}$ = replace $z^{n_3} \to n_3 z^{n_3-1}\left[-\left(\frac{x+iy}{\sqrt{2}}\right)\right]$

and $\left(\frac{x-iy}{\sqrt{2}}\right)^{n_-} \to n_-\left(\frac{x-iy}{\sqrt{2}}\right)^{n_--1} z$

$$= -n_3\left(\frac{x+iy}{\sqrt{2}}\right)^{n_++1}\left(\frac{x-iy}{\sqrt{2}}\right)^{n_-} z^{n_3-1} + n_-\left(\frac{x+iy}{\sqrt{2}}\right)^{n_+}\left(\frac{x-iy}{\sqrt{2}}\right)^{n_--1} z^{n_3+1}$$

Also $L_3 = i(x\partial_y - y\partial_x)$

To streamline notation, let $x_\pm = \frac{x \pm iy}{\sqrt{2}}$ and $\partial_\pm = \frac{\partial_x \mp i\partial_y}{\sqrt{2}}$

(so $\partial_\pm x_\pm = 1$, $\partial_\pm x_\mp = 0$).

Then $x = \frac{1}{\sqrt{2}}(x_+ + x_-)$ $\quad y = -\frac{i}{\sqrt{2}}(x_+ - x_-)$ $\quad \partial_x = \frac{1}{\sqrt{2}}(\partial_+ + \partial_-)$ $\quad \partial_y = \frac{i}{\sqrt{2}}(\partial_+ - \partial_-)$

so $L_3 = \frac{-i}{2}\left[i(x_+ + x_-)(\partial_+ - \partial_-) + i(x_+ - x_-)(\partial_+ + \partial_-)\right] = x_+\partial_+ - x_-\partial_-$

and in this notation $L_\pm = \mp x_\pm \partial_z \pm z\partial_\mp$ counts ± 1 for each power of $x_\pm$, 0 for z

This suggests $|\ell,\ell\rangle = N_\ell\, x_+^\ell$ with N_ℓ a normalization constant

Clearly $L_+|\ell,\ell\rangle = 0$, $L_3|\ell,\ell\rangle = \ell|\ell,\ell\rangle$

$$L^2|\ell,\ell\rangle = \left(L_+L_- + L_-L_+ + L_z^2\right) N_\ell\, x_+^\ell = N_\ell\left(L_+(-\ell z\, x_+^{\ell-1}) + 0 + \ell^2 x_+^\ell\right)$$

$$= N_\ell\left[-\ell(-x_+^\ell) + 0 + \ell^2 x_+^\ell\right] = \ell(\ell+1)\,|\ell,\ell\rangle$$

Bingo! We have $|\ell,\ell-k\rangle = N_{\ell-k}\, L_-^k\, x_+^\ell$

For example $|\ell,\ell-1\rangle = N_{\ell-1} L_- x_+^\ell = N_{\ell-1}(x_-\partial_z - z\partial_+)x_+^\ell = -\ell N_{\ell-1}\, z\, x_+^{\ell-1}$

and $|\ell,\ell-2\rangle = N_{\ell-2} L_-(-\ell z\, x_+^{\ell-1}) = N_{\ell-2}(-\ell)\left(x_- x_+^{\ell-1} - (\ell-1) z^2 x_+^{\ell-2}\right)$

In terms of θ, φ, $x_{\pm} = \frac{1}{\sqrt{2}} r \sin\theta (\cos\varphi \pm i \sin\varphi) = \frac{r}{\sqrt{2}} \sin\theta \, e^{\pm i\varphi}$

$$z = r\cos\theta$$

Since $|l,k\rangle$ defined above is proportional to r^l for all functions, we can take it out and replace $|l,k\rangle \to \frac{1}{r^l}|l,k\rangle$, a function of θ and φ only.

This gives Y_{lm}, up to normalization, $\boxed{Y_{lm}(\theta,\varphi) = \frac{1}{r^l} N_{lm} L_-^{l-m} x_+^l}$

Normalization: $\int d\Omega \, Y^*_{l'm'} Y_{lm} = \delta_{ll'} \delta_{mm'}$

Example: $l=1$. From the above $\frac{x_+}{r}$, z, $\frac{x_-}{r}$

So $Y_{1\pm1} = N_{\pm1} \sin\theta \, e^{\pm i\varphi}$ $\quad Y_{10} = N_0 \cos\theta$

$$\int d\Omega \, |Y_{1\pm1}|^2 = 2\pi |N_{1\pm1}|^2 \int_0^\pi \sin\theta d\theta \, \sin^2\theta = \frac{8\pi}{3} |N_{1\pm1}|^2 \qquad |N_{1\pm1}| = \sqrt{\frac{3}{8\pi}}$$

$$\int d\Omega \, |Y_{10}|^2 = 2\pi |N_{10}|^2 \int_0^\pi \sin\theta \, d\theta \, \cos^2\theta = \frac{4\pi}{3} |N_{10}|^2 \qquad |N_{10}| = \sqrt{\frac{3}{4\pi}}$$

The phase is by convention: $Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta$ and

get $Y_{1+1} = \frac{1}{|C_1|} L_+ Y_{10}$ $\quad Y_{1-1} = \frac{1}{|C_{-1}|} L_- Y_{10}$

$\Rightarrow \quad Y_{1\pm1} = \mp \sqrt{\frac{3}{8\pi}} \sin\theta \, e^{\pm i\varphi}$

* Note that the eigensystems with $2l$ = odd integer, i.e., $l = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ are periodic in φ with period 4π, not 2π: $e^{im(\varphi+2\pi)} \to -e^{im\varphi}$. Hence they do not play a role in solving Laplace's equation, but they do play a role in other physics (spinors).

* Note: derivation of eigenvalues only depends on commutation relations and normalizable vectors. Applies equally to $\vec{L}$ = matrices. Then $\vec{L}$ are $l \times l$ on space of $|l,m\rangle$ vectors.

This presentation emphasizes the connection to the rotation group and angular momentum. There are many other ways to introduce Y_{lm}'s and many additional developments. Here we list some facts

- $Y_{lm}(-\hat{r}) = (-1)^l \, Y_{lm}(\hat{r})$
- $Y^*_{lm}(\hat{r}) = (-1)^m \, Y_{l-m}(\hat{r})$
- Completeness: $\displaystyle\sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y^*_{lm}(\theta,\varphi) Y_{lm}(\theta',\varphi') = \delta(\cos\theta - \cos\theta')\,\delta(\varphi-\varphi')$
- Addition theorem: let $\cos\theta_{12} = \hat{r}_1 \cdot \hat{r}_2$ (and recall Y_{l0} is φ independent).
$$Y_{l0}(\theta_{12}, 0) = \sqrt{\frac{4\pi}{2l+1}} \sum_{m=-l}^{l} Y^*_{lm}(\hat{r}_1)\, Y_{lm}(\hat{r}_2)$$
- Relation to Legendre polynomials $Y_{l0}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}} \; P_l(\cos\theta)$

(and, more generally, to Associated Legendre functions
$$Y_{lm}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}} \; P_l^m(\cos\theta)\, e^{im\varphi} \quad)$$

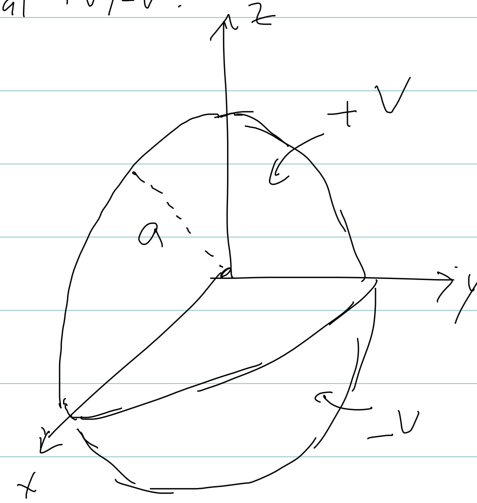
Generating function for $P_l(x)$:
$$\frac{1}{\sqrt{1+t^2-2tx}} = \sum_{l=0}^{\infty} t^l P_l(x) \qquad x \in [-1, 1]$$

- Generating function for P_l + addition theorem $\Rightarrow$
$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} \; Y^*_{lm}(\hat{r}\,')\, Y_{lm}(\hat{r})$$

where $r_< = \min(|\vec{r}|, |\vec{r}\,'|)$ and $r_> = \max(|\vec{r}|, |\vec{r}\,'|)$. I have switched notation from $\vec{x}$ to $\vec{r}$ to stay closer to textbook (Garg).

Example:

Conducting Sphere of radius a with upper/lower hemispheres at potential +V/−V.



Has azymuthal symmetry so only $m=0$ contributes to expansion of ϕ:

$$\phi(r,\theta) = \sum_{l=0}^{\infty} \left(c_{l0} r^l + d_{l0} r^{-l-1}\right) Y_{l0}$$

For ϕ inside sphere $d_{l0}=0$ (regularity at origin).

Inverting

$$c_{l0} r^l = \int d\Omega\, Y^*_{l0}\, \phi(r,\theta)$$

or, evaluating at $r=a$

$$c_{l0} = \frac{1}{a^l} 2\pi \int_{-1}^{1} d(\cos\theta) \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta) \times \begin{cases} +V & \cos\theta > 1 \\ -V & \cos\theta < 1 \end{cases}$$

$$= \frac{V}{a^l} 2\pi \sqrt{\frac{2l+1}{4\pi}} \int_0^1 dx \left[P_l(x) - P_l(-x) \right]$$

This vanishes for l = even, and we need to do the integral

$$\int_0^1 dx\, P_l(x) \quad \text{for odd } l.$$

From the generating function:

$$\int_0^1 dx \sum t^\ell P_\ell(x) = \sum_{\ell=0}^{\infty} t^\ell \int_0^1 dx\, P_\ell(x)$$

which equals

$$\int_0^1 dx \frac{1}{\sqrt{1+t^2-2tx}} = \frac{2}{(-2t)} \sqrt{1+t^2-2tx}\Big|_0^1 = \frac{1}{t}\left(\sqrt{1+t^2} - (1-t)\right)$$

Use the Taylor expansion

$$(1+x)^s = \sum_{k=0}^{\infty} \frac{1}{k!} s(s-1)\cdots(s-k+1)x^k \quad \text{with } s=\frac{1}{2},\ x=t^2$$

So

$$\frac{1}{t}\sqrt{1+t^2} - \frac{1}{t} = \sum_{k=1}^{\infty} \frac{1}{k!}\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-k+1\right) t^{2k-1}$$

and so $(\ell = 2k-1)$

$$\int_0^1 P_{2\ell-1}(x)\,dx = \frac{(-1)^{k+1}}{2^k k!} \underbrace{1\cdot 1\cdot 3 \cdots (2k-3)}_{(2k-3)!!}$$

So we have

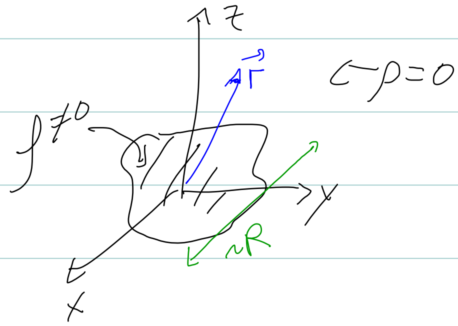
$$\phi(r,\theta) = V\sum_{\ell=1}^{\infty} \sqrt{4\pi}\sqrt{4\ell-1}\left(\frac{r}{a}\right)^{2\ell-1} \frac{(-1)^{\ell+1}}{2^\ell \ell!}(2\ell-3)!!\, Y_{2\ell-1,0}(\theta,\varphi)$$

$$= V\sum_{\ell=1}^{\infty} (4\ell-1)\left(\frac{r}{a}\right)^{2\ell-1} \frac{(-1)^{\ell+1}}{2^\ell \ell!}(2\ell-3)!!\, P_{2\ell-1}(\cos\theta)$$

$$= V\left[\frac{3}{2}\frac{r}{a}P_1(\cos\theta) - \frac{7}{8}\left(\frac{r}{a}\right)^3 P_3(\cos\theta) + \frac{11}{16}\left(\frac{r}{a}\right)^5 P_5(\cos\theta)\cdots\right]$$

Multipole expansion [Garg 19]

Localized charge distribution ρ:



Interested in $\phi(\vec{r})$ outside (and far from) ρ.

$$\phi(\vec{r}) = \sum_a \frac{q_a}{|\vec{r}-\vec{r}_a|} = \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

Physical idea of multipole expansion: at $r \gg R$ we should have $\phi \approx \frac{q}{r}$, where $q = \int d^3r' \rho(\vec{r}')$. Corrections are by an expansion in D/r: $\phi \sim \frac{q}{r}\left(1 + c_1 \frac{R}{r} + c_2 \frac{R^2}{r^2} + \cdots\right)$.

- - -

Expand in powers of $|\vec{r}'|/|\vec{r}| < 1$

We can do this in one swoop by using Y_{lm} expansion of $1/|\vec{r}-\vec{r}'|$. But let's get some intuition of what this is by expanding by hand first. We'll use a Taylor series for $\vec{r}'$ about $\vec{0}$:

$$f(\vec{x}) = f(0) + x_i \partial_i f(0) + \frac{1}{2!} x_i x_j \partial_i \partial_j f(0) + \cdots$$

$$\Rightarrow \frac{1}{|\vec{r}'-\vec{r}|} = \frac{1}{r} + x_i' \partial_i' \frac{1}{|\vec{r}'-\vec{r}|}\Big|_{\vec{r}'=0} + \frac{1}{2!} x_i' x_j' \partial_i' \partial_j' \frac{1}{|\vec{r}'-\vec{r}|}\Big|_{\vec{r}'=0} + \cdots$$

$$= \frac{1}{r} + x_i' f_{1i}(\vec{r}) + x_i' x_j' f_{2ij}(\vec{r}) + \cdots$$

Stick this into

$$\int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} d^3r' = \frac{1}{r} \underbrace{\int \rho(\vec{r}') d^3r'}_{\substack{\text{mono-pole} \\ = \text{charge}}} + f_{1i}(\vec{r}) \underbrace{\int \rho(\vec{r}') x_i' d^3r'}_{\text{"dipole" moment}} + f_{2ij} \underbrace{\int \rho x_i' x_j' d^3r'}_{\text{"quadrupole" moment}}$$

More precisely:

Define $q = \int \rho(\vec{r}')d^3r'$ monopole = charge

$\vec{d} = \int \rho(\vec{r}')\,\vec{r}'\,d^3r'$ "dipole moment"

$D_{ij} = \int \rho(\vec{r}')\,(3x_i' x_j' - r'^2 \delta_{ij})$ "quadrupole moment"

The extra term $\delta_{ij} r'^2$ in the definition of D_{ij} is included so that $\delta^{ij} D_{ij} = 0$ (ie, it's traceless). This can be done freely because the coefficient $f_{2ij}(\vec{r})$ is traceless as well, as we will see. The "3" is just an arbitrary normalization in the definition of D_{ij}.

Note that under rotations,

$q \leftrightarrow$ scalar ($l=0$) $\vec{d} \leftrightarrow$ vector ($l=1$), $D_{ij} \leftrightarrow l=2$

ie, 2-index symmetric tensors transform in 1-to-1 correspondance with $|l,m\rangle$ for $l=2$. Note both have 5 (independent) components. This is why we subtract the trace in the definition of D_{ij}: a 3×3 matrix M_{ij} in general has 9 components:

		components	
Trace:	$M_{ii} \longleftrightarrow l=0$	1	sum = 9
Anti-symmetric:	$M_{ij} - M_{ji} \longleftrightarrow l=1$	3	
Symmetric-traceless	$M_{ij} + M_{ji} - \frac{2}{3}\delta_{ij} M_{kk} \longleftrightarrow l=2$	5	$(= 3\times3\ \checkmark)$

Compute: $f_{1,2}$:

$$\partial_i' f(\vec{r}'-\vec{r})\Big|_{\vec{r}'=0} = -\partial_i f(\vec{r}'-\vec{r})\Big|_{\vec{r}'=0} = -\partial_i f(-\vec{r})$$

and $\partial_i \frac{1}{|\vec{r}|} = \partial_i \frac{1}{\sqrt{x_j x_j}} = -\frac{1}{2}\frac{2x_i}{|\vec{r}|^3}$ and $\partial_i \partial_j \frac{1}{|\vec{r}|} = -\frac{\delta_{ij}}{|r|^3} + \frac{3x_i x_j}{|r|^5}$

$f_{1i} = \frac{x_i}{r^3}$ $f_{2ij} = \frac{3x_i x_j - \delta_{ij} r^2}{r^5}$ note $\delta^{ij} f_{2ij} = 0$ as advertised

So we have

$$\phi(\vec{r}) = \frac{q}{r} + \frac{\vec{d}\cdot\vec{r}}{r^3} + \frac{1}{2}\frac{x_i x_j - \frac{1}{3}\delta_{ij} r^2}{r^5} D_{ij} + \ldots$$

or

$$\boxed{\phi(\vec{r}) = \frac{q}{r} + \frac{\vec{d}\cdot\vec{n}}{r^2} + \frac{n_i n_j}{2r^3} D_{ij} + \ldots}$$

where $\vec{n} = \frac{\vec{r}}{r}$ has $n^2 = 1$.

Clearly $|\vec{d}| \lesssim qR$ $|D_{ij}| \leq qR^2$ as expected.

Systematize:

$$\phi(\vec{r}) = \int d^3_{r'} \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} = \int d^3_{r'} \rho(\vec{r}') \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{r'^l}{r^{l+1}} Y^*_{lm}(\hat{r}') Y_{lm}(\hat{r})$$

$$= \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} \sqrt{\frac{4\pi}{2l+1}} \sum_{m=-l}^{l} q_{lm} Y_{lm}(\hat{r})$$

where $q_{lm} \equiv \sqrt{\frac{4\pi}{2l+1}} \int d^3_{r'} \rho(\vec{r}') r'^l Y^*_{lm}(\hat{r}')$ "2^l-pole moment"

Note that $\phi(\vec{r})^* = \phi(\vec{r}) \Rightarrow \sum_{m=-l}^{l} q_{lm} Y_{lm}(\hat{r}) = \sum_{m=-l}^{l} q^*_{lm} Y^*_{lm}(\hat{r}) = \sum_{m=-l}^{l} q^*_{lm} (-1)^m Y_{l,-m}(\hat{r})$

$\Rightarrow q_{lm} = (-1)^m q^*_{l,-m}$ which is verified by its definition in terms of the integral over the real charge density $\rho(\vec{r}')$.

Relation to $\vec{d}$: $r' Y^*_{1\pm1}(\hat{r}') = \mp\sqrt{\frac{3}{8\pi}}(x' \pm i y')^* = \mp\sqrt{\frac{3}{8\pi}}(x' \mp i y')$

$$\Rightarrow q_{1\pm1} = \mp\sqrt{\frac{4\pi}{3}\cdot\frac{3}{8\pi}} \int d^3_{r'} \rho(\vec{r}')(x' \mp i y') = \mp\frac{1}{\sqrt{2}}(d_x \mp i d_y)$$

Similarly $q_{10} = d_z$

Exercise: show 19.25 in Garg

$$q_{20} = D_{zz} \qquad q_{2\pm1} = \mp \frac{1}{\sqrt{6}} \left(D_{xz} \mp i D_{yz} \right)$$

$$q_{2\pm2} = \frac{1}{2\sqrt{6}} \left(D_{xx} - D_{yy} \mp 2 i D_{xy} \right)$$

Assignment: Read about Earnshaw's theorem in Garg.

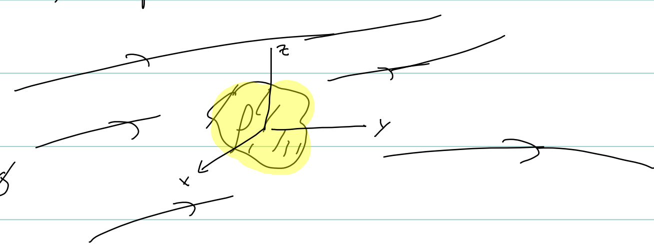
In a charge free region $\langle \phi \rangle$ over a sphere equals $\phi|_{center}$.

Charge distributions in external Fields [Garg 20]

A charge distribution characterized by $q, \vec{d}, D_{ij} \ldots$ that give its field away, in an external $\vec{E}$ field experiences forces and torques. What are they?

Take now $\phi(\vec{r})$ = potential due to external sources

$\vec{E} = -\vec{\nabla}\phi$



The energy in this configuration is (with $q_a = d^3r' \rho(\vec{r}\,')$)

$$U = \sum_a q_a \phi(\vec{r}_a) = \int d^3\vec{r}\,' \rho(\vec{r}\,')\phi(\vec{r}\,')$$

Now choose coordinate system with origin within ρ (heck, choose the same one as was used to define multi-poles): expanding $\phi(\vec{r}\,')$

$$\phi(\vec{r}\,') = \phi(0) + x_i' \partial_i \phi\Big|_0 + \frac{1}{2} x_i' x_j' \partial_i \partial_j \phi\Big|_0 + \cdots$$

so

$$U = \int d^3\vec{r}\,' \rho(\vec{r}\,') [\phi(0) + \cdots] = q\phi(0) + d_i \partial_i \phi\Big|_0 + \frac{1}{6} D_{ij} \partial_i \partial_j \phi\Big|_0 + \cdots$$

(using $\nabla^2\phi\big|_0 = 0$ since the external field is due to remote charges).

Examine result: contributions to U:

(i) Lowest: $q\phi(0)$

If we were to move the distribution to a new location, $\vec{r}$, we would have instead $q\phi(\vec{r})$. The force on this is

$$\vec{F} = -\vec{\nabla}U(\vec{r}) = q(-\vec{\nabla}\phi) = q\vec{E}(\vec{r})$$

No surprise!

(ii) 1^{st} correction: $d_i \partial_i \phi|_0 = -\vec{d}\cdot\vec{E}(0)$

As above, a rigid translation $\Rightarrow -\vec{d}\cdot\vec{E}(\vec{r})$

Force $\vec{F} = -\vec{\nabla}(-\vec{d}\cdot\vec{E}) = d_i \vec{\nabla} E_i$

or $F_j = d_i \partial_j E_i = d_i(\partial_j E_i - \partial_i E_j) + (\vec{d}\cdot\vec{\nabla})E_j$

But in static situation $\vec{\nabla}\times\vec{E} = 0 \Rightarrow \vec{F} = (\vec{d}\cdot\vec{\nabla})\vec{E}$

Even if $\partial_i E_j = 0$ ($\vec{E}$ = uniform) there is a torque

$$\vec{N} = \int \vec{r}\,' \times \left(d^3r'\rho(r')\vec{E}(\vec{r}\,')\right) = \int d^3r'\rho(\vec{r}\,')\,\vec{r}\,' \times \left(\vec{E}(0) + x_i' \partial_i \vec{E}\big|_0 + \cdots\right)$$

lowest term

$$\vec{N} = \vec{d} \times \vec{E}(0)$$

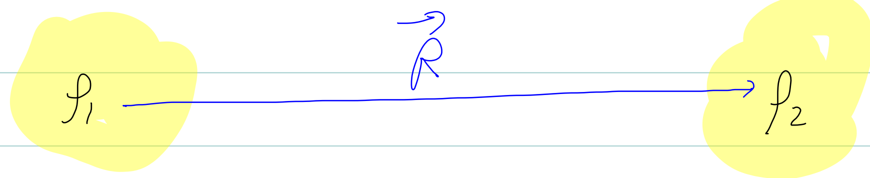
(iii) 2^{nd} corr: $\frac{1}{6} D_{ij}\,\partial_i\partial_j\phi\big|_0 = -\frac{1}{6} D_{ij}\,\partial_i E_j(0)$

$$F_k = \frac{1}{6} D_{ij}\,\partial_i\partial_k E_j$$

and $N_i = \epsilon_{ijk}\left(\int d^3r'\rho(r')\,x_j' x_m'\right)\partial_m E_k = \frac{1}{3}\epsilon_{ijk} D_{jm}\,\partial_m E_k$

where we used $\vec{\nabla}\times\vec{E} = 0$ to include the δ_{jm} term at no price.

Charge on charge:



Energy of configuration: use ρ_1 as source of external field and ρ_2 in presence of this

$$\phi = \frac{q_1}{r} + \frac{\vec{d}_1 \cdot \vec{r}}{r^3} + \frac{1}{2} D^{(1)}_{ij} \frac{r_i r_j}{r^5} + \cdots \qquad (1)$$

and

$$U = q_2 \phi(0) - \vec{d}_2 \cdot \vec{E}(0) - \frac{1}{6} D^{(2)}_{ij} \partial_i E_j(0) + \cdots \qquad (2)$$

Here "0" is in the ρ_2 distribution, so take that as $\vec{R}$ from "center" of ρ_1; set $\vec{r} = \vec{R}$ in (1). Then stick (1) into (2). Expanding in $1/R$ we have $U = U^{(0)} + U^{(1)} + U^{(2)} + \cdots$ with

$$U^{(0)} = q_2 \left[\frac{q_1}{R} + \cdots \right] = \frac{q_1 q_2}{R}$$

the potential energy of two point charges

Next

$$U^{(1)} = q_2\left[\frac{\vec{d}_1\cdot\vec{R}}{R^3}\right] + \left[-\vec{d}_2\cdot\vec{E}\right]$$

where $\vec{E} = -\vec{\nabla}\left(\frac{q_1}{r}\right)\Big|_{\vec{r}=\vec{R}}$

$$= q_1\frac{\vec{R}}{R^3}$$

$$\Rightarrow \quad U^{(1)} = q_2\frac{\vec{d}_1\cdot\vec{R}}{R^3} - q_1\frac{\vec{d}_2\cdot\vec{R}}{R^3}$$

(Looks asymmetric, but isn't: exchange $q_1 \Leftrightarrow q_2$ also exchanges $\vec{R}\leftrightarrow-\vec{R}$).

At order $1/R^3$ there are $q\,D_{ij}$ terms and $d_1 d_2$ terms.

The latter are

$$U^{(2)} = -\vec{d}_2\cdot\vec{E}$$

with $\vec{E} = -\vec{\nabla}\frac{\vec{d}_1\cdot\vec{r}}{r^3}\Big|_{\vec{R}} = \frac{3R_iR_j-\delta_{ij}R^2}{R^5}d_{1,i}$

$$\Rightarrow \quad U^{(2)} = \frac{\vec{d}_1\cdot\vec{d}_2R^2-3(\vec{R}\cdot\vec{d}_1)(\vec{R}\cdot\vec{d}_2)}{R^5}$$

Min for $\vec{d}_1\parallel\vec{d}_2\parallel\vec{R}$, max for $\vec{d}_1\parallel\vec{R}$, $\vec{d}_2\parallel(-\vec{R})$ or viceversa.

Garg Chap 10: Radiation From Localized Sources.

Private Notes

$\partial\cdot A = 0 \rightarrow \quad \partial^2 A_\mu = \frac{4\pi}{c} J_\mu \rightarrow A_\mu = \frac{1}{c}\int d^3x' \frac{J_\mu(\vec r', t_r)}{|\vec r - \vec r'|} \qquad t_r = t - \frac{|\vec r - \vec r'|}{c}$

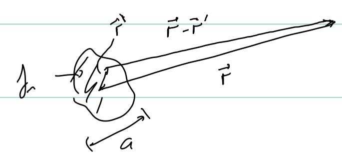
$\left(A_0 = \phi = \int d^3x' \frac{\rho(\vec r', t_r)}{|\vec r - \vec r'|} \right)$

FT's $\quad J_\mu = \int \frac{d\omega}{2\pi} e^{-i\omega t} \tilde J_\mu(\vec r, \omega) \qquad \left((c\rho_\omega(\vec r), \vec J_\omega(\vec r)) \text{ in Garg} \right).$

$= \int \frac{d^3q\, d\omega}{(2\pi)^4} e^{i(\vec q\cdot\vec r - \omega t)} \tilde J(\vec q, \omega) \qquad (J_{\vec q\omega} \text{ in Garg}).$

$\vec E = -\vec\nabla\phi - \frac{\partial \vec A}{c\,\partial t} \Rightarrow \tilde{\vec E} = -\vec\nabla\phi(\omega) + \frac{i\omega}{c}\tilde{\vec A} = -\vec\nabla\tilde\phi + ik\tilde A \qquad k \equiv \frac{\omega}{c}$,

Far



use $|\vec r - \vec r'| = r - \hat r\cdot\vec r' + O(\frac{1}{r})$

Garg $\vec r = $ "$\vec R$" $\quad \hookrightarrow = \vec R - \hat R\cdot\vec r' + O(\frac{1}{R})$

WHY DOES HE DO THIS? yikes.

Far zone approx: $a \ll R, \ \lambda \ll R$.

$\lambda = \frac{c}{\omega}$ and currents ~ qv then $v \sim a\omega$ so NR approx is $\frac{a\omega}{c} \ll 1 \Leftrightarrow a \ll \lambda$.

$$\tilde A_\mu(\omega, \vec r) = \int dt\, e^{i\omega t} A_\mu(\vec r, t) = \frac{1}{c}\int d^3x' \frac{1}{|\vec r - \vec r'|} \underbrace{\int dt\, e^{i\omega t} J_\mu(\vec r', t_r)\, e^{\frac{-i\omega|\vec r - \vec r'|}{c}}}_{\tilde J_\mu(\vec r', \omega)} \quad e^{ik|\vec r - \vec r'|}$$

$$= \frac{1}{c}\int d^3x' \frac{e^{ik|\vec r - \vec r'|}}{|\vec r - \vec r'|} \tilde J_\mu(\vec r', \omega)$$

$$\approx \frac{1}{c}\int d^3x' \frac{e^{ik(R - \hat R\cdot\vec r')}}{R} \tilde J_\mu(\vec r', \omega)$$

$$= \frac{1}{cR} e^{ikR} \tilde J_\mu(\vec k, \omega) \qquad \text{with } \vec k = k\hat R$$

(Note: LHS is $A_\mu(\vec r, \omega)$ (ie FT in time only)
while RHS is $\tilde J_\mu(\vec k, \omega)$ (ie FT in time & space)

Fields: $\vec{E}(\vec{r},\omega) = -\vec{\nabla}\phi(\omega) + ik\vec{A}(\omega)$

$$= -\vec{\nabla}\left(\frac{e^{ikR}}{R}\tilde{\rho}(k\hat{R},\omega)\right) + ik\frac{e^{ikR}}{R}\frac{1}{c}\tilde{\vec{j}}(\vec{k},\omega)$$

Since $\vec{\nabla}\frac{1}{R} \sim \frac{1}{R^2}$ and $\frac{1}{R}\vec{\nabla}\hat{R} \sim \frac{1}{R^2}$ keep only $\frac{1}{R}\vec{\nabla}e^{ikR} = \frac{e^{ikR}}{R}ik\hat{R}$

$$= ik\frac{e^{ikR}}{R}\left[-\hat{R}\rho(\vec{k},\omega) + \frac{1}{c}\vec{j}(\vec{k},\omega)\right]$$

$$\vec{B}(\vec{r},\omega) = \vec{\nabla}\times\vec{A}(\vec{r},\omega) = \frac{ik}{c}\frac{e^{ikR}}{R}\;\hat{R}\times\vec{j}(\vec{k},\omega)$$

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0 \quad\Rightarrow\quad -i\omega\,\rho(\vec{q},\omega) + i\vec{q}\cdot\vec{j}(\vec{q},\omega) = 0$$

Here is $\omega\rho(\vec{k},\omega) = k\hat{R}\cdot\vec{j}(\vec{k},\omega) = \omega\frac{1}{c}\hat{R}\cdot\vec{j}(\vec{k},\omega)$

$$\Rightarrow \vec{E}(\vec{R},\omega) = ik\frac{e^{ikR}}{cR}\underbrace{\left[\vec{j}(\vec{k},\omega) - \hat{R}\,\hat{R}\cdot\vec{j}(\vec{k},\omega)\right]}_{\equiv\,\vec{j}^{\perp}} = ik\frac{e^{ikR}}{cR}\tilde{\vec{j}}^{\perp}(\vec{k},\omega)$$

$$\vec{j} = \vec{j}^{\perp} + \vec{j}^{\parallel}$$

$$\vec{B} = ik\frac{e^{ikR}}{cR}\hat{R}\times\vec{j}^{\perp}(\vec{k},\omega) = \hat{R}\times\vec{E}(\vec{r},\omega)$$

(only used $R \gg r'$ so far, no assumption on ω (or λ) yet $\Rightarrow$ ok for relativistic sources $\Rightarrow$ apply to point particle recover old results).

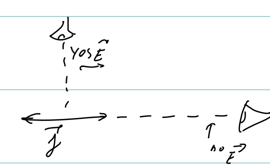
Time domain: $\vec{E}(\vec{R},t) = \int\frac{d\omega}{2\pi}e^{-i\omega t}\vec{E}(\vec{R},\omega)$

$$= \int\frac{d\omega}{2\pi}e^{-i\omega t}\,ik\frac{e^{ikR}}{cR}\int d^3r'\,e^{-ik\hat{R}\cdot\vec{r}'}\,\vec{j}^{\perp}(\vec{r}',\omega)$$

$$= \frac{1}{cR}\int d^3r'\int\frac{d\omega}{2\pi}\,ik\,e^{-i\omega(t - \frac{R}{c} + \hat{R}\cdot\vec{r}'/c)}\,\vec{j}^{\perp}(\vec{r}',\omega)$$

The factor of $ik = \frac{\omega}{c}$ is $-\frac{\partial}{\partial t}$ $\Rightarrow$ $= -\frac{1}{c^2R}\int d^3r'\,\frac{\partial}{\partial t}\vec{j}^{\perp}\left(\vec{r}', t - \frac{R}{c} + \frac{\hat{R}\cdot\vec{r}'}{c}\right)$

$$= -\frac{1}{c^2R}\int d^3r'\,\frac{\partial}{\partial t}\vec{j}^{\perp}(\vec{r}', t_r)$$



$\Rightarrow$ $\vec{E}$ is along $\vec{j}^{\perp}$, no $\vec{E}$ if $\vec{j}$ points along line of sight

$\Rightarrow$ Steady currents do not radiate ($j = qv\,\delta^{(3)}$ $\Rightarrow$ $\dot{j} = q\dot{v}\,\delta$ ↳ acceleration ✓)

Power $\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B} = \frac{c}{4\pi} \hat{n} E^2$

$$\frac{dP}{d\Omega} = R^2 \hat{n}\cdot\vec{S} = \frac{1}{4\pi c^3} \left[\frac{\partial}{\partial t} \int d^3r' \, \vec{J}^{\perp}(\vec{r}', t_r) \right]^2 =$$

Spectrum: We already have $\vec{E}(\hat{n},\omega) = \frac{ik e^{ikR}}{cR} \tilde{\vec{J}}^{\perp}(\vec{k},\omega)$

Now $$\frac{dI}{d\Omega} = \int_{-\infty}^{\infty} dt \frac{dP}{d\Omega} = \frac{c}{4\pi} R^2 \int_{-\infty}^{\infty} dt \, E^2(t) \underset{\text{Parseval's}}{=} \frac{c}{4\pi} R^2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |E(\omega)|^2 = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{c}{4\pi} \left(\frac{k^2}{c^2}\right) |\vec{J}^{\perp}(\vec{k},\omega)|^2$$
$$= 2\int_0^{\infty} \text{idem}$$

$$\Rightarrow \boxed{\frac{dI}{d\Omega\, d\omega} = \frac{\omega^2}{4\pi^2 c^3} |\tilde{\vec{J}}^{\perp}(\vec{k},\omega)|^2}$$

(I = intensity, is "$\mathcal{E}$" in Garg).

(This is for the case "Burst" in Garg, that must mean "localized in time").

Periodic: Fundamental $\omega_0 \to \frac{dP}{d\Omega} = \sum_n c_n e^{in\omega_0 t}$ (Note: ~~different~~ <u>different</u> treatment than in Garg)

$$\frac{1}{T}\int_0^T dt\, e^{-in\omega_0 t} \frac{dP}{d\Omega} = c_m \quad \left(T = \frac{2\pi}{\omega_0}\right).$$

Now, instead of $\psi(\vec{r},t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \psi(\vec{r},\omega) \xrightarrow{\text{have}} \sum_n e^{-in\omega_0 t} \psi_n(\vec{r})$

So formulas are translated ie $\vec{E}_n(\vec{R}) = i k_n \frac{e^{ik_n R}}{cR} \vec{J}_n^{\perp}(k_n \hat{n})$ $\quad k_n = \frac{n\omega_0}{c}$

And Parseval is now

$$\langle E^2 \rangle = \frac{1}{T}\int_0^T dt\, E^2(t) = \sum_{n,m} \int_0^T \frac{dt}{T} E_n E_m e^{-i(n+m)\omega_0 t} = \sum_n E_n E_{-n} = E_0^2 + \sum_{n=1}^{\infty} 2 E_n E_n^*$$

$$\Rightarrow \left\langle \frac{dP}{d\Omega} \right\rangle = \frac{cR^2}{4\pi} 2 \sum_{n=1}^{\infty} |\vec{E}_n|^2 = \frac{c}{2\pi} \sum_n \left| \frac{i k_n}{c} e^{ik_n R} \vec{J}_n^{\perp}(k_n \hat{R}) \right|^2$$

$$\text{or } \boxed{\frac{dP_n}{d\Omega} = \frac{n^2\omega_0^2}{2\pi c^3} \left|\vec{J}_n^{\perp}\left(\frac{n\omega_0}{c}\hat{R}\right)\right|^2}$$

Connect with Garg:

A Fourier exp can be written as a FT:

$$f(t) = \sum_n f_n e^{-in\omega_0 t} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \sum_n 2\pi\, \delta(\omega - n\omega_0) f_n$$

$$\Rightarrow \left\langle \frac{dP}{d\Omega} \right\rangle = \int d\omega\, e^{-i\omega t} \sum_n \frac{\omega^2}{2\pi c^3} \left|\tilde{\vec{J}}_n^{\perp}\left(\frac{\omega}{c}\hat{R}\right)\right|^2 \delta(\omega - \omega_0 n)$$

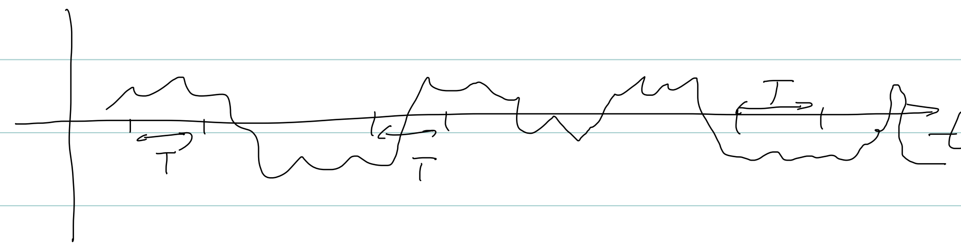
$$\Rightarrow \frac{dP}{d\Omega\, d\omega} = \frac{\omega^2}{2\pi c^3} \sum_n \left|\vec{J}_n^{\perp}\left(\frac{\omega}{c}\hat{R}\right)\right|^2 \delta(\omega - n\omega_0)$$

Garg has a $\cos\omega_0 t = \frac{1}{2} e^{i\omega_0 t} + \text{c.c.} \Rightarrow$ an extra $\left(\frac{1}{2}\right)^2 \to \frac{\omega^2}{8\pi c^3} \cdots$.

Stochastic

Underlying process (unspecified) gives rise to random movement of charges in the confined region. $\vec{J}$ becomes a random variable.

If $f(t)$ is random (assume $\langle f(t) \rangle = 0$) we can take one instance of the function $f(t)$ and look at widely separated intervals (t_1, t_1+T), (t_2, t_2+T), ... with $t_1 \ll t_2 \ll \ldots$ with T large, but $T \ll t_{n+1} - t_n$ and each segment will be a sample of $f(t)$ over an interval $(0,T)$ taken from a random distribution.



(This plot has some $\langle f \rangle \neq 0$, but if T gets larger then eventually $\langle f \rangle = 0$).

Then we can use a single function to compute correlations:

$$\langle f(t) f(t+\tau) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt\, f(t) f(t+\tau)$$

Note: I am not sure how to prove this. I'd like

$\langle f(t) f(t+\tau) \rangle = N \int [df]\, \mu(f)\, f(t) f(t+\tau)$ for some measure $\mu(f)$, say, $\mu = e^{-\int_{-\infty}^{\infty} \left(\frac{df}{dt}\right)^2 dt}$. End note.

We assume $J^\perp$ is stochastic. To obtain the spectrum, start from the above formula

$$\frac{dI}{d\Omega\, d\omega} = \frac{\omega^2}{4\pi^2 c^3} \langle |\tilde{\vec{J}}^\perp(\vec{k}, \omega)|^2 \rangle$$

where we have taken the expectation value of the stochastic variable.

Now we undo the time FT:

$$\langle |\vec{\tilde{j}}^{\perp}(\vec{k},\omega)|^2 \rangle = \left\langle \int_{-\infty}^{\infty} dt_1\, e^{i\omega t_1}\, \vec{\tilde{j}}^{\perp}(\vec{k},t_1) \cdot \left(\int_{-\infty}^{\infty} dt_2\, e^{i\omega t_2}\, \vec{\tilde{j}}^{\perp}(\vec{k},t_2) \right)^{*} \right\rangle$$

change variables $t_1 = t + \frac{1}{2}\tau \qquad t_2 = t - \frac{1}{2}\tau$

$$dt_1\, dt_2 = \left(\frac{\partial(t_1,t_2)}{\partial(t,\tau)} \right) dt\, d\tau = \begin{vmatrix} 1 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} dt\, d\tau = dt\, d\tau$$

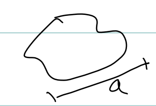
$$\ldots = \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} dt\; e^{i\omega\tau} \underbrace{\left\langle \vec{\tilde{j}}(\vec{k}, t+\tfrac{1}{2}\tau) \cdot \vec{\tilde{j}}^{*\perp}(\vec{k}, t-\tfrac{1}{2}\tau) \right\rangle}_{\equiv G^{\perp}_{jj}(\tau)}$$

So $$\int_{-\infty}^{\infty} dt\, \frac{dP}{d\Omega\, d\omega} = \int_{-\infty}^{\infty} dt \left[\frac{\omega^2}{4\pi^2 c^3} \int_{-\infty}^{\infty} d\tau\, e^{i\omega\tau}\, G^{\perp}_{jj}(\tau) \right]$$

and we take a leap of faith equating the integrands and interpreting the LHS as an instantaneous $\frac{dP}{d\Omega\, d\omega}(t)$:

$$\frac{dP}{d\Omega\, d\omega} = \frac{\omega^2}{4\pi^2 c^3} \int_{-\infty}^{\infty} d\tau\, e^{i\omega\tau}\, G^{\perp}_{jj}(\tau)$$

The long wavelength = non-relativistic = electric dipole approximation



if typical velocity is v then since motion is within size R we will have (fundamental) frequency $\omega R = v \Rightarrow$ the emitted spectrum has

$$\lambda = \frac{2\pi c}{\omega} \sim \frac{ac}{v} \Rightarrow a/\lambda \sim \beta. \text{ So } \beta \ll 1 \Leftrightarrow \lambda \gg a$$

Now $\tilde{\vec{J}}(\omega,\vec{k}) = \int d^3\vec{r}\, e^{i\vec{k}\cdot\vec{r}}\, \vec{J}(\vec{r},\omega)$

The multipole expansion is $e^{i\vec{k}\cdot\vec{r}} = 1 + \vec{k}\cdot\vec{r} + \cdots$, ie small $|\vec{k}\cdot\vec{r}| \sim \frac{R}{\lambda}$

Lowest order: $\tilde{\vec{J}}^{(0)}(\omega) = \int d^3r\, \tilde{\vec{J}}(\vec{r},\omega)$ ("0" means $(\vec{k}\cdot\vec{r})^0$).

Interpretation: with $\vec{J}(\vec{r},t) = \sum_a q_a \vec{v}_a\, \delta^{(3)}(\vec{r}-\vec{r}_a(t))$

$$\rightarrow \tilde{\vec{J}}^{(0)}(\omega) = \int dt\, d^3r\, e^{i\omega t} \sum_a q_a \dot{\vec{r}}_a\, \delta^3(\vec{r}-\vec{r}_a(t)) = \int dt\, e^{i\omega t} \sum_a q_a \dot{\vec{r}}_a = -i\omega \int dt\, e^{i\omega t} \sum_a q_a \vec{r}_a = -i\omega \int dt\, e^{i\omega t}\, \vec{d}(t)$$

or $\boxed{\tilde{\vec{J}}^{(0)}(\omega) = -i\omega\, \vec{d}(\omega)}$

Then $\vec{E}(\vec{R},\omega) = \frac{ik e^{ikR}}{cR}\, \tilde{\vec{J}}(\vec{k},\omega) \simeq k^2 \vec{d}^{\perp}(\omega)\, \frac{e^{ikR}}{R}$

(For discrete particles this is as before, from Lienard Wiechert in NR limit:

$$\vec{E}(\vec{r},t) = \int d\omega\, e^{-i\omega t}\left[\frac{\omega^2}{c^2}\frac{1}{R}\, e^{i\frac{\omega}{c}R}\, \vec{d}^{\perp}(\omega)\right] = -\frac{1}{R}\frac{1}{c^2}\frac{d^2}{dt^2}\int dt\, e^{-i\omega(t-\frac{R}{c})}\, \vec{d}^{\perp}(\omega) = -\frac{1}{Rc^2}\frac{d^2}{dt^2}\vec{d}^{\perp}(t_{ret})$$

Now $\vec{d}(t) = q\vec{r}(t) \rightarrow \vec{d}^{\perp} = -\hat{R}\times(\hat{R}\times\vec{d})$ and the correspondance follows).

Dipole Spectrum:

$$\frac{d^2 I}{d\Omega\, d\omega} = \frac{\omega^2}{4\pi^2 c^3}|\tilde{J}^{\perp}|^2 = \frac{\omega^4}{4\pi^2 c^3}|\vec{d}^{\perp}|^2 = \frac{\omega^4}{4\pi^2 c^3}|\hat{R}\times(\hat{R}\times\vec{d})|^2$$

and

$$\frac{dI}{d\omega} = \frac{\omega^4}{4\pi^2 c^3}\int d\Omega\, |\vec{d} - \hat{R}\,\hat{R}\cdot\vec{d}|^2 = \frac{\omega^4}{4\pi c^3}\, d_i d_j \int d\Omega\, (\delta_{ij} - \hat{R}_i\hat{R}_j) = \frac{\omega^4}{4\pi^2 c^3}\, d_i d_j\, 4\pi\left(\delta_{ij} - \frac{1}{3}\delta_{ij}\right)$$

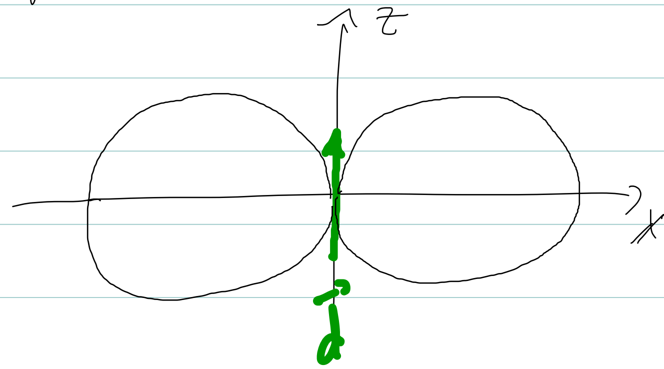
$$\Rightarrow \frac{dI}{d\omega} = \frac{2}{3\pi}\frac{\omega^4 d^2(\omega)}{c^3}$$

What kind of pattern? It depends on $\vec{d}$. Example: take $\vec{d} = \hat{z} d$,

$$\Rightarrow |\vec{d} - \hat{R}(\hat{R}\cdot\vec{d})|^2 = d^2 - (\hat{R}\cdot\vec{d})^2 = d^2(1-\cos^2\theta) = \sin^2\theta\, d^2$$

so $\displaystyle \frac{d^2 I}{d\Omega\, d\omega} = \frac{\omega^2}{4\pi^2 c^3}\, d^2 \sin^2\theta$

The radiation pattern (whereby $\frac{dI}{d\omega\, d\Omega}$ is represented as distance from origin) is



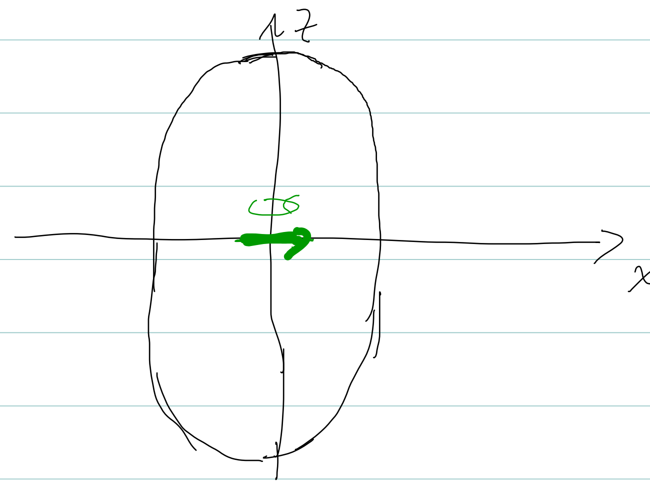
$\vec{d} \propto \hat{z}$

$l = 1$

$m = 0$

For $l=1$, $m=1$ $\quad \vec{d}(\omega) = d_\omega \frac{(\hat{x} + i\hat{y})}{\sqrt{2}}$ $\quad \hat{n}\cdot\vec{d} = \frac{d}{\sqrt{2}} \sin\theta\, (\cos\varphi + i \sin\varphi) = \frac{d}{\sqrt{2}} \sin\theta\, e^{i\varphi}$

$$|\vec{d}|^2 - |\hat{n}\cdot\vec{d}|^2 = |d_\omega|^2 \left(1 - \tfrac{1}{2}\sin^2\theta\right) = \tfrac{1}{2}|d_\omega|^2 \left(1 + \cos^2\theta\right)$$

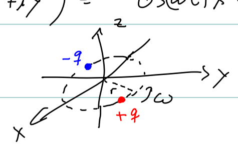


$\vec{d} \propto \hat{x} + i\hat{y}$

$l=1, m=1$

Note that $\mathrm{Re}\left(e^{-i\omega t}\, \hat{x} + i\hat{y}\right) = \cos(\omega t)\hat{x} + \sin(\omega t)\hat{y}$ $\rightarrow$ circular motion in xy plane.

Dipole:



Next order: $\vec{\tilde{J}}^{(1)}(\vec{k},\omega) = \int (-i\vec{k}\cdot\vec{r})\,\vec{J}(\vec{r},\omega)\,d^3r$

$$\tilde{J}^{(1)}_j(\vec{k},\omega) = -i\frac{\omega}{c}\hat{R}_i \underbrace{\int r_i J_j(\vec{r},\omega)\,d^3r}$$

what is this 2-index object?

Answ: use $\vec{J}(\vec{r},t) = \sum_a q_a \vec{v}_a \delta^3(\vec{r}-\vec{r}_a(t))$ again (postpone FT to ω):

$$\int r_i \sum_a q_a v_{aj} \delta^3(\vec{r}-\vec{r}_a) = \sum_a q_a r_{ai} v_{aj}$$

Now $\frac{d}{dt} r_i r_j = v_i r_j + r_i v_j$

$$\Rightarrow \quad r_i v_j = \frac{1}{2}(r_i v_j + r_j v_i) + \frac{1}{2}(r_i v_j - r_j v_i)$$

$$= \frac{1}{2}\frac{d}{dt}(r_i r_j) + \frac{1}{2}\epsilon_{ijk}\epsilon_{kmn} r_m v_n$$

The 1st term gives

$$\frac{d}{dt}\sum_a q_a r_{ai} r_{aj} = \frac{d}{dt}\frac{1}{3}\underbrace{\sum_a q_a(3r_{ai}r_{aj} - r_a^2\delta_{ij})}_{D_{ij}} + \frac{1}{3}\delta_{ij}\frac{1}{3}\frac{d}{dt}\sum_a q_a r_a^2$$

The 2nd gives $\sum_a q_a \epsilon_{kmn} r_{am} v_{an} = \sum_a q_a(\vec{r}_a\times\vec{v}_a) = 2c\vec{m}$ (magnetic dipole moment).

So, with $\vec{D} = \hat{R}_i D_{ij}$ and taking $-i\omega \to \frac{d}{dt}$

$$\vec{\tilde{J}}^{(1)}(\vec{k},\omega) = \int dt\, e^{i\omega t}\left[\dot{\vec{m}}\times\hat{R} + \frac{1}{6c}\ddot{\vec{D}} + \underbrace{\frac{1}{6c}\left(\frac{d^2}{dt^2}\sum_a q_a r_a^2\right)\hat{R}}_{\text{only } J^{\parallel}}\right]$$

When taking $J^{\perp}$, the last term never contributes: drop it.

$$\Rightarrow \quad \vec{E}^{(1)}(\vec{R},\omega) = \frac{ike^{ikR}}{cR}J^{\perp}(\vec{k},\omega) = \frac{i\omega e^{ikR}}{c^2R}\left[-\frac{\omega^2}{6c}\vec{D}(\omega) - i\omega\,\vec{m}(\omega)\times\hat{R}\right]$$

or in time domain $\omega \to i\frac{\partial}{\partial t}$ and e^{ikR} is $t \to t_{ret}$. Putting (0)+(1) together:

$$\vec{E}(\vec{R},t) = \frac{1}{c^2R}\left[\hat{R}\times(\hat{R}\times\ddot{\vec{d}}) + \hat{R}\times\ddot{\vec{m}} + \frac{1}{6c}\hat{R}\times(\hat{R}\times\dddot{\vec{D}})\right]_{ret}$$

$$\vec{B}(\vec{R},t) = \hat{R}\times\vec{E} = \frac{1}{c^2R}\left[-\hat{R}\times\ddot{\vec{d}} + \hat{R}\times(\hat{R}\times\ddot{\vec{m}}) - \frac{1}{6c}\hat{R}\times\dddot{\vec{D}}\right]_{ret}$$

Terminology radiation from $\vec{d}, \vec{D}, \ldots$ is E1, E2, ...

from $\vec{m}, \ldots$ is M1, M2, ..

$\frac{dP}{d\Omega}$: patterns have interference between those terms

$P = \int \frac{dP}{d\Omega} d\Omega$ is sum of individual powers?

To see this we need averages of powers of $\hat{R}$: let $\langle \cdot \rangle = \frac{1}{4\pi}\int d\Omega \,\cdot$

$\langle 1 \rangle = 1$, $\langle \hat{R} \rangle = 0 = \langle \hat{R}_{i_1} \hat{R}_{i_2} \cdots \hat{R}_{i_{2n+1}} \rangle$

$\langle \hat{R}_i \hat{R}_j \rangle = \frac{1}{3}\delta_{ij}$

(Proof: $\langle \hat{R}_i \hat{R}_j \rangle = c\,\delta_{ij}$ by rotational symmetry $\Rightarrow \langle 1 \rangle = 3c \Rightarrow c = 1/3$ ✓).

$\langle \hat{R}_i \hat{R}_j \hat{R}_k \hat{R}_l \rangle = \frac{1}{15}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$ $\quad \left(\langle 1 \rangle = \frac{1}{15}(9+3+3)\right)$

So for $\frac{dP}{d\omega}$ (or P) need $\langle E^2 \rangle$; keep in mind $\vec{D}_i = \hat{R}_j D_{ij}$. So the number of $\hat{R}_i$'s in $\vec{d}$ has $\vec{m}$ or $\vec{D}$ cross terms is odd. Moreover, the $\vec{m}$ crossterm with $\vec{D}$ has four $\hat{R}_i$'s $\rightarrow$ product of δ_{ij}'s and ϵ_{ijk}'s that need to contract with $m_k D_{ij}$: only possibility is $\epsilon^{ijk} m_k D_{ij} = 0$ $\quad (D_{ij} = D_{ji})$

$\Rightarrow \frac{dP}{d\omega} \sim d^2,\ m^2$ and D_{ij}^2 terms

In fact: $\quad P = \frac{2}{3c^2}\ddot{\vec{d}}^{\,2} + \frac{2}{3c^2}\ddot{\vec{m}}^{\,2} + \frac{1}{180c^5}\dddot{D}_{ij}\dddot{D}_{ij}$

Do $\vec{d}$ first: $\quad \frac{dP}{d\Omega} = R^2 \frac{c}{4\pi}\vec{E}^2 = \frac{1}{4\pi c^3}\left|\hat{R} \times \ddot{\vec{d}}\right|^2 = \frac{1}{4\pi c^3}\left(|\ddot{\vec{d}}|^2 - |\hat{R}\cdot\ddot{\vec{d}}|^2\right)$

$$\Rightarrow \quad P = \frac{1}{c^3}\left(|\ddot{\vec{d}}|^2 - \langle \hat{R}_i \hat{R}_j \rangle \ddot{d}_i^* \ddot{d}_j\right) = \frac{1}{c^3}\left(|\ddot{\vec{d}}|^2 - \frac{1}{3}\delta_{ij}\ddot{d}_i^* \ddot{d}_j\right) = \frac{2}{3c^3}\ddot{\vec{d}}$$

Clearly $\ddot{\vec{m}}$ is the same.

Exercise: Do $\dddot{D}^2$ term

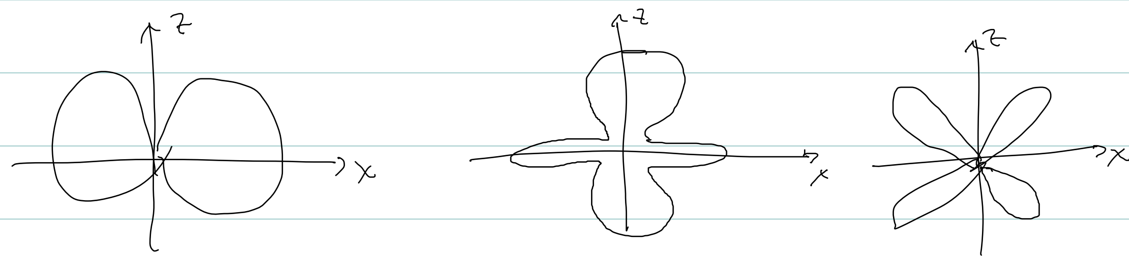
See text for pure quadrupole patterns.

$$\frac{dP}{d\Omega} = \frac{c}{4\pi}\left|\frac{1}{c^2}\cdot\frac{1}{6c}\hat{n}\times(\hat{n}\times\dddot{\vec{D}})\right|^2 = \frac{1}{144\pi c^5}\left(|\dddot{\vec{D}}|^2 - |\hat{n}\cdot\dddot{\vec{D}}|^2\right)$$

$$= \frac{1}{144\pi c^5}\left(\hat{n}_i\hat{n}_j\delta_{mn}\dddot{D}_{im}\dddot{D}_{jn} - \hat{n}_i\hat{n}_j\hat{n}_m\hat{n}_n\dddot{D}_{im}\dddot{D}_{jn}\right)$$

Then pick particular D_{ij} (traceless, symmetric). Text picks them to be q_{2m}'s.

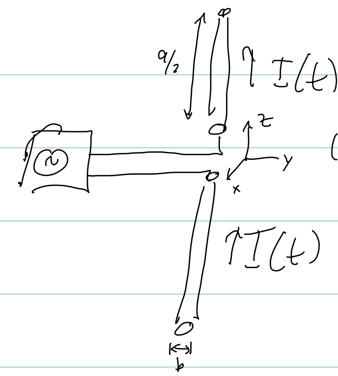
Pictures:



$l=2$ $m=2$ $\qquad$ $l=2$ $m=1$ $\qquad$ $l=2$ $m=0$

Antennas

Centerfed linear antenna:



(antenna on z-axis, origin at middle point).

Crude model: $I(t) = I_0 \cos\omega_0 t$ independent of z! (unrealistic).

$$\text{Recall} -i\omega\vec{d}(\omega) = \vec{J}^{(0)}(\omega) = \int d^3r\, \vec{J}(\vec{r},\omega)$$

and $\quad \vec{J}(\vec{r},\omega) = \int dt\, e^{i\omega t}\, J(\vec{r},t)$

so $\quad \vec{J}^{(0)}(\omega) = \int dt\, e^{i\omega t}\, \hat{z}\, I(t) a = \hat{z}\frac{1}{2} a I_0 \int dt\, e^{i\omega t}\left(e^{i\omega_0 t} - e^{-i\omega_0 t}\right)$

$$= \pi a I_0 \hat{z}\left(\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right)$$

$$\dot{\vec{d}} = \int dt\, e^{i\omega t}\, \vec{J}^{(0)}(\omega) = a I_0 \hat{z} \cos(\omega_0 t)$$

$$\Rightarrow \quad |\vec{E}(\vec{R},t)| = \left|\frac{1}{c^2 R}\ddot{\vec{d}}_{ret}\right| = \frac{a I_0 \omega_0}{c^2 R}\sin(\omega_0 t)\sin\theta \quad \left(\sin\theta = \sqrt{1-(\hat{z}\cdot\hat{R})^2} = \sqrt{1-\cos^2\theta}\right).$$

So the power follows:

$$\frac{dP}{d\Omega} = R^2 \frac{c}{4\pi}|\vec{E}|^2 = \frac{a^2 I_0^2 \omega_0^2}{4\pi c^3}\sin^2(\omega_0 t_{ret})\sin^2\theta$$

or in terms of wave-length $\lambda = \frac{2\pi c}{\omega_0}$, $\quad \frac{dP}{d\Omega} = \frac{\pi}{c}\left(\frac{a}{\lambda}\right)^2 I_0^2 \sin^2(\omega_0 t_{ret})\sin^2\theta$

and averaging over one cycle, $\sin^2(\omega_0 t) \to \frac{1}{2}$ $\quad \overline{\frac{dP}{d\Omega}} = \frac{\pi}{c}\left(\frac{a}{\lambda}\right)^2 I_0^2 \sin^2\theta$

We could have guessed the $\left(\frac{a}{\lambda}\right)^2$ (leading multipole, $d \sim \frac{a}{\lambda}$)

the $\sin^2\theta$ ($\vec{d}$ along $\hat{z}$) and the I_0^2 ($\dot{\vec{d}} \sim I_0$). The $\frac{1}{c}$ is from dimensional analysis. Left with "π", for which we needed a calculation.

A more realistic model needs $I(|z|=a/2)=0$. For a very thin antenna we take $\vec{J}\propto\delta(x)\delta(y)$. So we propose

$$\vec{J} = I_m \sin(\tfrac{1}{2}k_0 a - k_0|z|)\cos(\omega_0 t)\,\delta(y)\delta(x)\,\hat{z}$$

I_m is the max current: at center, the fed current is $I_0 = I_m \sin(\frac{1}{2}k_0 a)$

Calculate: $$\vec{J}^{(0)}(\omega) = \int dt\, e^{i\omega t}\int d^3r\, \vec{J}(\vec{r},t)$$

$$= \hat{z} I_m \int dt\, e^{i\omega t}\cos(\omega_0 t)\int_{-a/2}^{a/2} dz\, \sin(\tfrac{1}{2}k_0 a - k_0|z|)$$

$$= \hat{z} I_m \int dt\, e^{i\omega t}\cos(\omega_0 t)\,\frac{2}{k_0}\left(1-\cos(\tfrac{1}{2}k_0 a)\right)$$

$$\dot{\vec{d}} = \int\frac{d\omega}{2\pi}e^{-i\omega t}\vec{J}^{(0)}(\omega) = \hat{z} I_m \int \frac{d\omega}{2\pi}e^{-i\omega t}\int dt'\, e^{i\omega t'}\cos(\omega_0 t')\,\frac{2}{k_0}\left[1-\cos(\tfrac{1}{2}k_0 a)\right]$$

$$= \hat{z} I_m \cos(\omega_0 t)\,\frac{2}{k_0}\left[1-\cos(\tfrac{1}{2}k_0 a)\right]$$

$$\ddot{\vec{d}}_{ret} = -\hat{z} I_m \frac{2\omega_0}{k_0}\sin(\omega_0 t_{ret})\left[1-\cos(\tfrac{1}{2}k_0 a)\right]$$

Then $$|\vec{E}| = \left|\frac{1}{c^2R}\ddot{\vec{d}}_{ret}\right| = \frac{2 I_m}{cR}|\sin(\omega_0 t_{ret})|\left[1-\cos(\tfrac{1}{2}k_0 a)\right]\sin\theta$$

and $$\overline{\frac{dP}{d\Omega}} = R^2\frac{c}{4\pi}|E|^2 = \frac{1}{2\pi c} I_m^2\left(1-\cos(\tfrac{1}{2}k_0 a)\right)^2\sin^2\theta$$

For small $k_0 a$ (dipole approximation, which we are using) $1-\cos(\frac{1}{2}k_0 a)\simeq\frac{1}{2}(\frac{1}{2}k_0 a)^2$

$$\Rightarrow \overline{\frac{dP}{d\Omega}} = \frac{1}{128\pi c} I_m^2 (k_0 a)^4 \sin^2\theta$$

This problem can be solved without use of multipole expansion: recall

Recall for monochromatic source

$$\frac{dP_n}{d\Omega} = \frac{n^2\omega_0^2}{2\pi c^3}\left|\vec{J}_n^{\perp}(n\tfrac{\omega_0}{c}\hat{R})\right|^2$$

where we only need $n=1$, and $\vec{J}_n^{\perp}(\vec{k}) = \frac{1}{T}\int_0^T dt\, e^{in\omega_0 t}\vec{J}(\vec{k},t) = \frac{1}{T}\int_0^T dt\, e^{in\omega_0 t}\int d^3r\, e^{-i\vec{k}\cdot\vec{r}}\vec{J}(\vec{r},t)$

So $$\vec{J}_1^{\perp}(\vec{k}) = \frac{1}{T}\int_0^T dt\, e^{-i\omega_0 t}\int d^3r\, e^{-i\vec{k}\cdot\vec{r}}\,\hat{z} I_m \sin(\tfrac{1}{2}k_0 a - k_0|z|)\,\delta(x)\delta(y)\cos(\omega_0 t)$$

$$= \frac{1}{T}\int_0^T dt\, e^{i\omega_0 t}\cos(\omega_0 t)\;\hat{z} I_m \int_{-a/2}^{a/2} dz\, e^{-ik_z z}\sin(\tfrac{1}{2}k_0 a - k_0|z|)$$

The time integral gives $\frac{1}{2}$ (and there is a $\vec{J}_{-1}$ which also has $\frac{1}{2}$, but we have accounted for it, recall $2\sum_{n>0}E_n E_n^* \cdots$).

Need $\int_{-a/2}^{a/2} dz\, e^{-ik_z z} \sin\left(\tfrac{1}{2}k_0 a - k_0|z|\right)$

$$= \int_0^{a/2} dz\, e^{-ik_z z} \sin\left(\tfrac{1}{2}k_0 a - k_0 z\right) + \int_{-a/2}^{0} dz\, e^{-ik_z z} \sin\left(\tfrac{1}{2}k_0 a + k_0 z\right)$$

$$= \mathrm{Im}\left[\int_0^{a/2} dz\, e^{-ik_z z} e^{i(\frac{1}{2}k_0 a - k_0 z)} + \int_{-a/2}^{0} dz\, e^{-ik_z z} e^{i(\frac{1}{2}k_0 a + k_0 z)}\right]$$

$$= \mathrm{Im}\left[e^{i\frac{1}{2}k_0 a} \frac{i}{k_0+k_z}\left(e^{-i(k_z+k_0)\frac{a}{2}} - 1\right) + e^{ik_0\frac{a}{2}} \frac{i}{k_z-k_0}\left(1 - e^{-i(k_z-k_0)\frac{a}{2}}\right)\right]$$

$$= -\mathrm{Re}\left[\frac{1}{k_z+k_0}\left(e^{-ik_z\frac{a}{2}} - e^{ik_0 a/2}\right) + \frac{1}{k_z-k_0}\left(e^{ik_0\frac{a}{2}} - e^{-ik_z\frac{a}{2}}\right)\right]$$

$$= -\frac{1}{k_z+k_0}\left(\cos(k_z\tfrac{a}{2}) - \cos(k_0\tfrac{a}{2})\right) - \frac{1}{k_z-k_0}\left(\cos(\tfrac{k_0 a}{2}) - \cos(\tfrac{k_z a}{2})\right)$$

$$= \left(\cos(k_z\tfrac{a}{2}) - \cos(k_0\tfrac{a}{2})\right) \frac{-2k_0}{k_z^2-k_0^2}$$

This is for arbitrary $\vec{k}$. But we need only $\vec{k} = \hat{R}k_0 \rightarrow k_z = \cos\theta\, k_0$

$$= \left(\cos(\tfrac{k_0 a}{2}\cos\theta) - \cos(\tfrac{k_0 a}{2})\right) \frac{-2k_0}{k_0^2(\cos^2\theta - 1)} = \frac{2}{k_0\sin^2\theta}\left(\cos(\tfrac{k_0 a}{2}\cos\theta) - \cos(\tfrac{k_0 a}{2})\right)$$

so $\vec{J}_1(k_0\hat{R}) = \hat{z}\, I_m \frac{1}{k_0} \frac{\left(\cos(\frac{k_0 a}{2}\cos\theta) - \cos(\frac{k_0 a}{2})\right)}{\sin^2\theta}$

For $|\vec{J}^{\perp}|^2$ need $|\hat{z} - \hat{R}\cdot\hat{z}\,\hat{R}|^2 = 1 - \cos^2\theta = \sin^2\theta$

It follows that (use $\frac{\omega_0^2}{k_0^2} = c^2$)

$$\frac{dP}{d\Omega} = \frac{I_m^2}{2\pi c} \frac{\left(\cos(\frac{k_0 a}{2}\cos\theta) - \cos(\frac{k_0 a}{2})\right)^2}{\sin^2\theta}$$

For $k_0 a \ll 1$, $\cos(\frac{k_0 a}{2}\cos\theta) - \cos(\frac{k_0 a}{2}) = -\frac{1}{2}\left(\frac{k_0 a}{2}\cos\theta\right)^2 + \frac{1}{2}\left(\frac{k_0 a}{2}\right)^2 = \frac{(k_0 a)^2}{8}\sin^2\theta$

$\Rightarrow \frac{dP}{d\Omega} = \frac{I_m^2}{128\pi c}(k_0 a)^4 \sin^2\theta$ as before for dipole approximation.

Radiation pattern: we've seen in dipole approx we have $\frac{dP}{d\Omega} \sim \sin^2\theta$

But (for this model) in exact case we have

$$\frac{dP}{d\Omega} \propto \frac{\left(\cos\left(\frac{k_0 a}{2}\cos\theta\right) - \cos\left(\frac{k_0 a}{2}\right)\right)^2}{\sin^2\theta}$$

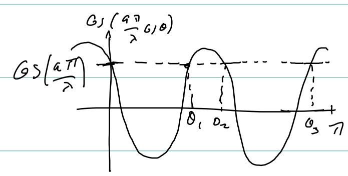
At small θ $\cos\left(\frac{k_0 a}{2}\left(1-\frac{1}{2}\theta^2\right)\right) = \cos\left(\frac{k_0 a}{2}\right)\cos\left(-\frac{k_0 a}{4}\theta^2\right) + \sin\left(\frac{k_0 a}{2}\right)\sin\left(\frac{k_0 a}{4}\theta^2\right)$

$$\simeq \cos\left(\frac{k_0 a}{2}\right) + \sin\left(\frac{k_0 a}{2}\right)\frac{k_0 a}{4}\theta^2$$

So $\frac{dP}{d\Omega} \propto \theta^2$ so $\theta = 0$ (and π) is a null-direction just as in dipole approximation.

Additional null directions arise if $k_0 a$ is not small

that is $\frac{k_0 a}{2} = \frac{1}{2}\left(\frac{2\pi}{\lambda}\right)a = \frac{\pi a}{\lambda}$:



Define radiation resistance R_{rad} of antenna such that the power radiated is the "dissipated" power: $P = \frac{1}{2} I_0^2 R_{rad}$

In the dipole approximation, $I_0 = I_m \sin(\frac{k_0 a}{2}) \approx I_m \frac{k_0 a}{2}$

and $P = \int \frac{dP}{d\Omega} d\Omega = \int \frac{I_m^2}{128\pi c} (k_0 a)^4 \sin^2\theta \, d\Omega$

$$(k_0 a)^2 \frac{4 (I_m \frac{k_0 a}{2})^2}{128 \pi c} 2\pi \int_{-1}^{1} d\xi \, (1 - \xi^2)$$

$$= \frac{(k_0 a)^2}{12 c} I_0^2 \qquad R_{rad} = \frac{(k_0 a)^2}{6c}$$

[Units? : recall $F = \frac{q^2}{r^2} \Rightarrow [P] = [q]^2 [L]^{-2} \cdot [L][T]^{-1} = [q]^2 [L]^{-1} [T]^{-1} = [I]^2 [T][L]^{-1} \checkmark$]

So what is $\frac{1}{c}$ in ohms? We can look in tables, or use our translation instructions.

Recall: $q_G^2 = \frac{q_{SI}^2}{4\pi\epsilon_0} \rightarrow I_G^2 = \frac{I_{SI}^2}{4\pi\epsilon_0} \qquad c^2 = \frac{1}{\epsilon_0 \mu_0}$

$$P = \frac{(k_0 a)^2}{12} \sqrt{\epsilon_0 \mu_0} \frac{I_0^2}{4\pi\epsilon_0} = \frac{(k_0 a)^2}{48\pi} \sqrt{\frac{\mu_0}{\epsilon_0}} I_0^2$$

The quantity $Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377$ ohms is known as the "impedance of the vacuum."

and $R_{rad} = Z_0 \frac{(k_0 a)^2}{24\pi} = \frac{Z_0}{24\pi} \left(\frac{2\pi a}{\lambda}\right)^2 = \frac{\pi}{6} Z_0 \left(\frac{a}{\lambda}\right)^2 = 197\Omega \left(\frac{a}{\lambda}\right)^2$

Since $a \ll \lambda$ in this approximation, this is a fairly small number, hence low radiation efficiency. For $\frac{a}{\lambda} \simeq 0.2$ $R_{rad} \simeq 8\Omega$, while $\frac{a}{\lambda} \simeq \frac{1}{2}$ ("half-wave antenna") $R_{rad} \simeq 49\Omega$. Better yet $a \simeq \lambda$ ("full-wave antenna") but then need to integrate $\int \frac{dP}{d\Omega} d\Omega$ without $\frac{a}{\lambda} \ll 1$ approximation.

Near Zone Fields: Very brief word on this. The region $R \ll \lambda$ is called "near-zone". For NR sources one has in addition $a \ll \lambda$.

Then in $A_\mu(\vec{r},t) = \frac{1}{c}\int d^3r' \, \frac{J_\mu(\vec{r}',t_{ret})}{|\vec{r}-\vec{r}'|}$

one has $t_{ret} = t - \frac{R}{c} \to t$ and $A_\mu(\vec{r},t) = \frac{1}{c}\int d^3r' \, \frac{J_\mu(\vec{r}',t)}{|\vec{r}-\vec{r}'|}$

the instantaneous field.

One can justify this ($t_{ret} \to t$) more precisely by going to Fourier space:

Recall $\tilde{A}_\mu(\vec{r},\omega) = \frac{e^{ikR}}{cR} \, \vec{\tilde{j}}(\vec{k},\omega)$

Now $e^{ikR} \simeq 1 + ikR + \cdots$ with $kR = 2\pi \frac{R}{\lambda} \ll 1$. Replace $e^{ikR} \to 1$. Done.

Maxwell equations in media (Garg: Chap 13, Sec 81).

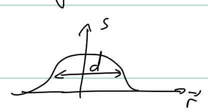
Model media as made up of charges that are fixed (as in molecules, which may not be neutral as they may have deficit or excess of electrons) plus free charges (as in conduction electrons in conductors, or added charges in insulators).

Denote by $\vec{e}$ & $\vec{b}$ the microscopic fields, ie, the fundamental fields. These change over atomic distance scales. So we have

$$\vec{\nabla}\cdot\vec{e} = 4\pi\rho_{micro} \qquad \nabla\times\vec{e} + \frac{1}{c}\frac{\partial\vec{b}}{\partial t} = 0 \qquad \vec{\nabla}\times\vec{b} - \frac{1}{c}\frac{\partial\vec{e}}{\partial t} = \frac{4\pi}{c}\vec{j}_{micro} \qquad \vec{\nabla}\cdot\vec{b} = 0$$

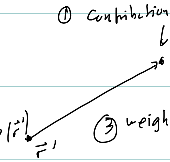
Now, smooth these out over "macroscopic" distances (where "macroscopic" depends on context, but can be as short as $\sim\times 10$ atomic distance, say). To this end use a smoothing (averaging) function $s(\vec{r})$: we want

$$\int d^3r\, s(\vec{r}) = 1$$

and  with $d \gg$ typical fast variation of $\vec{e}$ & $\vec{b}$, on "atomic" scale

Then let $\vec{E}(\vec{r},t) = \int d^3r'\, s(\vec{r}-\vec{r}\,')\, \vec{e}(\vec{r}\,',t)$

① contribution to $\vec{E}(\vec{r})$



② from $e(\vec{r}\,')$ ③ weighed by s(vector from $\vec{e}(\vec{r}\,')$ to point $\vec{r}$).

and $\vec{B}(\vec{r},t) = \int d^3r'\, s(\vec{r}-\vec{r}\,')\, \vec{b}(\vec{r}\,',t)$

Now $\partial_i E_j = \int d^3r'\, \partial_i s(\vec{r}-\vec{r}\,')\, e_j(\vec{r}\,',t) = \int d^3r' (-\partial_i' s) e_j = \int d^3r'\, s\, \partial_i' e_j$ (int. by parts + s has local support)

So all diff ops go through and:

$$\vec{\nabla}\cdot\vec{E} = 4\pi\langle\rho\rangle_{micro} \qquad \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0 \qquad \vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\langle\vec{j}\rangle_{micro} \qquad \vec{\nabla}\cdot\vec{B} = 0$$

where $\langle(\cdots)\rangle = \int d^3r'\, s(\vec{r}-\vec{r}\,')(\cdots)$ as above.

Now task is to break $\langle\rho\rangle$ & $\langle\vec{j}\rangle$ into pieces in a useful/convenient way.

Text discusses nuances of physics at atomic scales (requires QM). This is one of those cases that "it's not relevant, except when it is". So ignore for now and deal with QM as needed.

(Sec 82): ρ and $\vec{P}$.

ρ_{micro} →
- charges bound to atoms/molecules
- charges not bound to atoms/molecules →
  - conduction e's in metals
  - externally supplied charge in insulators.
  - other ...

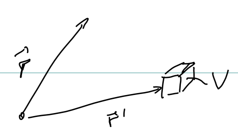
Bound charge: molecules are neutral; effect at distances large compared to size ⟶ multipole expansion ⟶ usually dipole $\vec{d}$ suffices. BOTH for its field and its response to applied fields.

let $\vec{P}$ = dipole moment/volume, a local quantity (ie $\vec{P} = \vec{P}(\vec{r})$)

in this δV, $\vec{P} = \dfrac{\sum_{\text{molecules}} \vec{d}}{\delta V}$: δV is large enough to smooth out atomic scale fluctuations, yet small enough that multipole expansion makes sense.

Alternatively, text says $\vec{P} = n\, d$ $\quad n = \#$ density

Now $\vec{P} \Rightarrow$ charge distribution. Recall $\phi(\vec{r}) = \dfrac{\vec{d}\cdot\vec{r}}{r^3}$ from dipole at origin.



$$\Rightarrow \quad \phi(\vec{r}) = \int_V (d^3r'\, \vec{P}(\vec{r}\,')) \cdot \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

Use $\vec{\nabla}' \dfrac{1}{|\vec{r}-\vec{r}\,'|} = \dfrac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$, integrate by parts, keep surface term:

$$\Rightarrow \quad \phi(\vec{r}) = \underbrace{\int_{\partial V} \frac{\vec{P}(\vec{r}\,')\cdot\hat{n}'}{|\vec{r}-\vec{r}\,'|} d^2s}_{\vec{P}\cdot\hat{n} = \sigma_{pol}} - \underbrace{\int_V \frac{\vec{\nabla}'\cdot\vec{P}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} d^3r'}_{\vec{\nabla}\cdot\vec{P} = -\rho_{pol}}$$

σ_{pol} is cancelled by that of adjacent volume

cancelling σ_{pol} because $\hat{n}_1 = -\hat{n}_2$

. Not so for surface of material.

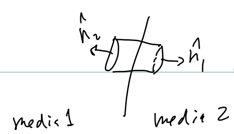
So break $\langle \rho_{micro} \rangle$ into $\rho_{pol} + \rho_{free}$

$$\Rightarrow \quad \vec{\nabla}\cdot\vec{E} = 4\pi(\rho_{pol} + \rho_{free}) = 4\pi(-\vec{\nabla}\cdot\vec{P} + \rho_{free})$$

$$\Rightarrow \quad \vec{\nabla}\cdot(\underbrace{\vec{E} + 4\pi\vec{P}}_{\text{"}\vec{D}\text{" electric displacement}}) = 4\pi\rho_{free} \quad \Rightarrow \quad \boxed{\vec{\nabla}\cdot\vec{D} = 4\pi\rho_{free}}$$

With this definition then also

$$\vec{E}_2\cdot\hat{n}_1 + \vec{E}_1\cdot\hat{n}_2 = (\vec{E}_2 - \vec{E}_1)\cdot\hat{n}_{21} = 4\pi(\sigma_{pol} + \sigma_{free}) = 4\pi(\vec{P}_1\cdot\hat{n}_1 + \vec{P}_2\cdot\hat{n}_2 + \sigma_{free})$$



$(\hat{n}_1 = -\hat{n}_2 \equiv \hat{n}_{21})$

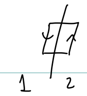
$$\Rightarrow \left[(\vec{E}_2 + 4\pi\vec{P}_2) - (\vec{E}_1 + 4\pi\vec{P}_1)\right]\cdot\hat{n}_{21} = 4\pi\sigma_{free}$$

$$\boxed{(\vec{D}_2 - \vec{D}_1)\cdot\hat{n}_{21} = 4\pi\sigma_{free}}$$

(We used this last quarter in wave propagation in media).

To complete boundary conditions at interfaces, $\vec{\nabla}\times\vec{E} = 0$

$\Rightarrow \oint \vec{E}\cdot d\vec{l} = 0 \Rightarrow$



$\vec{E}_{1t} = \vec{E}_{2t}$ "t" = tangential.

Notes:

- $\vec{D}$ is not sourced by ρ_{free}: in addition to $\vec{\nabla}\cdot\vec{D} = 4\pi\rho_{free}$ we have $\vec{\nabla}\times\vec{D} = \vec{\nabla}\times(\vec{E} + 4\pi\vec{P}) = 4\pi\,\vec{\nabla}\times\vec{P}$. That is, we have to look at $\vec{D}$, $\vec{E}$ and some way of determining $\vec{P}$ to get the whole picture.
- Dimensions & Units: in Gaussian things are simple and make sense $\vec{D}$, $\vec{E}$ and $\vec{P}$ have same units. In SI $\vec{D} = \epsilon_0\vec{E} + \vec{P}$, so $\vec{D}$ has unit of $\vec{P}$ $(= [Q][L]^{-2})$, different than $\vec{E}$.

    See text for translation (or work it out).

(Sec 83): Macroscopic current density

Break into components:

$$\langle \vec{J}_{micro} \rangle = \vec{J}_{free} + \vec{J}_{pol} + \vec{J}_{conv} + \vec{J}_{mag}$$

$\vec{J}_{free}$ is from motion of ρ_{free} so $\frac{\partial \rho_{free}}{\partial t} + \vec{\nabla}\cdot\vec{J}_{free} = 0$

$\vec{J}_{pol}$ is from t-dependence on $\vec{P}$: $\rho_{pol} = -\vec{\nabla}\cdot\vec{P} \Rightarrow \frac{\partial \rho_{pol}}{\partial t} = -\vec{\nabla}\cdot\frac{\partial \vec{P}}{\partial t} \Rightarrow \vec{J}_{pol} = \frac{\partial \vec{P}}{\partial t}$

The convection current $\vec{J}_{conv}$ (gases & liquids only), charges carried by overall motion of fluid. Artificial breakdown, but useful. If $\vec{v}$ is fluid velocity field

$$\vec{J}_{conv} = (\rho_{free} + \rho_{pol})\,\vec{v}$$

Then one has to make sure of no double counting so this is subtracted from $\vec{J}_{free}$ & $\vec{J}_{pol}$ which then measure current in a comoving fluid element.

<u>Most interesting</u>: $\vec{J}_{mag}$ magnetization current.

$\vec{M}(\vec{r})$ = magnetic dipole moment / volume.

Let's recall field due to magnetic dipole: for this need:

Brief review of magnetostatics: recall $\partial^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) = \frac{4\pi}{c} J_\nu$. Spatial components steady state: $-\vec{\nabla}^2\vec{A} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) = \frac{4\pi}{c}\vec{J}$, choose $\vec{\nabla}\cdot\vec{A} = 0$ gauge, and

$$\vec{A} = \frac{1}{c}\int \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3r'$$

Magnetostatics: $\frac{\partial \rho}{\partial t} = 0 \Rightarrow \vec{\nabla}\cdot\vec{J} = 0$.

Multipole expansion:

$$\vec{A}(\vec{r}) = \frac{1}{c}\int d^3r' \; \vec{j}(\vec{r}')\left[\frac{1}{|r|} + \frac{\vec{r}'\cdot\vec{r}}{|\vec{r}|^3} + \dots\right]$$

Trick: With $\vec{j}$ localized, given two functions $f(\vec{r})$ & $g(\vec{r})$ we have

$$\int d^3r \; \vec{\nabla}(fg\vec{j}) = 0 \quad \Rightarrow \quad \int d^3r \,(f\vec{j}\cdot\vec{\nabla}g + g\vec{j}\cdot\vec{\nabla}f) = 0 \quad (\text{used } \vec{\nabla}\cdot\vec{j} = 0).$$

With $f = 1$, $g = r_i$ so $\vec{\nabla}f = 0$, $\nabla_j g = \delta_{ij}$ we have

$$\int \vec{j}\, d^3r = 0 \Rightarrow \text{monopole term vanishes.}$$

Next $f = x_i$, $g = x_j$ $\Rightarrow \int d^3r\,(r_i j_j + r_j j_i) = 0$

For next (dipole) term need

$$\int d^3r' \, j_i(\vec{r}')\, r_j' = \int d^3r' \left[\underbrace{\tfrac{1}{2}(j_i r_j' + j_j r_i')}_{0 \text{ (above)}} + \tfrac{1}{2}(j_i r_j' - j_j r_i')\right]$$

$$\Rightarrow A_i(\vec{r}) = \frac{1}{2c}\frac{r_j}{|r|^3}\int d^3r' \,(j_i r_j' - j_j r_i')$$

Now $j_i r_j' - j_j r_i' = \epsilon_{ijk}\epsilon_{kmn} j_m r_n' = \epsilon_{ijk}(\vec{j}\times\vec{r}')_k$

$$\vec{A}(\vec{r}) = \frac{\vec{m}\times\vec{r}}{|r|^3} \quad \text{with } \vec{m} = \frac{1}{2c}\int d^3r' \; \vec{r}'\times\vec{j}(\vec{r}') \quad \text{the magnetic dipole of } \vec{j}.$$

$\vec{B} = \vec{\nabla}\times\vec{A}$ was an assignment (earlier in course 203A).

Gives $\vec{B}$ in terms of $\vec{m}$ just as $\vec{E}$ in terms of $\vec{d}$.

Back to $\vec{J}_{mag}$: from $\vec{A} = \frac{\vec{m} \times \vec{r}}{r^3}$ we have

$$\vec{A}(\vec{r}) = \int_V d^3 r' \, \frac{\vec{M}(\vec{r}\,') \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3}$$

Now repeat steps we did for ϕ: $\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} = \vec{\nabla}' \frac{1}{|\vec{r}-\vec{r}\,'|}$, integrate by parts:

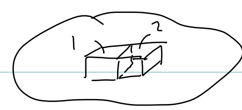
$$\vec{A}(\vec{r}) = \int_S d^2 s \, \frac{\vec{M}(\vec{r}\,') \times \hat{n}}{|\vec{r} - \vec{r}\,'|} + \int_V d^3 r' \, \frac{\vec{\nabla}' \times \vec{M}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}$$

The 2nd term, cu , ih $\vec{A}(\vec{r}) = \frac{1}{c} \int d^3 r' \, \frac{\vec{J}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} \Rightarrow \boxed{\vec{J}_{mag}(\vec{r}) = c \, \vec{\nabla} \times \vec{M}(\vec{r})}$

The 1st term is similar but for a surface current density $\vec{K}$:

$$\boxed{\vec{K}_{mag} = c \, \vec{M} \times \hat{n}}$$

Again, in the interior of the material adjacent volume elements give cancelling contributions



$\hat{n}_1 = -\hat{n}_2 \Rightarrow \vec{K}_{mag\,1} + \vec{K}_{mag\,2} = 0$.

But not so for bounding surface.

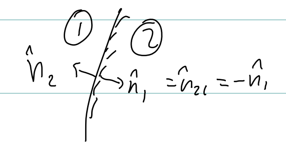
More generally, $\vec{M}$ should include $\vec{m}$'s from intrinsic magnetic dipole moments from particle spin.

Now $\vec{\nabla} \times \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c} \left(\vec{J}_{free} + \underbrace{\vec{J}_{pol}}_{\frac{\partial \vec{P}}{\partial t}} + \vec{J}_{conv} + \underbrace{\vec{J}_{mag}}_{c \vec{\nabla} \times \vec{M}} \right)$

$$\Rightarrow \vec{\nabla} \times (\vec{B} - 4\pi \vec{M}) - \frac{1}{c} \frac{\partial}{\partial t} (\vec{E} + 4\pi \vec{P}) = \frac{4\pi}{c} (\vec{J}_{free} + \vec{J}_{conv})$$

Define $\vec{H} = \vec{B} - 4\pi \vec{M} \Rightarrow \boxed{\vec{\nabla} \times \vec{H} - \frac{1}{c} \frac{\partial \vec{D}}{\partial t} = \frac{4\pi}{c} (\vec{J}_{free} + \vec{J}_{conv})}$

For Garg, $\vec{H}$ = "magnetizing field", $\vec{B}$ = "magnetic field" which I like.

Interface 

$$\vec{B}_{1n} = \vec{B}_{2n} \quad (\text{from } \vec{\nabla}\cdot\vec{B} = 0)$$

$$(\vec{H}_2 - \vec{H}_1)\times\hat{n}_{21} = \frac{4\pi}{c}\vec{K}_{free}$$

(We have given up convection currents here).

For Dimensions ($\vec{H}$ & $\vec{M}$ same as $\vec{B}$ same as $\vec{E}$) and units (including translation to SI) see Garg.

Constitutive Relations (Garg Sec 84).

To solve the macroscopic Maxwell equations we need additional relations (giving eg, $\vec{P}$ in terms of $\vec{E}$ or $\vec{D}$, and $\vec{M}$ in terms of $\vec{B}$ or $\vec{H}$). We also need to know something about separating current/charge into free part.

Conductors: Ohm's law $\quad \vec{J} = \sigma\vec{E}$

σ = "conductivity"

Not a "law". Fails in semiconductors, or at large fields in conductors. Often frequency dependent, so in Fourier space $\vec{J}(\omega) = \sigma(\omega)\,\vec{E}(\omega)$

Dielectrics: For insulators

$$\vec{D} = \epsilon \vec{E}$$

ϵ = "dielectric constant"

Only at small fields. Sometimes need different ϵ in different directions $D_i = \epsilon_{ij} E_j$ (non-isotropic materials)

Also frequency dependent

$$\vec{D}(\omega) = \epsilon(\omega) \vec{E}(\omega)$$

Permeability: $\vec{B} = \mu \vec{H}$

Not for ferromagnets, nor superconductors.

For ferromagnets, complicated functional relation.

For superconductors $\vec{B} = 0$ in bulk ("Meissner" effect).

(type I superconductors, and Meissner phase of type II).

Energetics (Garg 85)

Issues:

- Is $E^2+B^2 = e^2+b^2$?

No! There is a lot of E&M energy in binding charges to form molecules, in making the structure of a solid, and so on. None of this is captured by $\vec{E}^2+\vec{B}^2$ not even when compared to averages $\langle e^2+b^2\rangle$ because these are averages over positive definite quantities.

- So there is some internal energy that is not in $\vec{E}$ & $\vec{B}$.
- Dissipation: lose energy, need time averages over small enough times to consider internal energy

So calculate work on free charges. That on bound charges goes into internal energy or lost to heat.

$$\frac{\text{work on free charges}}{\text{time}} = \vec{J}_{free}\cdot\vec{E} = \frac{c}{4\pi}\left(\vec{\nabla}\times\vec{H} - \frac{1}{c}\frac{\partial\vec{D}}{\partial t}\right)\cdot\vec{E}$$

Now $\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = \epsilon_{ijk}\,\partial_i\,(E_jH_k) = \epsilon_{ijk}(\partial_i E_j)H_k + \epsilon_{ijk}E_j\,\partial_i H_k$

$$= \vec{H}\cdot\vec{\nabla}\times\vec{E} - \vec{E}\cdot(\vec{\nabla}\times\vec{H})$$

$$= \vec{H}\cdot\left(-\frac{1}{c}\frac{\partial\vec{B}}{\partial t}\right) - \vec{E}\cdot(\vec{\nabla}\times\vec{H})$$

$$\Rightarrow \boxed{-\vec{\nabla}\cdot\left[\frac{c}{4\pi}\vec{E}\times\vec{H}\right] = \vec{J}_{free}\cdot\vec{E} + \frac{1}{4\pi}\left(\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right)}$$

$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{H}$ is the macroscopic version of Poynting vector.

Recall $\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S} = \frac{\text{work}}{\text{time}}$ microscopically, but here

we have $\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}$ instead of $(2x)\ \frac{\partial}{\partial t}(e^2+b^2)$.

To get beyond this we need constitutive relations.

Then the last term is

$$\frac{1}{8\pi}\frac{\partial}{\partial t}\left(\epsilon E^2 + \mu H^2\right)$$

But this has limited use/validity.

- $\frac{1}{4\pi}\left(\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right)$ includes both energy increase plus power that goes into heat. Cannot generally disentangle. $\Rightarrow$ not $\frac{\partial}{\partial t}$ of a single quantity.

- For validity: ϵ & μ are time independent, really meaning, frequency independent $\epsilon = \epsilon(\omega) \approx$ constant, $\mu = \mu(\omega) \approx$ constant

- For validity, must be in linear regime so the simple constitutive relations apply.

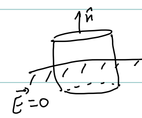
Keep this in mind!

Electrostatics with (around?) conductors ((Garg Chap 14).

We have already covered much of this chapter. These notes focus on new material, and even there, are mostly supplemental → read text!

Summary of main ingredients:

- $\vec{E}=0 \Leftrightarrow \phi =$ constant in conductors
- Charges on conductors live on surface



$E_n \cdot A = 4\pi Q_{enc}$
$E_n = \frac{4\pi Q_{enc}}{A} = 4\pi\sigma$

$$4\pi\sigma = E_n = -\frac{\partial \phi}{\partial n} \qquad \hat{n} = \text{normal to surface}$$

(fields just outside).

- Uniqueness: if ϕ_1 & ϕ_2 solve the boundary value problem (ϕ or $\frac{\partial\phi}{\partial n}$ specified at boundaries) then $\phi_2 = \phi_1$ up to constant (and constant $=0$ if ϕ specified anywhere at a boundary).

Electrostatic energy: From $u = \frac{1}{8\pi}(E^2+B^2)$, for $\vec{B}=0$ and $\vec{E} = -\vec{\nabla}\phi$

one has $$\mathcal{E} = \int_V d^3r\, \frac{1}{8\pi}(\vec{\nabla}\phi)^2 = \int_{\partial V} d^2s\, \frac{1}{8\pi}\phi\frac{\partial\phi}{\partial n} - \frac{1}{8\pi}\int d^3r\, \phi\nabla^2\phi$$

Then: * if the boundary is at ∞, and the fields vanish there, using $\nabla^2\phi = -4\pi\rho$

$$\mathcal{E} = \frac{1}{2}\int d^3r\, \phi(\vec{r})\rho(\vec{r})$$

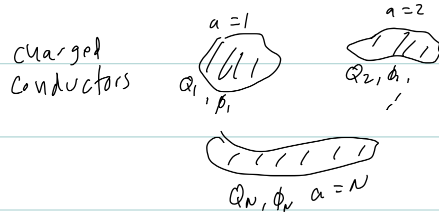
(which you have seen as $\mathcal{E} = \frac{1}{2}\sum_{i\neq j}\frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|} = \frac{1}{2}\sum_{i\neq j} q_i \phi_i(\vec{r}_i)$ where $\phi_i(\vec{r}) = \sum_{j\neq i}\frac{q_j}{|\vec{r}-\vec{r}_j|}$

is the potential due to all charges but q_i at $\vec{r}$)

* If $\rho = 0$ in V but there are conductors bounding V, then $\phi = \phi_a$ $a = 1,\dots,N$ in each of the N surfaces bounding V, and (i) $\phi_a =$ constant on the surface $(\partial V)_a$, and (ii) $\frac{\partial\phi}{\partial n} = 4\pi\sigma$ on that surface (the sign change is because this n points into volume)

$$\Longrightarrow \mathcal{E} = \int_{\partial V} d^2s\, \frac{1}{8\pi}\phi\frac{\partial\phi}{\partial n} = \frac{1}{8\pi}\sum_a \phi_a \int_{\partial V_a} d^2s\,(4\pi\sigma) = \frac{1}{2}\sum_a \phi_a Q_a \quad \text{where } Q_a = \int_{\partial V_a} d^2s\,\sigma .$$

Capacitance:



Problem: given ϕ_a's what are Q_a's?
OR, given Q_a what are ϕ_a (up to additive constant: assume $\phi(\vec{r}) \xrightarrow[|\vec{r}|\to\infty]{} 0$).

The basic result is: this is a linear relation

$$Q_a = \sum_b C_{ab}\, \phi_b$$

C_{ab} = capacitance

or C_{aa} = "capacity" or "capacitance"

C_{ab}, $b \neq a$ = coefficient of electrostatic induction.

AND: C_{ab} depend only on geometry (ie, not on ϕ_a nor Q_a).

This is proved in a wishy-washy manner in textbook (and not at all in Jackson). Here is my argument: consider the Green's function for the Poisson eq.

$$\nabla^2 G = -4\pi\, \delta^{(3)}(\vec{r}-\vec{r}\,')$$

with appropriate boundary conditions (we need Dirichlet, but keep it general for now).

[Note $G = \frac{1}{|\vec{r}-\vec{r}\,'|} + F(\vec{r},\vec{r}\,')$ where $\nabla^2 F = 0$ is chosen to fix boundary conditions).]

Then from Green's 2nd Identity:

$$\int_V d^3r \left(\psi_1 \nabla^2 \psi_2 - \psi_2 \nabla^2 \psi_1 \right) = \int_{\partial V} d^2s \left(\psi_1 \frac{\partial \psi_2}{\partial n} - \psi_2 \frac{\partial \psi_1}{\partial n} \right)$$

with $\psi_1 = \phi$ and $\psi_2 = G$ we have

$$-4\pi\, \phi(\vec{r}) - \int d^3r'\, G(\vec{r},\vec{r}\,')\, \nabla^2 \phi = \int_{\partial V} d^2s' \left(\phi \frac{\partial G}{\partial n'} - G \frac{\partial \phi}{\partial n'} \right)$$

In the case of interest $\nabla^2\phi = 0$ (no charge in V), $G|_{\partial V} = 0$ (Dirichlet), so

$$\phi(\vec{r}) = -\frac{1}{4\pi} \int_{\partial V} d^2r'\, \phi(\vec{r}\,') \frac{\partial G}{\partial n'} = \sum_a \phi_a\, F_a(\vec{r})$$

where $F_a(\vec{r}) = -\frac{1}{4\pi} \int_{\partial V} d^2r' \frac{\partial G}{\partial n'}(\vec{r},\vec{r}\,')$ depends on Geometry but not on ϕ.

From this one can compute $\sigma_b = -\frac{1}{4\pi}\frac{\partial\phi}{\partial n}\Big|_{\partial V_b}$ and $Q_b = \int_{\partial V_b} d^2r\, \sigma_b$

which gives the C_{ba} in terms of $\int_{\partial V_b} d^2r \frac{\partial}{\partial n} F_a(\vec r)$ $\Rightarrow$ purely geometric. END of "proof".

While we are proving things not shown in text nor Jackson: $C_{ab} = C_{ba}$

For this we use "Green's reciprocity": Consider two different charge distributions ρ_1 & ρ_2 and associated potentials $\phi_1(\vec r)$ & $\phi_2(\vec r)$ (for same boundary conditions, including $\phi \to 0$ at ∞).

$$\Rightarrow \int_V d^3r \left(\phi_1 \nabla^2\phi_2 - \phi_2 \nabla^2\phi_1\right) = \int_{\partial V} d^2r \left(\phi_1 \frac{\partial\phi_2}{\partial n} - \phi_2 \frac{\partial\phi_1}{\partial n}\right)$$

For our case, $\rho = 0$ in V, and $\frac{\partial\phi}{\partial n} \propto \sigma$. Moreover $\phi_{1,2}$ on ∂V is constant:

$$0 = \sum \left(\phi_{1a} Q_{2a} - \phi_{2a} Q_{1a}\right)$$

The textbook obtains this in a different way by considering point charges and ignoring singular terms. The rest is as in text:

$$\Rightarrow \quad 0 = \sum_{a,b} \phi_{1a}\phi_{2b}\left(C_{ab} - C_{ba}\right)$$

and arbitrariness in ϕ_1 & ϕ_2 $\Rightarrow$ $C_{ab} = C_{ba}$.

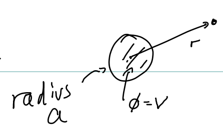
Computations: $Q_a = \sum_b C_{ab}\phi_b$, so one may set $\phi_c = 0$ for all c except $c = b$.

and then compute $Q_a = C_{ab}\phi_b$. This requires solving the boundary value problem $\nabla^2\phi = 0$ (with $\phi|_{\partial V}$ as explained); Q_a is computed from $\sigma_a \propto \frac{\partial\phi}{\partial n}\Big|_{\partial V_a}$.

But only simple geometries can be done analytically.

Example.

(a) Single sphere:



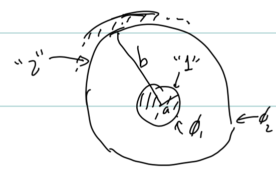
$\phi(\vec r) = \frac{K}{r}$: K set by $\phi\big|_{\partial V} = \phi(\vec r)\big|_{|\vec r| = a} = V \; (= \frac{K}{a})$

$\Rightarrow \phi(\vec r) = V\frac{a}{r}$. Charge on sphere $Q = Va \Rightarrow \boxed{C = a}$

((Can verify $Q = Va$: $\frac{\partial\phi}{\partial n}\Big|_{\partial V} = \frac{\partial\phi}{\partial r}\Big|_{r=a} = -\frac{Va}{a^2} \Rightarrow \sigma = -\frac{1}{4\pi}\left(-\frac{V}{a}\right)$

Then $Q = \int d^2s\, \sigma = (4\pi a^2)\left(\frac{1}{4\pi}\frac{V}{a}\right) = Va$ ✓✓).

(iii) Concentric spheres: $\nabla^2\phi = 0 \Rightarrow \phi = \frac{k_1}{r} + k_2$



So $\phi_1 = \frac{k_1}{a} + k_2$, $\phi_2 = \frac{k_1}{b} + k_2$

$$\Rightarrow k_1 = \frac{\phi_2 - \phi_1}{b^{-1} - a^{-1}} \quad k_2 = \frac{b\phi_2 - a\phi_1}{b-a}$$

Now $\sigma_1 = -\frac{1}{4\pi}\frac{\partial\phi}{\partial r}\Big|_a = \frac{1}{4\pi}\frac{k_1}{a^2}$, $\sigma_2 = \frac{1}{4\pi}\frac{\partial\phi}{\partial r}\Big|_{r=b} = -\frac{1}{4\pi}\frac{k_1}{b^2}$

so that $Q_1 = k_1 = -Q_2$

(Note, we knew this all along since $\vec{E} = 0$ in the interior of conductor "2" so that Gauss's law gives charge enclosed in gaussian surface within "2" $= 0$).

Compute: $Q_1 = C_{11}\phi_1$ (set $\phi_2 = 0$). $C_{11} = \frac{Q_1}{\phi_1} = \frac{k_1}{\phi_1} = \frac{1}{\phi_1}\frac{\phi_2 - \phi_1}{b^{-1} - a^{-1}}\Big|_{\phi_2 = 0} = \frac{1}{a^{-1} - b^{-1}} = \frac{ab}{b-a}$

$Q_1 = C_{12}\phi_2$ (set $\phi_1 = 0$) $\quad C_{12} = \frac{Q_1}{\phi_2} = \frac{k_1}{\phi_2} = -\frac{ab}{b-a}$

$C_{21} = C_{12}$

$Q_2 = C_{22}\phi_2$ (set $\phi_1 = 0$) $\quad C_{22} = \frac{Q_2}{\phi_2} = -\frac{Q_1}{\phi_2} = -C_{12}$

$$\Rightarrow C_{11} = C_{22} = -C_{12} = -C_{21} = \frac{ab}{b-a}$$

Note that these have charges $\pm Q$, so the definition of "capacitance" $Q = C\Delta\phi$ applies:

$$C = \frac{|Q|}{|\phi_2 - \phi_1|} = \left|\frac{k_1}{\phi_2 - \phi_1}\right| = \frac{ab}{b-a}.$$

Some additional comments:

(i) Electrostatic energy $\mathcal{E} = \frac{1}{2}\sum_a Q_a\phi_a = \frac{1}{2}\sum_{a,b} C_{ab}\phi_a\phi_b$

Or with $\phi_a = \sum_b (C^{-1})_{ab} Q_b$, $\mathcal{E} = \frac{1}{2}\sum_{a,b}(C^{-1})_{ab} Q_a Q_b$ (where $C^{-1}\cdot C = \mathbb{1}$ as matrices).

(ii) We have found charges when potentials are specified (Dirichlett problem)

If charges are specified instead, find C_{ab} as before, then $\phi_a = \sum_b (C^{-1})_{ab} Q_b$.

(iii) One can use C_{ab}'s to solve problem with $\rho \neq 0$ and b.c.'s on conducting ∂V. Just solve $\nabla^2\phi = -4\pi\rho$ with grounded conductors 1st, and then add to this $\nabla^2\phi = 0$ with appropriate b.c.'s.

- For two conductors with potential <u>difference</u> V and with charges $\pm Q$ the "capacitance" $\mathcal{C}$ (confusion of terminology ‼) is $Q = \mathcal{C} V$.

(Exercise 88.2)

Relation to C_{ab}: Use $\phi_a = \sum_b (C^{-1})_{ab} Q_b$ and $Q_1 = Q$, $Q_2 = -Q$

so $V = \phi_1 - \phi_2 = (C^{-1})_{11} Q_1 + (C^{-1})_{12} Q_2 - (C^{-1})_{21} Q_1 - (C^{-1})_{22} Q_2$

$= Q\left((C^{-1})_{11} + (C^{-1})_{22} - 2(C^{-1})_{12}\right)$ (used $C^{-1}_{12} = C^{-1}_{21}$).

$\Rightarrow \dfrac{1}{\mathcal{C}} = C^{-1}_{11} + C^{-1}_{22} - 2C^{-1}_{12}$

To write this in terms of C_{ab},

$$(C^{-1}) = \frac{1}{\det C}\begin{pmatrix} C_{22} & -C_{12} \\ -C_{12} & C_{11}\end{pmatrix}$$

so $\dfrac{1}{\mathcal{C}} = \dfrac{1}{\det C}\left(C_{11} + C_{22} + 2C_{12}\right)$

or $\mathcal{C} = \dfrac{C_{11}C_{22} - C_{12}^2}{C_{11} + C_{22} + 2C_{12}}$

The energy stored in the capacitor is $\mathcal{E} = \frac{1}{2}\sum_{a,b} (C^{-1})_{ab} Q_a Q_b$

$= \frac{1}{2} Q^2 \left(C^{-1}_{11} + C^{-1}_{22} - 2C^{-1}_{12}\right)$

$= \frac{1}{2}\dfrac{Q^2}{\mathcal{C}}$

Note added: I just realized C_{ab} is NOT invertible. Why does the text (Garg) as well as the biblical Landau & Lifshitz treat it as such is a mystery. Here is the argument:

We can solve the problem $Q_a = \sum_b C_{ab}\phi_b$ for the case $\phi(\vec{r}) \underset{r\to\infty}{\to} \phi_\infty = \text{arbitrary}$. This just correspond to shifting ϕ of the previous, $\phi_\infty = 0$, solution by a constant. This leaves Q_a unaffected, since it is obtained from a derivative, $\left.\dfrac{\partial\phi}{\partial n}\right|_{\partial V_a}$.

So $Q_a = \sum_b C_{ab}(\phi_b + \phi_\infty)$ is independent of $\phi_\infty \Rightarrow \sum_b C_{ab} = 0 \Rightarrow \det C = 0$

(To see that $\det C = 0$, recall that, considering columns of M as vectors, then $\det M \neq 0$ $\Leftrightarrow$ the vectors are linearly independent. So the columns of C_{ab} are vectors $(\vec{V}^{(b)})_a = C_{ab}$ then $\sum_b C_{ab} = 0$ is $\sum_b \vec{V}^{(b)} = 0$).

For the 2×2 case $C_{ab} = C_{ba}$ and $\sum_b C_{ab} = 0$ implies

$$(C)_{ab} = c\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{for some } c > 0$$

Then $Q = Q_1 = C_{11}\phi_1 + C_{12}\phi_2 = c(\phi_1 - \phi_2) \Rightarrow c = \frac{Q}{\phi_1 - \phi_2}$ <u>is</u> the capacitance

<u>wow!</u>

Methods for solving boundary value problems.

(i) Solve PDE with separation of variables; special functions DONE

(ii) Images

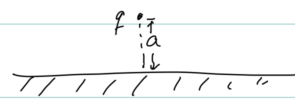
(iii) Green functions (combine the above)

(iv) Numerical

(v) Variational.

Method of Images

By example: point charge with infinite plane conductor:

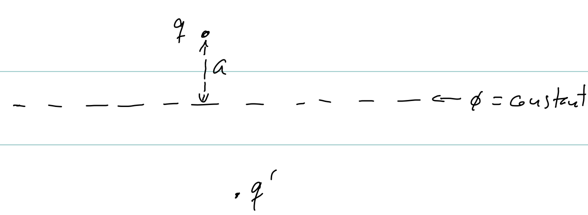


Note: if conductor is finite but ends at distance $L \gg a$, we expect this to be a good approximation

Consider a problem with charge q, a second "image" charge q', and no conductor. We seek to find magnitude of q' and location so that

(i) there exists an equipotential $\phi =$ constant that is a plane a distance a from q

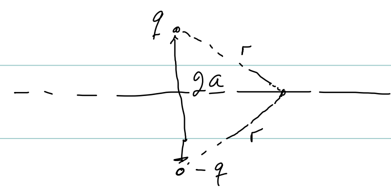
(ii) q' is on the other side of this plane



Then $\phi(\vec{r})$ for this problem is a solution to our problem: it satisfies $\nabla^2\phi = -4\pi\rho$ and the b.c. $\phi =$ const at plane.

In this case the solution is obvious: make $q' = -q$ a distance $2a$ from q

(figure next page)



The points on the mid-plane have potential $\phi = \frac{q}{r} + \frac{(-q)}{r} = 0$

More explicitly, place q at $\vec{r}_0 = (0, 0, a)$ and $-q$ at $-\vec{r}_0$. Then

$$\phi(\vec{r}) = q\left(\frac{1}{|\vec{r}-\vec{r}_0|} - \frac{1}{|\vec{r}+\vec{r}_0|}\right) \qquad (A)$$

Then

$$\phi(\vec{r}) = 0$$

determines a surface: $|\vec{r}-\vec{r}_0| = |\vec{r}+\vec{r}_0| \iff x^2+y^2+(z-a)^2 = x^2+y^2+(z+a)^2$

$\iff \boxed{z=0}$

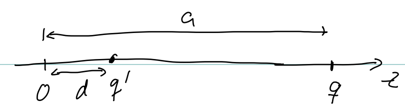
So (A) is a $\phi(\vec{r})$ that gives $\nabla^2\phi = -4\pi\rho$ ($\rho = q\,\delta^3(\vec{r}-\vec{r}_0)$) with

$\phi(\vec{r}) = 0$ on $z=0$.

One can (see text) compute $\sigma = -\frac{1}{4\pi}\frac{\partial\phi}{\partial n}$ to find charge distribution on conductor. Clearly, $\int d^2s\,\sigma = -q$ (from Gauss's law). One can check this.

One may consider some charges, look for an equipotential of some desired shape, and use the charges on one side as "images".

Example:

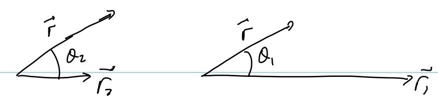


$$\phi(\vec{r}) = \frac{q}{|\vec{r}-\vec{r}_1|} + \frac{q'}{|\vec{r}-\vec{r}_2|} \qquad \vec{r}_1 = (0,0,a) \qquad \vec{r}_2 = (0,0,d)$$

On $|\vec{r}| = R$ (a sphere about origin) we have $\phi = 0$ if

$$q|\vec{r}-\vec{r}_2| = -q'|\vec{r}-\vec{r}_1| \quad \text{ie} \quad q\sqrt{R^2+d^2-2Rd\cos\theta_2} = -q'\sqrt{R^2+a^2-2Ra\cos\theta_1}$$

where $\theta_{1,2}$ are



$\Rightarrow \theta_2 = \theta_1 \equiv \theta$

Take, say $\theta = 0$. Then $-\frac{q'}{q} = \frac{R-d}{a-R}$. If $\theta = \pi$ $-\frac{q'}{q} = \frac{R+d}{R+a}$

$$\Rightarrow \frac{R-d}{a-R} = \frac{R+d}{R+a} \Rightarrow R^2 + R(a-d) - ad = -R^2 + R(a-d) + ad \Rightarrow R^2 = ad \Rightarrow -\frac{q'}{q} = \frac{R-R^2/a}{a-R} = \frac{R}{a}$$

Does this work for general θ? Squaring $q\sqrt{} = -q'\sqrt{}$:

$$a^2(R^2+d^2-2Rd\cos\theta) \stackrel{?}{=} R^2(R^2+a^2-2Ra\cos\theta)$$

$$\Rightarrow a^2R^2 + R^4 - 2\frac{R^3}{a}\cos\theta \stackrel{?}{=} R^4 + R^2a^2 - 2R^3a\cos\theta \quad \underline{\text{Yes}}!$$

So $\phi(\vec{r}) = q\left[\frac{1}{|\vec{r}-\vec{r}_1|} - \frac{R/a}{|\vec{r}-\vec{r}_2|}\right]$ has $\phi = 0$ or $|\vec{r}| = R$ and satisfies

$$\nabla^2\phi = -4\pi q\,\delta^{(3)}(\vec{r}-\vec{r}_1)$$

Third example: conducting sphere in uniform external field $\vec{E}_0$

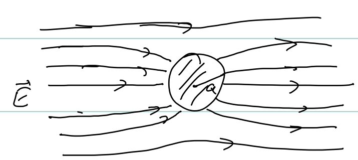
Consider a dipole $\vec{d} = d\hat{z}$ plus a field (superposition) $\vec{E}_0 = E_0\hat{z}$ so that

$\phi(\vec{r}) = \frac{dz}{|\vec{r}|^3} - E_0 z$ (we have put $\vec{d}$ at the origin).

Then the surface $|\vec{r}|=a$ (a sphere of radius a) has $\phi=0$ if $\frac{d}{a^3}=E_0$

So with our image "charge" being a dipole ($\vec{d}$) we have a conducting sphere of radius a in a field $\vec{E}_0 = E_0\hat{z}$ has potential

$$\phi(\vec{r}) = E_0 z\left(\frac{a^3}{|\vec{r}|^3} - 1\right) = -\vec{E}_0\cdot\vec{r}\left(1 - \frac{a^3}{|\vec{r}|^3}\right) \qquad (|\vec{r}| \geq a).$$



The charges have redistributed themselves, $\sigma = -\frac{1}{4\pi}\frac{\partial\phi}{\partial n}$, to create a dipole.

The dipole moment above is $\vec{d} = a^3\vec{E}_0$.

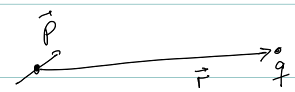
More generally, the σ on a conductor placed in an external field $\vec{E}_0$ produces an induced field that can be expanded in a multipole expansion. The leading term is the dipole (the charge on the conductor is assumed to vanish). The corresponding dipole moment $\vec{p}$ is linear in $\vec{E}_0$, but in general geometries the linearity means $p_i = \alpha_{ij} E_{0j}$, α_{ij} = "polarizability" tensor

In the case above $\alpha_{ij} = \delta_{ij} a^3$.

Moreover, the potential energy of the uncharged conductor in the external field, in the dipole approximation, is

$$\mathcal{E} = -\frac{1}{2}\vec{P}\cdot\vec{E}_0$$

To see this, consider the uncharged conductor in the presence of a point charge q at $\vec{r}$ in a frame with $\vec{P}$ at the origin.



For large $\vec{r}$, the field at the conductor is approximately uniform, $\vec{E}_0 = -\frac{q\vec{r}}{r^3}$

Now $\mathcal{E} = \frac{1}{8\pi}\int_V d^3r\, \nabla\phi\cdot\nabla\phi = \frac{1}{8\pi}\int_{\partial V} d^2r\, \phi\frac{\partial\phi}{\partial n} - \frac{1}{8\pi}\int_V d^3r\, \phi\nabla^2\phi$

Assuming fields vanish at infinity, and using $\nabla^2\phi = -4\pi\rho$

$$\mathcal{E} = \frac{1}{2}\sum_a Q_a\phi_a + \frac{1}{2}q\phi(\vec{r})$$

a generalization of our previous expression for $\mathcal{E}$ that now includes q. Now, we are assuming 1 conductor, with $Q=0 \Rightarrow \mathcal{E} = \frac{1}{2}q\phi(\vec{r}) = \frac{1}{2}q\frac{\vec{P}\cdot\vec{r}}{r^3} = -\frac{1}{2}\vec{P}\cdot\left(-\frac{q\vec{r}}{r^3}\right) = \underline{-\frac{1}{2}\vec{P}\cdot\vec{E}_0}$.

NOTE:

This derivation, taken from L&L, and in Exercise 87.5 of Garg, (implicitly) subtracts a divergence from ϕ of q at q, ie, $\frac{q}{0}$ (coming from $\int \phi\nabla^2\phi$.)

This is why the result is negative even if $u = \frac{1}{8\pi}E^2 > 0$.

I believe this makes this general sounding argument somewhat questionable. In Appendix B of this Unit I compute the total energy of grounded sphere in $\vec{E}_0$ take away the energy of the no-conductor case. The result is

$\mathcal{E} = \frac{1}{6}a^3E_0$

Variational Method

Good for analytic approximation, but for precision look at numerical methods.

It is used in other areas of physics → worth taking a look.

Consider the functional

$$W[\psi] = \int_V d^3r \left[\frac{1}{8\pi} (\vec\nabla\psi(\vec r))^2 - \psi(\vec r)\rho(\vec r) \right]$$

where $\psi(\vec r)$ is piecewise smooth, satisfying Dirichlet b.c. on ∂V.

Then W is minimized by the solution to Poisson, $\nabla^2\psi = -4\pi\rho$ satisfying the b.c.'s.

Trivial to show

$$\delta W = \int d^3r \left[\frac{1}{4\pi} \vec\nabla\psi \cdot \vec\nabla \delta\psi(\vec r) - \rho(\vec r)\,\delta\psi(\vec r) \right]$$

$$= \underbrace{\int_{\partial V} d^2s\, \frac{1}{4\pi} \hat n \cdot (\delta\psi \vec\nabla\psi)}_{=0 \text{ since } \delta\psi = 0 \text{ on } \partial V} - \int_V d^3r\, \delta\psi(\vec r) \left[\frac{1}{4\pi} \nabla^2\psi + \rho \right]$$

at extremum $\Rightarrow \frac{1}{4\pi}\nabla^2\psi + \rho = 0$, as advertised

To see that it is a minimum (not a maximum or saddle point) expand $W[\psi+\delta\psi]$ to order $\delta\psi^2$

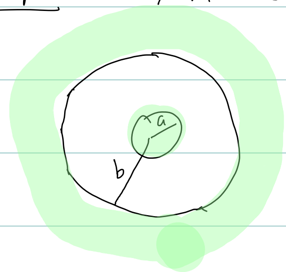
$$\delta^{(2)} W = \int d^3r \frac{1}{8\pi} (\vec\nabla \delta\psi)^2 \geq 0 \quad .$$ Done!

Notes

- To use this, find some functions $\psi_1, \psi_2, \psi_3, \ldots$ possibly with adjustable parameters, that satisfy the b.c.'s. Then minimize $W[\alpha_1\psi_1 + \alpha_2\psi_2 + \cdots]$ w.r.t. $\alpha_1, \alpha_2, \alpha_3, \ldots$ and adjustable parameters.
- If $\rho = 0$ in V, then $W[\psi] = \frac{1}{8\pi}\int d^3r\, \vec E \cdot \vec E$ = electrostatic energy.
- For 2 conductors, if $\phi_1 = 0$, $\phi_2 = 1$ $\Rightarrow$ $W[\psi_{min}] = \frac{1}{2}C(\Delta\phi)^2 = \frac{1}{2}C$

Exercise 91.1

Example: Cylindrical capacitor (circular cross section):



Some trial functions

(i) $\alpha(r-a)$

(ii) $\alpha(r-a)+\beta(r-a)^2$

We need to satisfy $\phi(r=b)=V$.

(i) $\alpha(b-a)=V \Rightarrow \alpha = V/(b-a) \Rightarrow$ no freedom for variation

$$\frac{1}{2}\frac{C}{l} = \frac{W}{l}\left[\frac{V}{b-a}(r-a)\right] = \frac{2\pi}{8\pi}\int_a^b r\,dr\left[\frac{1}{b-a}\right]^2 = \frac{1}{8}\frac{b+a}{b-a} \Rightarrow \frac{C}{l} = \frac{1}{4}\frac{b+a}{b-a}$$

(ii) $\alpha(b-a)+\beta(b-a)^2=V \Rightarrow \alpha = \frac{V}{b-a}-\beta(b-a)$

So Now $(\vec{\nabla}\psi)^2 = \left(\frac{\partial\psi}{\partial r}\right)^2 = (\alpha+2\beta(r-a))^2$

$$W[\beta] = \frac{1}{8\pi}\int_V d^3r\,(\alpha+2\beta(r-a))^2 = \frac{1}{8\pi}\int dz\cdot 2\pi\cdot\int_a^b r\,dr\,(\alpha+2\beta(r-a))^2$$

$$= \frac{1}{4}\int dz\left[\alpha^2\tfrac{1}{2}(b^2-a^2) + 4\alpha\beta\left(\tfrac{1}{3}(b^3-a^3)-\tfrac{1}{2}a(b^2-a^2)\right) + 4\beta^2\left(\tfrac{1}{4}(b^4-a^4) - \tfrac{2a}{3}(b^3-a^3) + a^2\tfrac{1}{2}(b^2-a^2)\right)\right]$$

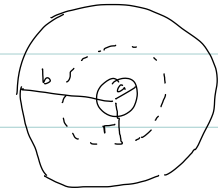
$$\frac{dW}{d\beta} = \frac{\partial W}{\partial\alpha}(-(b-a)) + \frac{\partial W}{\partial\beta} = 0 \Leftrightarrow -(b-a)(b^2-a^2)\left(\frac{V}{b-a}-\beta(b-a)\right) + 4\left(-(b-a)\beta + \frac{V}{b-a}-(b-a)\beta\right)\left(\tfrac{1}{3}(b^3-a^3)-\tfrac{1}{2}a(b^2-a^2)\right)$$
$$+ 8\beta\left[\tfrac{1}{4}(b^2-a^2)(b^2+\tfrac{3}{2}a^2)\right]$$

Solving $\beta = -\frac{V}{b^2-a^2} \Rightarrow \alpha = \frac{2bV}{b^2-a^2}$

$$\Rightarrow \psi(r) = \frac{V}{b^2-a^2}(r-a)\left[2b-r+a\right]$$

$$\frac{1}{2}\frac{C}{l} = \frac{W}{l} = \frac{2\pi}{8\pi}\int_a^b r\,dr\left[\frac{2(b+a-r)}{b^2-a^2}\right]^2 = \frac{1}{12}\frac{a^2+4ab+b^2}{b^2-a^2} \qquad \frac{C}{l} = \frac{1}{6}\frac{a^2+4ab+b^2}{b^2-a^2}$$

The exact solution is elementary: use a gaussian surface



$$E(r)\, l\, 2\pi r = 4\pi Q_{enc} = 4\pi \lambda l$$

$$\Rightarrow \quad E(r) = \frac{2\lambda}{r} \quad \Rightarrow \quad \phi = -2\lambda \ln(r/a)$$

$$Q = C\,\Delta\phi = C\left(-2\lambda \ln(b/a)\right) \qquad (Q = -\lambda l)$$

$$C = \frac{1}{2\ln(b/a)}$$

One may compare the approximate solutions to the exact one by, say, plotting $\frac{C_{approx}}{C_{exact}}$ as a function of $x = \frac{b}{a}$

For $x = 1+\epsilon$, $C_{exact} = \frac{1}{2\ln(1+\epsilon)} \simeq \frac{1}{2\epsilon}$

while $C^{(i)}_{approx} = \frac{1}{4}\frac{x+1}{x-1} = \frac{1}{4}\frac{2}{\epsilon} = \frac{1}{2\epsilon}$

and $C^{(ii)}_{approx} = \frac{1}{6}\frac{6}{(1+\epsilon)^2-1} = \frac{1}{2\epsilon}$

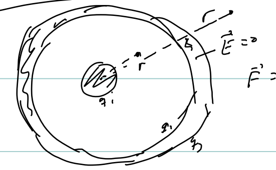
but as $x \gg 1$, $C_{exact} = \frac{1}{2\ln(x)}$

while $C^{(i)}_{approx} \simeq \frac{1}{4}$ and $C^{(ii)}_{approx} = \frac{1}{6}$

Next: Appendices

Not for lecture: unfinished business

Appendix A | Exercise 88.7

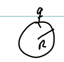


$\vec{E}=0$

$\vec{F} = -\nabla\phi$

$\phi = r^{-k}$ $\quad \nabla\phi = -k\frac{\vec{r}}{r^{k+2}}$

Concentric conducting spheres.

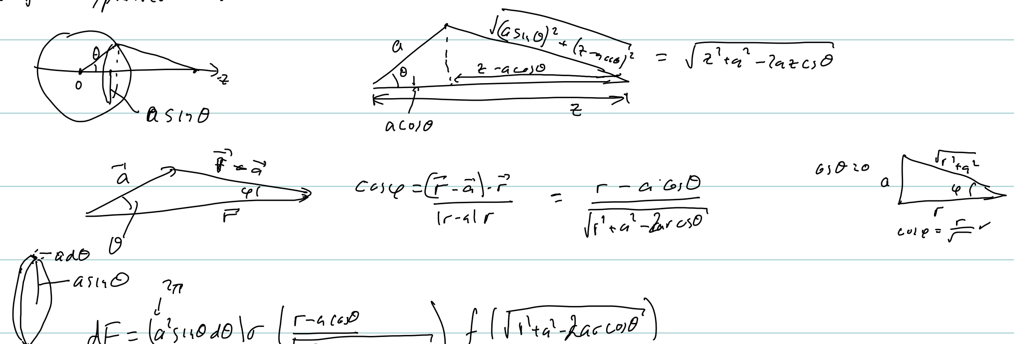


q on surface.

$$\vec{F} = \underset{\uparrow}{K}\frac{q\vec{r}}{r^{3+\eta}} = K_0\left(\frac{R}{r}\right)^{\eta}\frac{q\vec{r}}{r^3}$$

depends on R

Force from spherical shell with uniform density



$\sqrt{(a\sin\theta)^2+(z-a\cos\theta)^2} = \sqrt{z^2+a^2-2az\cos\theta}$

$$\cos\varphi = \frac{(\vec{r}-\vec{a})\cdot\vec{r}}{|r-a|r} = \frac{r-a\cos\theta}{\sqrt{r^2+a^2-2ar\cos\theta}}$$

$\cos\theta=0$ $\quad \cos\varphi = \frac{r}{\sqrt{\;}}$ ✓

$$dF = \overset{2\pi}{(}a^2\sin\theta\, d\theta)\sigma\left(\frac{r-a\cos\theta}{\sqrt{r^2+a^2-2ar\cos\theta}}\right) f\left(\sqrt{r^2+a^2-2ar\cos\theta}\right)$$

$$F = 2\pi a^2\sigma\int_{-1}^{1}dx\,\frac{r-ax}{\sqrt{r^2+a^2-2arx}}\,f\left(\sqrt{r^2+a^2-2arx}\right)$$

$$d\phi = (2\pi a^2\sin\theta\, d\theta)\,\sigma\, V(\sqrt{\;---})$$

$$\phi = 2\pi a^2\sigma\int_{-1}^{1}dx\, V(\sqrt{\;})$$

$$= \frac{2\pi a^2\sigma}{ar}\int_{r-a}^{r+a}du\, u\,V(u)$$

$$= \frac{2\pi a^2\sigma}{ar}\int du\,\frac{u}{u^{1+\eta}}$$

$\frac{u^{1-\eta}}{1-\eta}$

$$= \frac{2\pi a\sigma}{r}\frac{1}{1-\eta}\left[(r+a)^{1-\eta}-(r-a)^{1-\eta}\right)$$

For $\eta=0$

$$= \frac{4\pi a^2\sigma}{r}\ ✓$$

$$u = \sqrt{r^2+a^2-2arx}$$

$$du = \frac{1}{2}\frac{1}{\sqrt{r^2+a^2-2arx}}\,-2ar\,dx \qquad \frac{dx}{\sqrt{r^2+a^2-2arx}} = -\frac{du}{ar} \quad \text{nice.}$$

$$u^2 = r^2+a^2-2arx \Rightarrow x = \frac{r^2+a^2-u^2}{2ar}$$

$$F = \frac{\overset{2\pi}{\vee}a^2\sigma}{ar}\int_{u_{min}}^{u_{max}}du\left(r-\frac{1}{2r}(r^2+a^2-u^2)\right)f(u)$$

Need limits $U_\pm = \sqrt{r^2+a^2\pm2ar} = (r\pm a)$

$$F = \frac{2\pi a\sigma}{2r^2}\int_{r-a}^{r+a}du\left(r^2-a^2+u^2\right)f(u)$$

If $f(u) = \frac{1}{u^p}$

$$F = \frac{2\pi a\sigma}{2r^2}\int_{r-a}^{r+a}du\left[\frac{r^2-a^2}{u^p}+\frac{1}{u^{p-2}}\right] = \frac{a\sigma}{4r^2}\left[(r^2-a^2)\frac{u^{1-p}}{1-p}+\frac{u^{3-p}}{3-p}\right]_{r-a}^{r+a}$$

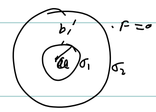
$$= \frac{2\pi a\sigma}{2r^2}\left[\frac{(r^2-a^2)}{1-p}\left((r+a)^{1-p}-(r-a)^{1-p}\right)+\frac{1}{3-p}\left((r+a)^{3-p}-(r-a)^{3-p}\right)\right]$$

$$F = \frac{2\pi a\sigma}{2r^2}\left[\frac{(r^2-a^2)}{1-p}\left((r+a)^{1-p}-(r-a)^{1-p}\right) + \frac{1}{3-p}\left((r+a)^{3-p}-(r-a)^{3-p}\right)\right]$$

For $p=2$

$$F = \frac{2\pi a\sigma}{2r^2}\left[\frac{(r^2-a^2)}{-1}\left(\frac{1}{r+a}-\frac{1}{r-a}\right) + \frac{1}{1}\left((r+a)-(r-a)\right)\right]$$

$$= \frac{2\pi a\sigma}{2r^2}\left[2a+2a\right] = \frac{4\pi a^2\sigma}{r^2} = \frac{Q}{r^2} \checkmark$$

So 

For $V = \frac{q}{r^{1+\eta}} \Rightarrow F = \frac{(1+\eta)q}{r^{2+\eta}}$ $\quad p = 2+\eta, \quad \sigma \to \sigma(1+\eta)$ (irrelevant in ratio)

$$F = \frac{\pi a\sigma}{r^2}\left[\frac{(r^2-a^2)}{-1-\eta}\left[(r+a)^{-1-\eta}-(r-a)^{-1-\eta}\right] + \frac{1}{1-\eta}\left((r+a)^{1-\eta}-(r-a)^{1-\eta}\right)\right]$$

$$\phi = \frac{q}{r^k} \quad F = -\frac{d\phi}{dr} = \frac{kq}{r^{k+1}}$$

$$d\phi = 2\pi a^2\sigma \sin\theta\, d\theta\, f\left(\sqrt{a^2+r^2-2ar\cos\theta}\right)$$

$$\phi = \frac{2\pi a\sigma}{r}\frac{1}{1-\eta}\left[(r+a)^{1-\eta}-(r-a)^{1-\eta}\right) \simeq \frac{2\pi a\sigma}{r}(1+\eta)\left[(r+a)(1-\eta\ln(r+a)) - (r-a)(1-\eta\ln(r-a))\right]$$

$$= \frac{2\pi a\sigma}{r}\left[2a + \eta\left[2a - (r+a)\ln(r+a) + (r-a)\ln(r-a)\right]\right]$$

$$= \frac{2\pi a\sigma}{r}\left[2a + \eta\left[2a - r\ln\frac{r+a}{r-a} - a\ln(r^2-a^2)\right]\right]$$

$2a\ln(2b) - (b+\epsilon)\ln(b+\epsilon) + (b-\epsilon)\ln(b-\epsilon)$ $\quad b = a+\epsilon$

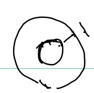
$2\epsilon\ln(2a+2\epsilon) - (2a+\epsilon)\ln(2a+\epsilon) + \epsilon\ln\epsilon$

$2a\left[\cancel{\ln 2a} + \frac{\epsilon}{a}\right] - 2a\cancel{\ln 2a} - \epsilon\ln 2a - 2a\frac{\epsilon}{2a} + \epsilon\ln\epsilon$

$= \epsilon(1-\ln 2\epsilon) + \epsilon\ln\epsilon \quad \times\frac{1}{\epsilon} \to \ln\epsilon + 1 - \ln 2a$!

Set $\phi_{inner} = \phi_{outer}$ ("connected by a wire")

$$\phi(r=a) = \frac{2\pi a\sigma}{a}\left[2a + \eta\left[2a - (a+\epsilon)\ln\frac{2a}{\epsilon} - a\ln(2a\epsilon)\right]\right]$$



Appendix B: Energy of conducting sphere in $\vec{E}_0$ relative to no sphere.

The energy density with conductor relative to that without it is

$$u = \frac{1}{8\pi}\left((\vec{E}_0 + \vec{E}_{ind})^2 - \vec{E}_0^{\,2}\right) \qquad \text{(where } \vec{E}_{ind} \text{ is the induced field)}$$

$$= \frac{1}{8\pi} \vec{E}_{ind} \cdot (2\vec{E}_0 + \vec{E}_{ind})$$

Integrating over all space $\int d^3r\, \vec{E}_0 \cdot \vec{E}_{ind} = \vec{E}_0 \cdot \int d^3r\, \vec{E}_{ind} = 0$

Since $\vec{E}_{ind} = \dfrac{3\vec{p}\cdot\vec{r}\,\vec{r} - r^2\vec{p}}{r^5}$ $\quad \int \vec{E}_{ind} = 0$ (see spherical symmetry argument below)

We are left with $= \frac{1}{8\pi} \int_V d^3r\, \vec{E}_{ind}^{\,2}$

Now $\vec{E}_{ind}^{\,2} = \frac{1}{r^{10}}\left[(\vec{p}\cdot\vec{r})^2 r^2 (9-3-3) + r^4 p^4\right] = \frac{1}{r^8}\left(3(\vec{p}\cdot\vec{r})^2 + r^2 p^2\right)$

Now the volume V is the space exterior to $|\vec{r}| = a$, so it is spherically symmetric around $\vec{r} = 0$; so that

$$\int_V d^3r\, f(r)\, r_i r_j = \int d^3r\, f(r) \frac{1}{3}\delta_{ij} r^2$$

Using this, $\frac{1}{8\pi}\int_V d^3r\, \vec{E}_{ind}^{\,2} = \frac{1}{8\pi}\int_V d^3r \frac{1}{r^8}\left(3 p_i p_j \frac{1}{3}\delta_{ij} r^2 + r^2 p^2\right) = \frac{p^2}{4\pi}\int_V d^3r \frac{1}{r^6}$

$$= \frac{p^2}{4\pi} \cdot 4\pi \int_a^\infty r^2 dr \frac{1}{r^6} = p^2 \frac{1}{3}\frac{1}{a^3} = \frac{1}{3}\vec{p}\cdot\vec{E}_0$$

In subtracting $\vec{E}_0^{\,2}$ from $(\vec{E}_0 + \vec{E}_{ind})^2$ we must also subtract $\int u$ inside the ball (which vanishes for conductor):

$$\int_{|\vec{r}|<a} d^3r \frac{1}{8\pi}E_0^2 = \frac{4\pi}{3}a^3\left(\frac{1}{8\pi}E_0^2\right) = \frac{1}{6}a^3 E_0^2$$

$$\Rightarrow \quad \Delta\mathcal{E} = \left(\frac{1}{3} - \frac{1}{6}\right)\vec{p}\cdot\vec{E}_0 = \frac{1}{6}\vec{p}\cdot\vec{E}_0$$

This disagrees with textbook and Landau & Lifshitz. Ugh!

Furthermore...

The textbook models $\vec{E}_0$ as a charge q at $z=s$ with $q\to\infty$ and $s\to\infty$ keeping $q/s^2 = E_0$ fixed, and including an image charge $q' = -\frac{a}{s}q$ at $z=\frac{a^2}{s}$.

So

$$\phi(\vec{r}) = q\left[\frac{1}{|\vec{r}-s\hat{z}|} - \frac{a/s}{|\vec{r}-\frac{a^2}{s}\hat{z}|}\right]$$

On surface:

$$\sigma = -\frac{1}{4\pi}\left.\frac{\partial\phi}{\partial r}\right|_{r=a} = \frac{q}{4\pi}\left[\frac{a-s\cos\theta}{(a^2+s^2-2as\cos\theta)^{3/2}} - \frac{\frac{a}{s}\left(a-\frac{a^2}{s}\cos\theta\right)}{\left(a^2+\frac{a^4}{s^2}-2\frac{a^3}{s}\cos\theta\right)^{3/2}}\right]$$

$$= \frac{q}{4\pi}\frac{1}{(a^2+s^2-2as\cos\theta)^{3/2}}\left[a-s\cos\theta - \frac{s^2}{a^2}\left(a-\frac{a^2}{s}\cos\theta\right)\right]$$

$$= -\frac{q}{4\pi}\frac{(s^2-a^2)}{a}\frac{1}{(a^2+s^2-2as\cos\theta)^{3/2}}$$

Potential energy = interference term

$$\int d^2r'\,\frac{\sigma(\vec{r}')\,q}{|\vec{r}'-s\hat{z}|} = -\frac{q^2}{4\pi}\frac{(s^2-a^2)}{a}\int_{-1}^{1} 2\pi a^2\,d\cos\theta\,\frac{1}{(a^2+s^2-2as\cos\theta)^2}$$

$$= -\frac{1}{2}q^2 a(s^2-a^2)\cdot\frac{1}{2as}\cdot\left[\frac{1}{(s-a)^2} - \frac{1}{(s+a)^2}\right]$$

$$= -\frac{1}{2}q^2 a(s^2-a^2)\frac{1}{2as}\frac{4as}{(s^2-a^2)^2} = -\frac{q^2 a}{s^2-a^2}$$

Note that this is the same as for the two point charges, $\frac{qq'}{s-a^2/s} = -\frac{q^2\frac{a}{s}}{s-a^2/s}$.

With $q = E_0 s^2$, $\quad \mathcal{E} = -\frac{E_0^2 s^4 a}{s^2-a^2} = -E_0^2 s^2 a - E_0^2 a^3 - O\left(\frac{a^2}{s^2}\right)$.

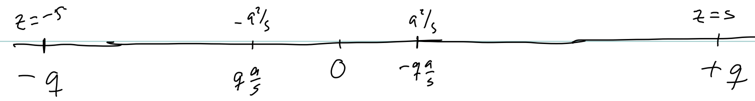
This is not $-\frac{1}{2}\vec{P}\cdot\vec{E}_0$ of exercise 89.5, nor energy in previous exercise.

It diverges. It has not $\frac{1}{2}$. It is negative.

The self-interaction term $\int d^2r\,d^2r'\,\frac{\sigma(\vec{r})\sigma(\vec{r}')}{|\vec{r}-\vec{r}'|} = \frac{q^2}{(4\pi)^2}\frac{(s^2-a^2)^2}{a^2}a^4\int d\Omega\,d\Omega'\,\frac{1}{(a^2+s^2-2as\cos\theta)^{3/2}}\frac{1}{(a^2+s^2-2as\cos\theta')^{3/2}}\frac{1}{|\vec{r}-\vec{r}'|}$

seems convergent and positive, and should be added. No time to compute right now

Another approach: this also has $\phi = 0$ on $|\vec{r}| = a$ and $\vec{E} \simeq$ constant as $s \to \infty$ & $q \to \infty$.



Take $E_0 = \frac{2q}{s^2}$ Then energy to bring in images:

$$\mathcal{E} = \frac{2q(-q\frac{a}{s})}{s - a^2/s} + \frac{2q(q\,a/s)}{s + a^2/s} + \frac{(q\frac{a}{s})(-q\frac{a}{s})}{2a^2/s}$$

$$= q^2 a \left[\frac{-4a^2}{s^4 - a^4} - \frac{1}{2}\frac{1}{sa} \right]$$

$$= -\frac{1}{4} E_0^2 s^4 a \left[\frac{4a^2}{s^4}\left(1 + \frac{a^4}{s^4} + \cdots\right) + \frac{1}{2sa} \right]$$

$$= -\frac{1}{8} E_0^2 s^3 - E_0^2 a^3$$

Check sign: is $\vec{p} = a^3 \vec{E}_0$ or $-a^3 \vec{E}_0$?

$$\phi = -E_0 z\left(1 - \frac{a^3}{r^3}\right) \qquad \vec{E}_{ind} = -\vec{\nabla}\, E_0 a^3 \frac{z}{r^3} = -E_0 a^3 \left(\frac{-3zx}{r^5}, \frac{-3zy}{r^5}, \frac{-3z^2}{r^5} + \frac{1}{r^3} \right)$$

$$= \frac{3\vec{p}\cdot\vec{r}\,\vec{r} - r^2\vec{p}}{r^5} \quad \text{with } \vec{p} = a^3 \vec{E}_0$$

Electrostatics with Dielectrics (Garg: Chap 15)

Basics (from previous chapters):

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho \qquad (\rho = \rho_{free}) \qquad\qquad \vec{\nabla}\times\vec{E} = 0$$

where $\vec{D} = \vec{E} + 4\pi\vec{P}$

We'll use $\vec{D} = \epsilon\vec{E}$ (not a law, not general, good enough for now)

Then, equivalently

$$\epsilon\vec{E} = \vec{E} + 4\pi\vec{P} \quad \text{or} \quad \vec{P} = \frac{\epsilon - 1}{4\pi}\vec{E} \equiv \chi_e\vec{E} \qquad \chi_e = \frac{\epsilon - 1}{4\pi}$$

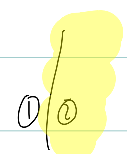
"electric suceptibility"

Boundary Value Problems with Dielectrics:

Since $\vec{\nabla}\times\vec{E} = 0 \Rightarrow \vec{E} = -\nabla\phi$ still.

Then $\vec{\nabla}\cdot\vec{D} = \epsilon\vec{\nabla}\cdot\vec{E}$ (assume ϵ = uniform in media) $\Rightarrow \underline{\nabla^2\phi = -4\pi\rho/\epsilon}$

At interfaces:



ϕ is continuous (so $\vec{E} = -\nabla\phi$ is finite)

$$\phi_1 = \phi_2$$

And $(\vec{D}_2 - \vec{D}_1)\cdot\hat{n}_{21} = 4\pi\sigma \qquad (\sigma = \sigma_{free})$

$$\Rightarrow \epsilon_2\frac{\partial\phi_2}{\partial n} = \epsilon_1\frac{\partial\phi_1}{\partial n}$$

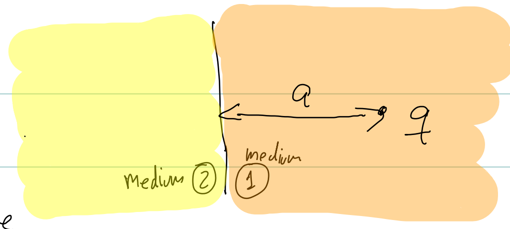
Note $E_{t1} = E_{t2}$ (tangential), from $\vec{\nabla}\times\vec{E} = 0$).

follows from continuity $\phi_1 = \phi_2$

(Note that this does not imply $\frac{\partial\phi_1}{\partial n} = \frac{\partial\phi_2}{\partial n}$).

Example:

Find $\vec{E}$ (or $\vec{D}$) everywhere.

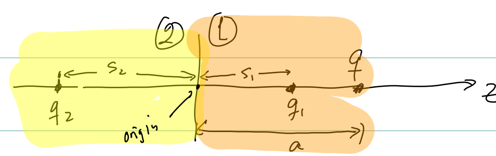


At the expense of just doing busy work, we do 1 example. This one has a solution using method of images. In contrast to analogous conductor case — where we know $\vec{E}=0$ in the conductor, so only need to determine $\vec{E}$ outside, where the free charge is located — here we want $\vec{E}$ (or, equivalently ϕ) both in regions ① & ② of the figure. So now we need

(i) image charge in ② plus q in ① to give ϕ in ①. Call image q_2

(ii) image charge in ①, say q_1, to give ϕ in ②

By symmetry place all in same axis, $\perp$ to plane interface:



In ① $\quad \phi_1(\vec{r}) = \frac{1}{\epsilon_1}\left[\frac{q}{|\vec{r}-a\hat{z}|} + \frac{q_2}{|\vec{r}+s_2\hat{z}|}\right]$

In ② $\quad \phi_2(\vec{r}) = \frac{1}{\epsilon_2}\frac{q_1}{|\vec{r}-s_1\hat{z}|}$

The factors of $\frac{1}{\epsilon}$ are so that $\left(\vec{\nabla}\cdot\vec{D}=4\pi\rho,\ \vec{D}=\epsilon\vec{E},\ \vec{E}=-\vec{\nabla}\phi \Rightarrow \vec{\nabla}^2\phi=-4\pi\rho/\epsilon\right)$.

Continuity at interface $\Rightarrow \phi_1(\vec{r})=\phi_2(\vec{r})$ on $\vec{r}=(x,y,0)$.

No surface charge: $\epsilon_1\frac{\partial\phi_1}{\partial z}=\epsilon_2\frac{\partial\phi_2}{\partial z}$ on $\vec{r}=(x,y,0)$

$$\frac{q}{\sqrt{x^2+y^2+a^2}} + \frac{q_2}{\sqrt{x^2+y^2+s_2^2}} = \frac{\epsilon_1/\epsilon_2\, q_1}{\sqrt{x^2+y^2+s_1^2}}$$

and

$$\frac{aq}{(x^2+y^2+a^2)^{3/2}} - \frac{s_2 q_2}{(x^2+y^2+s_2^2)^{3/2}} = \frac{s_1 q_1}{(x^2+y^2+s_1^2)^{3/2}}$$

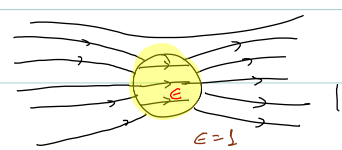
To satisfy for arbitrary $\rho^2 = x^2+y^2 \Rightarrow s_1^2 = s_2^2 = a^2 \Rightarrow s_1 = s_2 = a$ since we chose $s_1>0$ & $s_2>0$.

Then $q + q_2 = \frac{\epsilon_1}{\epsilon_2} q_1$ and $q - q_2 = q_1$

$$\Rightarrow q_1 = \frac{2}{1+\epsilon_1/\epsilon_2}\, q \qquad q_2 = \frac{1-\epsilon_1/\epsilon_2}{1+\epsilon_1/\epsilon_2}\, q$$

(Now one can compute $\vec{E}, \vec{D}, \vec{P} \ldots$

Dielectric sphere in uniform external electric field



radius : a

Much like for conductor.

Sphere suggest use spherical coordinates : center at center of sphere, $\hat{z} = \hat{E}_0$.

Azymuthal symmetry $\Rightarrow$ $Y_{\ell m}$ with $m=0$ only $\to P_\ell$'s.

inside sphere $\phi_{in}(\vec{r}) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$ (no $\frac{1}{r^{\ell+1}}$ terms because $r=0$ included in this region)

outside sphere $\phi_{out}(\vec{r}) = \underbrace{-E_0 r\cos\theta}_{\text{external applied } \vec{E}_0 = \text{constant}} + \sum_{\ell=0}^{\infty} B_\ell \frac{1}{r^{\ell+1}} P_\ell(\cos\theta)$

Conditions:

$\phi_{in} = \phi_{out}\big|_{r=a}$ $\quad A_\ell a^\ell = B_\ell \frac{1}{a^{\ell+1}}$ except $\ell=1$: $\quad A_1 a = -E_0 a + \frac{B_1}{a^2}$

$\epsilon \frac{\partial \phi_{in}}{\partial r}\Big|_a = \frac{\partial \phi_{out}}{\partial r}$ $\Rightarrow$ $\epsilon \ell A_\ell a^{\ell-1} = -(\ell+1) B_\ell a^{-(\ell+2)}$ except $\ell=1$: $\quad \epsilon A_1 = -E_0 - 2B_1 a^{-3}$

$\Rightarrow A_\ell = B_\ell = 0$ except $\ell=1$: $\quad \begin{array}{l} aA_1 - a^{-2}B_1 = -aE_0 \\ \epsilon A_1 + 2a^{-3}B_1 = -E_0 \end{array} \Rightarrow \begin{array}{l} A_1 = -\frac{3}{\epsilon+2}E_0 \\ B_1 = \frac{\epsilon-1}{\epsilon+2}\, a^3 E_0 \end{array}$

Hence $\quad \phi_{in}(\vec r) = -\frac{3}{\epsilon+2} E_0 r\cos\theta \quad \rightarrow$ uniform $\vec E = \frac{3}{\epsilon+2}\vec E_0$ inside sphere!

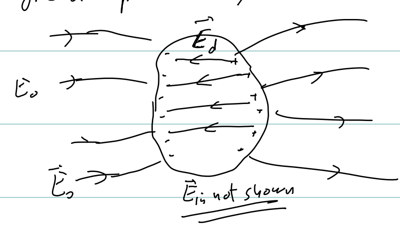
$$\phi_{out} = \underbrace{-E_0 r\cos\theta}_{\text{external } \vec E_0} + \underbrace{\frac{\epsilon-1}{\epsilon+2} E_0 a\left(\frac{a}{r}\right)^2 \cos\theta}_{\phi \text{ of dipole-induced}} \quad : \quad \vec d_{ind} = \frac{\epsilon-1}{\epsilon+2} a^3 \vec E_0$$

Note $\quad \vec P = \frac{\epsilon-1}{4\pi}\vec E = \frac{\epsilon-1}{4\pi}\cdot\left(\frac{3}{\epsilon+2}\right)\vec E_0 = \frac{1}{\frac{4\pi}{3}a^3}\left(\frac{\epsilon-1}{\epsilon+2}a^3\vec E_0\right) = \frac{\vec d_{ind}}{\text{Volume}}$

Depolarization: If $\vec E_{in} = \vec E_0 + \vec E_d$ $\rightarrow$ "depolarization"

$\Rightarrow$ we get $\quad \vec E_d = \left(\frac{3}{\epsilon+2} - 1\right)\vec E_0 = -\frac{\epsilon-1}{\epsilon+2}\vec E_0 = -\frac{4\pi}{3}\vec P$ $\quad \frac{4\pi}{3} \rightarrow$ "depolarization coefficient"

$\vec E_d$ is the field produce by surface charges from aligning dipoles



In our example one can check this: use that $\sigma_{pol} = \vec P\cdot\hat n$ on the surface to compute $\vec E_d$ directly: first $\phi_d(\vec r) = \int_{\partial V} \frac{\sigma(\vec r')}{|\vec r - \vec r'|} d^2r'$ and then take $\vec E_d = -\vec\nabla\phi_d$. The result gives $\vec E_d$ as above.

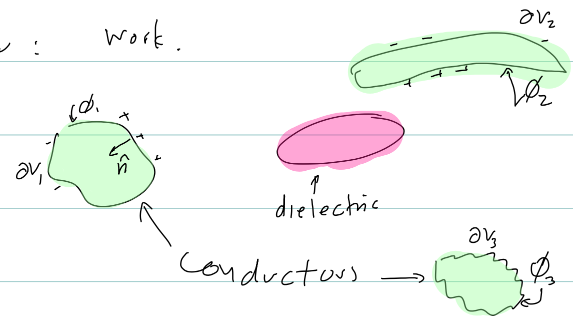
(The calculation uses $\frac{1}{|\vec r - \vec r'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi}{2l+1}\frac{r^l}{r'^{l+1}} Y^*_{lm}(\hat r) Y_{lm}(\hat r')$ at $r' = a > r$

and the integral, with $\sigma(\vec r') = \vec P\cdot\hat n = P\cos\theta$ picks up only $l=1, m=0$ term, so

$\phi(\vec r) = \frac{4\pi}{3} P r\cos\theta$)

Thermodynamics with dielectrics

For 1^{st} law: work.



Add charge δQ_a to conductors held at fixed ϕ_a, adiabatically (no heat exchange)

$$\delta W = \sum_a \phi_a \, \delta Q_a$$

$$= -\sum_a \frac{1}{4\pi} \int_{\partial V_a} \phi \, \delta\vec{D}\cdot\hat{n} \, d^2s \qquad \text{(recall } \sigma = \vec{D}\cdot(-\hat{n})$$

with $(-\hat{n})$ pointing away from conductor)

$$= -\frac{1}{4\pi} \int_V \vec{\nabla}(\phi \, \delta\vec{D}) \, d^3r \qquad \text{(by Gauss's theorem)}$$

Use $\vec{\nabla}\cdot\delta\vec{D} = 0$ $(\rho_{free} = 0)$ and $\vec{E} = -\vec{\nabla}\phi$ $\Rightarrow$

(Note: recall $\vec{\nabla}\times\vec{S} = \frac{1}{4\pi}\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \cdots$)

$$\boxed{\delta W = \frac{1}{4\pi}\int \vec{E}\cdot\delta\vec{D} \, d^3r}$$

Thermodynamic potentials:

let S = entropy, T = temperature, U = internal energy

$\Rightarrow$ 1^{st} law is $\boxed{\delta U = \underbrace{T\,\delta S}_{\text{heat!}} + \frac{1}{4\pi}\int_V \vec{E}\cdot\delta\vec{D}\, d^3r}$ $\qquad U = U(S, \vec{D})$

Now, $F = F(T, \vec{D})$ "free energy" with $F = U - TS$

(so that $\delta F = -S\,\delta T + \frac{1}{4\pi}\int_V \vec{E}\cdot\delta\vec{D}\, d^3r$

Or $\tilde{U} = \tilde{U}(S, \vec{E})$ with $\tilde{U} = U - \frac{1}{4\pi}\int \vec{E}\cdot\vec{D}\, d^3x$

and $\tilde{F} = \tilde{F}(T, \vec{E})$ with $\tilde{F} = \tilde{U} - TS$

Now, $\int \vec{E}\cdot\vec{D}$ in $U - \tilde{U}$ has interesting interpretation.

$$\frac{1}{4\pi}\int_V \vec{E}\cdot\vec{D}\,d^3r = -\frac{1}{4\pi}\int \vec{\nabla}\phi\cdot\vec{D}\,d^3r = -\frac{1}{4\pi}\int_V \vec{\nabla}(\phi\vec{D})\,d^3r \quad (\text{since } \rho_{free}=0)$$

$$= -\frac{1}{4\pi}\int_{\partial V} \phi\vec{D}\cdot\hat{n}\,d^2s$$

$\hat{n}$ this points out of V as in figure previous page

$$= \sum_a \phi_a Q_a$$

2x the energy in the conductors.

So $(\delta F)_T = (\delta U)_S = \sum_a \phi_a\,\delta Q_a$

ie. "at fixed T"

And $\tilde{F} = F - \sum_a \phi_a Q_a \Rightarrow (\delta\tilde{F})_T = (\delta\tilde{U})_S = -\sum_a \delta\phi_a\, Q_a$

Hold $\begin{pmatrix} T \,\&\, Q_a \\ T \,\&\, \phi_a \\ S \,\&\, Q_a \\ S \,\&\, \phi_a \end{pmatrix}$ fixed, system relaxes to minimum of $\begin{pmatrix} F \\ \tilde{F} \\ U \\ \tilde{U} \end{pmatrix}$ in equilibrium

Free energy in linear media: For $D_i = \epsilon_{ij}E_j$

$\Rightarrow \vec{E}\cdot\delta\vec{D} = \frac{1}{2}\delta(\vec{E}\cdot\vec{D}) = (\delta\vec{E})\cdot\vec{D}$ so $\delta U \sim \int \vec{E}\cdot\delta\vec{D}$ can be integrated

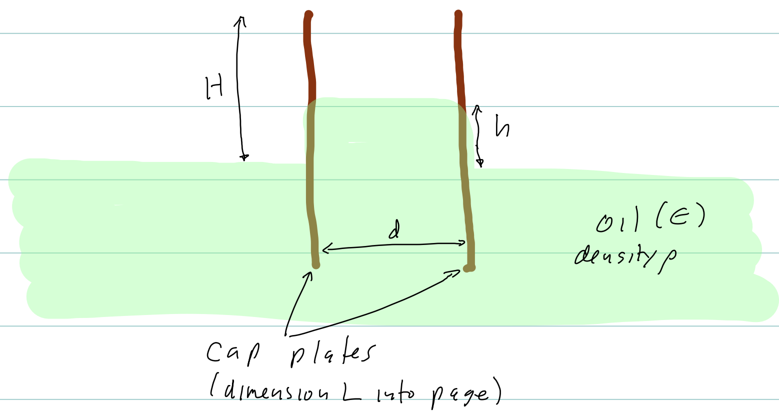
If F_0 = free energy at zero field $\Rightarrow$ $F - F_0 = \int_V \frac{1}{8\pi}\vec{E}\cdot\vec{D}\,d^3r = \frac{1}{2}\sum_a \phi_a Q_a$

similarly $\tilde{F} - \tilde{F}_0 = -\frac{1}{2}\sum_a \phi_a Q_a$

The difference in sign is the familiar effect that the plates of a capacitor held at fixed charge attract ($F = F(T, Q_a)$ decreases – goes towards equilibrium – for ϕ_a decreasing $\rightarrow$ move conductors closer) but capacitors held at fixed potential repel ($\tilde{F} = \tilde{F}(T, \phi_a)$ decreases for Q increasing, but larger Q requires moving plates appart to keep ϕ_a fixed).

What about the dielectric? We used $\vec{D}$ throughout and even assumed linearity $(D_i = \epsilon_{ij} E_j)$, but it is hidden in $\sum \phi_a Q_a$ which depends on free charge Q_a — but ϕ_a's actually will depend on ϵ.

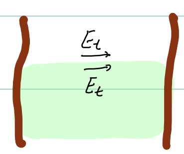
Example: see text for details, here only rough:



Isolated: S = constant. V fixed $\Rightarrow$ minimize $\tilde{U}$ to find equilibrium.

Need gravitational potential energy* $= mgh = \frac{1}{2}\rho(Ldh)gh$

and subtract $\frac{1}{8\pi}\int \vec{E}\cdot\vec{D}$. Need $\vec{E}$.



$E_t|_{air} = E_t|_{oil} \Rightarrow E = \frac{V}{d}$ everywhere

So $\int_V \vec{E}\cdot\vec{D}\,d^3x = [(H-h)Ld]\,\vec{E}\cdot(1\vec{E}) + [hLd]\,\vec{E}\cdot(\epsilon\vec{E}) = \left(\frac{V}{d}\right)^2 Ld\left[h(\epsilon-1) + \text{constant}\right]$

(h-independent)

Minimize (w.r.t. h): $\frac{\tilde{U}}{Ld} = \frac{1}{2}\rho g h^2 - \frac{1}{8\pi}\left(\frac{V}{d}\right)^2(\epsilon-1)h$

$$h_{eq} = \frac{\frac{1}{8\pi}\left(\frac{V}{d}\right)^2(\epsilon-1)}{\rho g}$$

* $\frac{1}{2}$ from $\int_0^h dz\, \rho L d z$.

If ϵ depends on T and on its volume $\epsilon = \epsilon(T, V)$ (depends on V because it may change if compressed, ie if under pressure) then we can write (assuming $\vec{D} = \epsilon \vec{E}$)

$$\tilde{F}(V,T,E) = \tilde{F}(V,T,0) - \frac{1}{8\pi} V_0 E^2 \epsilon(V,T)$$

assumed $\vec{E}$ = uniform and only on volume V_0

Since $\delta \tilde{F} = -S\,\delta T - \frac{1}{8\pi}\int \vec{D}\cdot\vec{E} \Rightarrow S = -\frac{\partial \tilde{F}}{\partial T}$

and $\tilde{U} = \tilde{F} + ST$

$$\Rightarrow S(V,T,E) = S(V,T,0) + \frac{1}{8\pi} V_0 E^2 \frac{\partial \epsilon}{\partial T}$$

and $$\tilde{U}(V,T,E) = \tilde{U}(V,T,0) - \frac{1}{8\pi} V_0 E^2 \left(\epsilon - T\frac{\partial \epsilon}{\partial T}\right)$$

Exercise 97.2 (electrorestriction)

ϵ depends on p (for some materials)

Apply uniform E to volume V of dielectric; in such cases

what is the change in volume ΔV of material?

$G = F + pV$ Gibbs free energy

$\tilde{G}$ idem with E as variable.

As above $d\tilde{G} = V\,dp + \cdots$ $\quad V = \frac{\partial \tilde{G}}{\partial p}$

$$V(p, T, E) = V(p, T, 0) - \frac{1}{8\pi} V_0 E^2 \left(\frac{\partial \epsilon}{\partial p}\right)_T$$

$$\Rightarrow \frac{V(p,T,E) - V(p,T,0)}{V_0} = \frac{\Delta V}{V} = -\frac{1}{8\pi} E^2 \left(\frac{\partial \epsilon}{\partial p}\right)_T$$

Models of ϵ

(Rarefied) gases. First look at colections of molecules that are so far apart (small number density n) that the $\vec{E}$ from some molecules on any one molecule is negligible compared to the applied field (for numbers quantifying this see text).

① Non-polar molecules/atoms (like He, H_2)

If $\vec{d} = \alpha \vec{E}$ for one molecule $\Rightarrow$ $\vec{P} = n\vec{d} = n\alpha\vec{E}$

$\Rightarrow$ $\chi_e = n\alpha$ $\quad \epsilon = 1 + 4\pi n\alpha$

At STP, $n = 2.7 \times 10^{19}\ cm^{-3}$. And $\alpha \sim 1\ \AA^3$

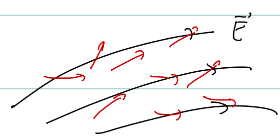
(For H one can use QM to calculate, $\alpha = \frac{9}{2} a_0^3$ $\quad a_0 =$ Bohr's radius)

and $\alpha_H = 6.7 \times 10^{-25}\ cm^3 = 0.67\ \AA^3$).

Then $n\alpha \sim 10^{-5}$ and $\epsilon - 1 \sim 10^{-4}$

② Polar molecules (like H_2O).

These can be polarized too, so have an $\alpha\vec{E}$ contribution as above. But also have a permanent $\vec{d}_0$ dipole moment



Thermal fluctuations about alignment with $\vec{E}$. Need thermal average $\langle \vec{d} \rangle$.

Assume $\vec{E}$ = uniform (on scale of interest). $U(\theta) = -\vec{d}_0 \cdot \vec{E} = -d_0 E \cos\theta$

$$\langle \vec{d} \rangle = \frac{\int d\Omega\, (d_0 \cos\theta\, \hat{E})\, e^{d_0 E \cos\theta / k_B T}}{\int d\Omega\, e^{d E_0 \cos\theta / k_B T}} \quad ; \quad \frac{d_0 E}{k_B T} \ll 1 \to \text{expand in powers to linear order}$$

since $\int d\Omega \cos\theta = 0$

$$\Rightarrow \quad \langle \vec{d} \rangle = \hat{E} \quad \frac{\int d\Omega \, d_0 \cos\theta \left(\frac{d_0 E \cos\theta}{k_B T} \right)}{\int d\Omega} = \frac{d_0^2 \vec{E}}{3 k_B T}$$

The rest is as above : $\chi = n\left(\alpha + \frac{d_0^2}{3k_BT}\right)$

$$\epsilon = 1 + 4\pi n \left(\alpha + \frac{d_0^2}{3k_BT}\right)$$

Plot vs $1/T$ $\rightarrow$ get both α and d_0 !

Typical numbers

steam at 400K, $n = 10^{19}\,\text{cm}^{-3}$, $\frac{d_0^2}{3k_BT} \simeq 2.1 \times 10^{-23}\,\text{cm}^3$

so $\epsilon - 1 \simeq 4\pi \times 2.1 \times 10^{-4} \sim 2 \times 10^{-3}$

and $d_0 \sim 2 \times 10^{-18}$ esu$\cdot$cm

Back to rarefied approximation.

Dipole field $E_{dip} \sim \frac{d}{r^3}$. With density n, typical distance

is $n^{-1/3}$ so $E_{dip} \sim \frac{d}{(n^{-1/3})^3} = nd = n\alpha E$

So $\frac{E_{dip}}{E} = n\alpha$ (or $n\left(\alpha + \frac{d_0^2}{3k_BT}\right)$ for polarizable case)

So approximation is good provided $\frac{\epsilon - 1}{4\pi} \ll 1$.

Dense dielectrics.

If $n\alpha \gtrsim 1$ the field of nearby molecules on a molecule cannot be neglected. (compared to the ambient $\vec{E}$). We need to go back to the microscopic description in terms of $\vec{e}$.

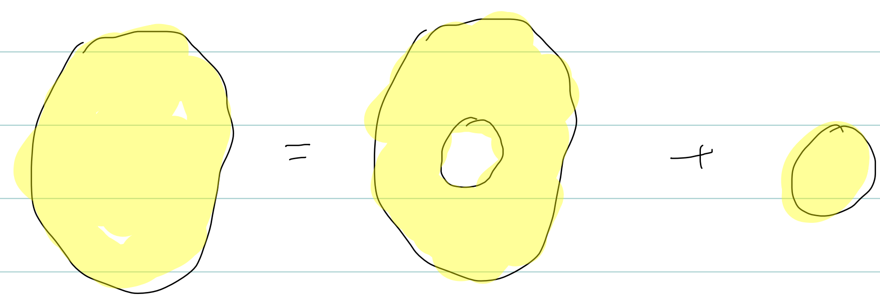
Now, we break the effect of $\vec{e}$ into two pieces

$$\vec{e} = \vec{e}_{near} + \vec{e}_{far}$$

where $\vec{e}_{near}$ is from a ball of radius a around the molecule of interest, and $\vec{e}_{far}$ from outside the ball. The ball is large enough that the effect of molecules outside the ball can be averaged as the field due to a polarizable material from which the ball has been removed, plus the applied field:

$$\vec{e}_{far} = \vec{E}_{far}$$

Here is the picture:



Yet a is small enough that the field $\vec{E}_{far}$ is uniform in it.

If $\vec{E}$ is the macroscopic field at the center of the ball we can write :

$$\vec{E} = \vec{E}_{far} + \vec{E}_{near}$$

where $\vec{E}_{near}$ is the field we would have if the ball is in an applied uniform external field $\vec{E}_{far}$.

But from the example in pp. 3–4 of this note, the field in the ball is $\vec{E} = \vec{E}_0 + \vec{E}_d$ where $\vec{E}_0$ is the applied uniform field and $\vec{E}_d$ is the depolarization field $\vec{E}_d = -\frac{4\pi}{3}\vec{P}$. In our case $\vec{E}_0 = \vec{E}_{far}$ and $\vec{E} = \vec{E}_{near}$ so

so $$\vec{E} = \vec{E}_{far} + \vec{E}_{near} = \vec{E}_{far} - \frac{4\pi}{3}\vec{P}$$

$$\Rightarrow \vec{E}_{far} = \vec{E} + \frac{4\pi}{3}\vec{P}$$

We still have to account for $\vec{e}_{near}$. This depends on the specific arrangement of molecules inside the ball.

Suppose the molecules are electric dipoles all of same magnitude an all aligned, and placed in a cubic lattice.

$$\vec{e}_{near} = \sum_n \frac{3(\vec{r}_n\cdot\vec{d})\vec{r}_n - r_n^2\vec{d}}{r_n^5}$$

where $\vec{r}_n$ are the locations of the vertices on the cubic lattice centered at the molecule of interest. This vanishes. To see this in a pedestrian way, take $\vec{r} = b(n,m,k)$ and consider $\sum \frac{3r_ir_j - \delta_{ij}r^2}{r^5}$:

$i=j=1$ $$\frac{1}{b^3}\sum_{n,m,k}\frac{3n^2 - (n^2+m^2+k^2)}{(n^2+m^2+k^2)^{5/2}} = \frac{1}{b^3}\sum_{n,m,k}\frac{2n^2-m^2-k^2}{(n^2+m^2+k^2)^{5/2}}$$

$n^2+m^2+k^2=1 \Rightarrow 8\frac{1}{b^3}\left[(2-0)+(0-1)+(0-1)\right]=0$ (8 is from ±1's in $(n,m,k) = (\pm1,0,0), (0,\pm1,0), (0,0,\pm1)$).

$n^2+m^2+k^2=2 \Rightarrow \frac{1}{b^3}\left[(2-1-0)+(2-0-1)+(0-1-1)\right]=0$ (from here on, ignore multiplicity).

$n^2+m^2+k^2=3 \Rightarrow \frac{1}{b^3}\left[(2-1-1)\right]=0$

and so on

For $i\neq j$ $$\frac{1}{b^3}\sum_{n,m,k}\frac{3nm}{(n^2+m^2+k^2)^{5/2}} = \frac{1}{b^3}\sum_{n,k}3n\sum_m\frac{m}{(n^2+m^2+k^2)^{5/2}} = 0$$ from $m=\pm|m|$ pairs.

Whether $\vec{e}_{near}$ vanishes depends on the specific lattice.

For a liquid with dipoles at random locations $\langle \vec{e}_{near} \rangle = 0$ too. This follows from

$$\langle r_i r_j \rangle = \frac{1}{3} \delta_{ij} r^2$$

on average.

Assuming $\vec{e}_{near}$ vanishes, then $\vec{e} = \vec{e}_{near} + \vec{e}_{far} = \vec{e}_{far} = \vec{E}_{far} = \vec{E} + \frac{4\pi}{3}\vec{P}$

Using $\vec{P} = \frac{\epsilon - 1}{4\pi}\vec{E}$ then

$$\vec{e} = \left(1 + \frac{1}{3}(\epsilon - 1)\right)\vec{E} = \frac{\epsilon + 2}{3}\vec{E}$$

To finish the calculation take α = molecular polarizability

n = number density, so $\vec{P} = n\alpha\vec{e} = n\alpha\frac{\epsilon+2}{3}\vec{E}$.

This gives $\chi_e = n\alpha\frac{\epsilon+2}{3}$ and therefore

$$\epsilon = 1 + 4\pi n\alpha \frac{\epsilon+2}{3}$$

or $\epsilon\left(1 - \frac{4\pi}{3}n\alpha\right) = 1 + \frac{8\pi n\alpha}{3} \Rightarrow \epsilon = \frac{1 + \frac{8\pi n\alpha}{3}}{1 - \frac{4\pi n\alpha}{3}}$

or $\boxed{\epsilon = 1 + \frac{4\pi n\alpha}{1 - \frac{4\pi n\alpha}{3}}}$ "Clausius-Mossotti"

One can determine α from a dilute gas of a material and then compute ϵ for a liquid (dense) of the same material. It works pretty well. See table 15.3. For example for CS_2 ϵ(Clausius-Mossotti) = 2.75 vs $\epsilon(exp)$ = 2.64 (at $n = 10^{22}\,cm^{-3}$).

The CM formula fails as $\frac{4\pi n\alpha}{3} \to 1$. For polar materials this is the case. But this is far beyond the scope of our study here.

Frequency Dependent Response of Materials

Preamble: General treatment of response functions, or "The generalized susceptibility"

(Taken from Landau & Lifshitz, Stat. Phys. 1, Sec 123).

Let $s(t)$ describe the state of a system, on which a "force" $f(t)$ acts through a "susceptibility" $\chi(t)$. In our applications we have cases where this terminology is very appropriate, e.g., for $s(t)$ could be $\vec{J}(t)$, and $f(t)$ the electric field $\vec{E}(t)$ (literally a force).

We will use Fourier transforms and their inverses for all quantities:

eg $\quad s(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{s}(\omega) \quad , \quad \tilde{s}(\omega) = \int_{-\infty}^{\infty} dt \, e^{i\omega t} s(t)$

Now $$s(t) = \int_{-\infty}^{t} dt' \, \chi(t-t') f(t')$$

Note that the integral goes up to $t'=t$ and no further because causality dictates that the force $f(t')$ does not affect $s(t)$ for times $t'>t$.

Alternatively, $$s(t) = \int_{-\infty}^{\infty} dt' \, \chi(t-t') f(t')$$

with $\chi(t-t') = 0$ for $t-t'<0$, i.e., $\chi(t)=0$ for $t<0$.

Now $\int_{-\infty}^{\infty} dt' f(t')\chi(t-t') = \int_{-\infty}^{\infty} dt' \int \frac{d\omega_1}{2\pi}\frac{d\omega_2}{2\pi}\, e^{i\omega_1 t'} e^{i\omega_2(t-t')} \tilde{f}(\omega_1)\tilde{\chi}(\omega_2)$

$$= \int \frac{d\omega_1}{2\pi}\int \frac{d\omega_2}{2\pi}\, \tilde{f}(\omega_1)\tilde{\chi}(\omega_2)\, e^{i\omega_2 t} \underbrace{\int dt'\, e^{i(\omega_1-\omega_2)t'}}_{=2\pi\,\delta(\omega_1-\omega_2)}$$

$$= \int \frac{d\omega_1}{2\pi}\, e^{i\omega_1 t}\, \tilde{f}(\omega_1)\tilde{\chi}(\omega_1)$$

So that $\quad \tilde{s}(\omega) = \tilde{\chi}(\omega)\tilde{f}(\omega)$

Examples:

For conductors $\tilde{\vec{j}}(\omega) = \tilde{\sigma}(\omega)\tilde{\vec{E}}(\omega)$, for dielectrics $\tilde{\vec{D}}(\omega) = \tilde{\epsilon}(\omega)\tilde{\vec{E}}(\omega)$

We want to study properties of $\tilde{\chi}(\omega)$. Let $\tilde{\chi}(\omega) = \tilde{\chi}_1(\omega) + i\,\tilde{\chi}_2(\omega)$

that is $\tilde{\chi}_1 = \mathrm{Re}\,\tilde{\chi}$ and $\tilde{\chi}_2 = \mathrm{Im}\,\tilde{\chi}$. Since $\chi(t) = 0$ for $t<0$ we have

$$\tilde{\chi}(\omega) = \int_0^{\infty} dt\, e^{i\omega t}\chi(t)$$

Moreover, $\chi(t)$ is real. Therefore

1. $\tilde{\chi}^*(\omega) = \tilde{\chi}(-\omega)$

That is $\chi_1(-\omega) = \chi_1(\omega)$ and $\chi_2(-\omega) = -\chi_2(\omega)$ (even & odd functions).

For $\chi(t)$ having support over some interval of size T we expect $\tilde{\chi}(\omega)$ to have support over $\Delta\omega \sim \frac{1}{T}$, so $\tilde{\chi}(\omega) \to 0$ as $\omega\to\infty$. Some care is required for $T\to\infty$ (as in $\chi(t) = 1$) and for $T\to 0$ (as is $\chi(t) = \delta(t-t_1)$) but we'll assume $\lim_{\omega\to\infty}\tilde{\chi}(\omega) = 0$.

2. $\omega \chi_2(\omega) > 0$

That is $\chi_2(\omega) > 0$ for $\omega > 0$.

The proof relies on the 2nd Law of Thermodynamics and the interpretation of f as a generalized force and s as a generalized displacement. In the absence of $f(t)$, the evolution of s is determined by a Hamiltonian H_0, and the effect of f is described as a perturbation $H' = -sf(t)$. The sign is so that $\dot{p} = -\frac{\partial H}{\partial s} = f$.

Now f acts on body, and changes to the state of the body are accompanied by dissipation (heat lost in the process). Then

$$\frac{dE}{dt} = \frac{\partial H}{\partial t} = -s\frac{df(t)}{dt}$$

Now for any two functions

$$\int_{-\infty}^{\infty} dt\, a(t) b(t) = \int \frac{d\omega}{2\pi} \tilde{a}(\omega)\tilde{b}(-\omega) = \frac{1}{2}\int \frac{d\omega}{2\pi}\left[\tilde{a}(\omega)\tilde{b}(-\omega) + \tilde{a}(-\omega)\tilde{b}(\omega)\right]$$

So $$\Delta E = \int \frac{dE}{dt} dt = -\frac{1}{2}\int \frac{d\omega}{2\pi}\left[\tilde{s}(\omega)\left[-i(-\omega)\tilde{f}(-\omega)\right] + \tilde{s}(-\omega)\left(-i\omega\tilde{f}(\omega)\right)\right]$$

now use $\tilde{s} = \tilde{\chi}\tilde{f}$ $\rightarrow$ $$= -\frac{i}{2}\int \frac{d\omega}{2\pi}\left[\left(\tilde{\chi}(\omega) - \tilde{\chi}(-\omega)\right)\omega\, \tilde{f}(-\omega)\tilde{f}(\omega)\right]$$

$$= -\frac{i}{2}\int \frac{d\omega}{2\pi}\left(2i\chi_2(\omega)\right)\omega\,|\tilde{f}(\omega)|^2$$

Now, $\tilde{f}(\omega)$ is arbitrary and $\Delta E > 0$ $\Rightarrow$ $\omega\chi_2(\omega) > 0$.

Analytic continuation: extend definition of $\tilde{\chi}(\omega)$ to complex argument, $\omega = \omega_1 + i\omega_2$.

3. $\tilde{\chi}(\omega)$ is analytic for $Im(\omega) > 0$.

Because
$$\tilde{\chi}(\omega) = \int_0^\infty dt\, e^{i\omega_1 t}\, e^{-\omega_2 t}\, \chi(t)$$

and the integral converges provided $\omega_2 > 0$ (since $\tilde{\chi}(\omega)$ for real ω is assumed to exist for some range of ω, we need not worry about the integral not converging because $\chi(t) \sim \exp(ct)$). Moreover
$$\frac{d^n \tilde{\chi}(\omega)}{d\omega^n} = i^n \int_0^\infty dt\, e^{i\omega_1 t}\, e^{-\omega_2 t}\, t^n \chi(t)$$
also converges ($e^{-\omega_2 t} t^n \to 0$ as $t \to \infty$ for any n).

Note that this is a consequence of <u>causality</u> (we used $\chi(t) = 0$ for $t < 0$).

1'. $\tilde{\chi}(-\omega^*) = \tilde{\chi}^*(\omega)$

Is the generalization of (1) to complex argument.

Then $\tilde{\chi}_1(-\omega_1 + i\omega_2) = \chi_1(\omega_1 + i\omega_2)$ and $\tilde{\chi}_2(-\omega_1 + i\omega_2) = -\chi_2(\omega_1 + i\omega_2)$

In particular, on the imaginary axis $\chi_2(i\omega_2) = 0 \Rightarrow \chi(i\omega_2)$ is real.

5. For $\omega_2 > 0$ (upper half plane), $\tilde{\chi} \neq 0$, except on $\omega_1 = 0$ (imaginary axis). For $\omega_1 = 0$, $\tilde{\chi}_1(i\omega_2)$ is monotonically decreasing from $\chi_0 = \tilde{\chi}(i0)$ to $\tilde{\chi}(i\infty) = 0$.

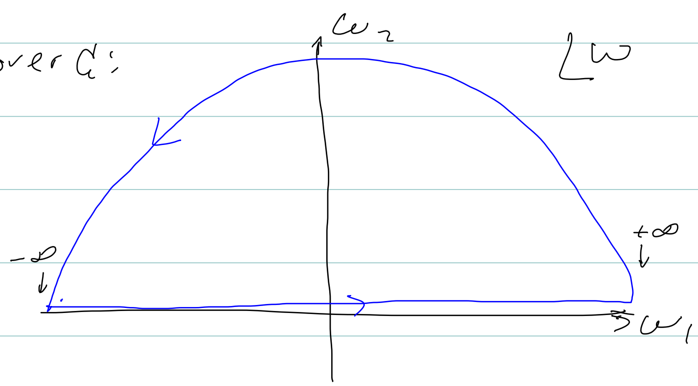
Therefore $\tilde{\chi}(\omega)$ has no zeroes in upper half plane.

Proof: From complex analysy $\frac{1}{2\pi i}\oint_c dz \frac{f'(z)}{f(z)}$.

$N_{z(p)}$ = number of zeroes (poles) of $f(z)$ in region interior to C.

Consider $I \equiv \frac{1}{2\pi i}\oint_c d\omega \frac{d\tilde{\chi}(\omega)}{d\omega} \frac{1}{\tilde{\chi}(\omega) - X}$ where X is real and

the integral is over C:



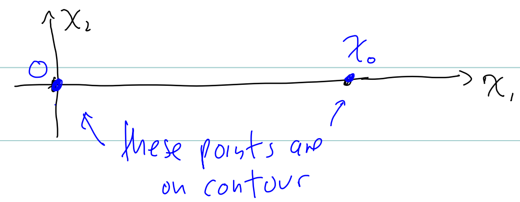
Now, for upper half plane, χ is analytic $\Rightarrow$ so is $\frac{d\chi}{d\omega}$. So in the statement about complex analysis above, $f = \chi(\omega) - X$ is analytic ($N_p = 0$) and therefore I = number of zeroes of $\chi(\omega) - X$ = number of times $\chi(\omega)$ takes on the real value X.

Now compute I: change variables:

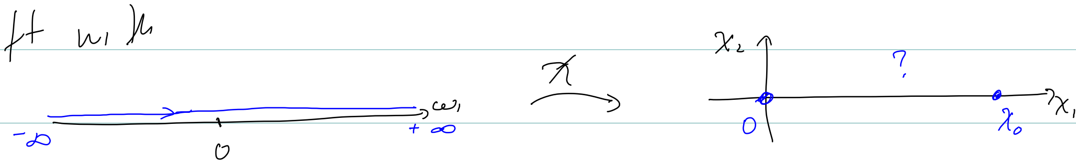
$$I = \frac{1}{2\pi i}\oint_{C'} d\chi \frac{1}{\chi - X}$$

Let's figure out C'

Since C goes through $i\omega_2 = 0+$ and $i\omega_2 = +i\infty$, we start with those $\chi(i0+) = \chi_0$ and $\chi(i\infty) = 0$

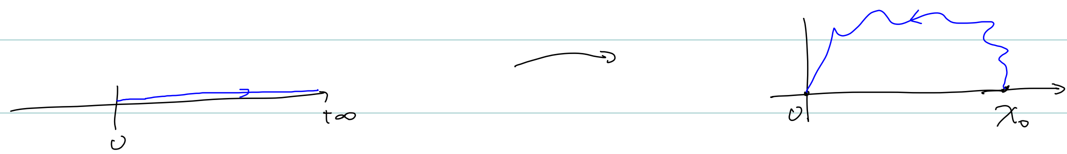


Actually, all of the semi-circle of C maps to 0, so we are left with

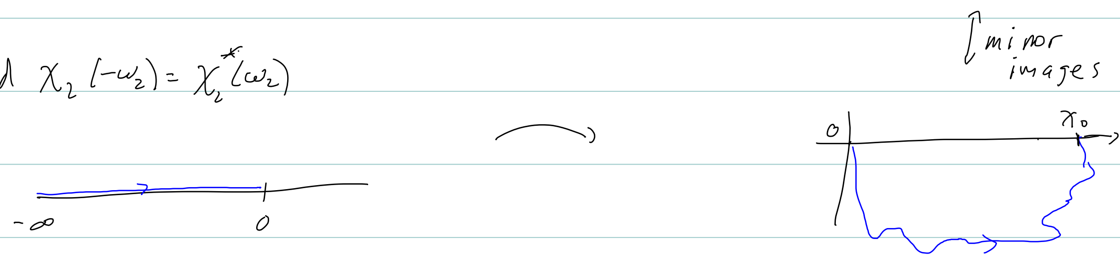


Now $\chi_2 > 0$ for $\omega_1 > 0$ and $\chi_2 < 0$ for $\omega_1 < 0$ by (2).

So



and $\chi_2(-\omega_2) = \chi_2^*(\omega_2)$



The point is C' crosses the real axis only at 0 and χ_0.

So, $I = 1$ for all values of χ, $0 < \chi < \chi_0$ and $I = 0$ otherwise.

To complete the argument (1) Since χ is real on the positive imaginary axis, and it is analytic and goes from χ_0 to 0 on the axis, it must take on every value in the interval $(0, \chi_0)$ along the axis. But it takes on each value only once. It must have $\chi_2 \neq 0$ everywhere else on the upper half plane $\Rightarrow \tilde{\chi}_2 \neq 0$ except at $+i\infty$.

(ii) Since it takes on every value in $(0, \chi_0)$ only once, $\chi(i\omega_2) = \chi_1(i\omega_2)$ cannot have a local minimum or maximum along the line: it is monotonic.

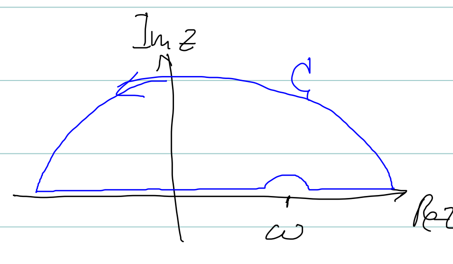
(iii) Since $\tilde{\chi}_2 \neq 0$ everywhere (on upper half plane) except on the imaginary axis, and there $\tilde{\chi}_1(\omega) \neq 0$ except at $\omega = +i\infty$, we have $\tilde{\chi}(\omega) \neq 0$ (except at $+i\infty$).

—— , ——

6. Kramers - Kronig relation.

Consider

$$\frac{1}{2\pi i} \oint_C dz \, \frac{\tilde{\chi}(z)}{z-\omega} \quad \text{for } C:$$



Since $\tilde{\chi}(z)$ is analytic for $\mathrm{Im}(z) > 0$ and ω is outside C, there are no poles inside C: by Cauchy's theorem the integral vanishes.

The integral over the small semicircle is, with $z = \omega + \epsilon e^{i\phi}$

$$\lim_{\epsilon \to 0} \int_{\pi}^{0} \epsilon e^{i\phi} \, i \, d\phi \, \frac{\tilde{\chi}(\omega + \epsilon e^{i\phi})}{\epsilon e^{i\phi}} = -i\pi \tilde{\chi}(\omega)$$

The integral over the real axis is the principal value integral. So

$$0 = -i\pi \tilde{\chi}(\omega) + P \int_{-\infty}^{\infty} d\omega' \, \frac{\chi(\omega')}{\omega' - \omega}$$

or

$$\tilde{\chi}(\omega) = -\frac{i}{\pi} P \int_{-\infty}^{\infty} d\omega' \, \frac{\chi(\omega')}{\omega' - \omega}$$

Taking Im or Re of this equation:

$$\tilde{\chi}_2(\omega) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \frac{\tilde{\chi}_1(\omega')}{\omega' - \omega}$$

Kramers-Kronig relations

$$\tilde{\chi}_1(\omega) = \frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \frac{\tilde{\chi}_2(\omega')}{\omega' - \omega}$$

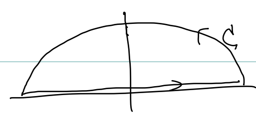
Then, $\tilde{\chi}_1(\omega)$ completely fixes $\tilde{\chi}_2(\omega)$, and viceverza.

Many additional results follow.

Exercises:

(i) Show $\tilde{\chi}_1(\omega) = \frac{2}{\pi} P \int_0^{\infty} d\omega' \frac{\omega' \tilde{\chi}_2(\omega')}{\omega'^2 - \omega^2}$

(ii) By considering $\oint_C dz \frac{z\tilde{\chi}(z)}{z^2+\omega^2}$ for real ω, along a contour



, show $\int_{-\infty}^{\infty} d\omega' \frac{\omega' \tilde{\chi}(\omega')}{\omega'^2 + \omega^2} = i\pi \tilde{\chi}(i\omega)$

Use this to show $\tilde{\chi}(i\omega) = \frac{2}{\pi} \int_0^{\infty} \frac{\omega' \tilde{\chi}_2(\omega')}{\omega'^2 + \omega_0^2}$

(iii) Use the previous result (integrate over ω) to show

$$\int_0^{\infty} d\omega\, \tilde{\chi}(i\omega) = \int_0^{\infty} \tilde{\chi}_2(\omega)\, d\omega$$

To derive these formulae (including Kramers-Kronig) we only used the fact that $\tilde{\chi}(\omega)$ is regular in the upper half-plane. If in addition we know $\tilde{\chi}^*(\omega) = \tilde{\chi}(-\omega^*)$ so $\tilde{\chi}(i\omega) = \tilde{\chi}_1(i\omega)$ we have $\int_0^{\infty} d\omega\, \tilde{\chi}_1(i\omega) = \int_0^{\infty} \tilde{\chi}_2(\omega)\, d\omega$

Frequency dependent conductivity

Ohm's Law: $\tilde{\vec{j}}(\omega) = \tilde{\sigma}(\omega)\tilde{\vec{E}}(\omega)$

$\tilde{\sigma}(\omega)$: frequency dependent conductivity.

where, e.g. $\vec{j}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \tilde{\vec{j}}(\omega)\, e^{-i\omega t}$

Aside on Fourier Transform of a product.

with $a(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t}\, \tilde{a}(\omega)$ and $b(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t}\, \tilde{b}(\omega)$

where $\tilde{a}(\omega) = \int dt\, e^{i\omega t}\, a(t)$

Then, the (I)FT of the product is

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t}\, a(\omega) b(\omega) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \int_{-\infty}^{\infty} dt_1\, e^{i\omega t_1} a(t_1) \int_{-\infty}^{\infty} dt_2\, e^{i\omega t_2}\, b(t_2)$$

$$= \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2\, a(t_1)\, b(t_2) \underbrace{\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega(t_1+t_2-t)}}_{=\delta(t_1+t_2-t)}$$

$$= \int_{-\infty}^{\infty} dt_1\, a(t_1)\, b(t-t_1)$$

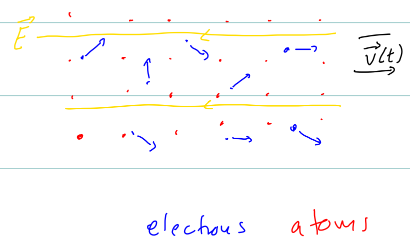
Ohm's law in time domain

$$\vec{j}(t) = \int_{-\infty}^{\infty} dt'\, \sigma(t-t')\, \vec{E}(t')$$

The "response" function $\sigma(t-t')$ must vanish for $t'>t$ since $\vec{E}(t')$ cannot influence the current $\vec{j}(t)$ at prior times (ie for $t<t'$): this follows from causality.

So $\sigma(t) = 0$ for $t<0$.

Drude Model:



- electrons move with average velocity $\vec{v}(t)$
- They accelerate ($\vec{F} = m\vec{a}$) due to electric field $\vec{E}$
- They bounce off fixed atoms.

Simple model: probabilistic

- each electron has probability per unit time $\frac{1}{\tau}$ of colliding
- after collision velocity is randomized

$\Rightarrow \vec{v}(t+\Delta t) - \vec{v}(t) =$ decreased by fraction of electrons $\left(\frac{\Delta t}{\tau}\right)$ that collide, since randomized directions average to zero $+$ (acceleration) $\times \Delta t$

$$= -\frac{\Delta t}{\tau}\vec{v}(t) + \frac{q\vec{E}(t)}{m}\Delta t$$

(textbook uses $q = -e$)
(τ = "relaxation" or "collision" time).

$\Rightarrow \frac{\partial \vec{v}}{\partial t} = -\frac{1}{\tau}\vec{v} + \frac{q\vec{E}}{m}$; with n = number density $\vec{j}(t) = nq\vec{v}(t) \Rightarrow \frac{\partial \vec{j}}{\partial t} = -\frac{1}{\tau}\vec{j} + \frac{nq^2}{m}\vec{E}$

or, after Fourier transform $-i\omega\tilde{\vec{j}} = -\frac{1}{\tau}\tilde{\vec{j}} + \frac{nq^2}{m}\tilde{\vec{E}}$

Solving for $\tilde{\vec{j}}$

$$\boxed{\tilde{\sigma}(\omega) = \frac{\frac{nq^2}{m}}{\frac{1}{\tau} - i\omega} = \frac{nq^2\tau}{m}\frac{1}{1-i\omega\tau}}$$

Notes:

- $\tau \sim 10^{-14}$ sec is typical. So for frequencies $\omega \ll \frac{1}{\tau} \sim 10^{14}$ Hz $\tilde{\sigma}(\omega) \approx \sigma_0 = \frac{nq^2\tau}{m}$

Using $n \sim 10^{22}$ cm^{-3} and q, m for electron $\rightarrow \sigma_0 \sim 10^{18}$ sec^{-1} or $\frac{1}{\sigma_0} \sim 10^{-6}$ ohm cm

• Opposite limit : $\omega >> \tau^{-1}$ $\Rightarrow$ $\boxed{\tilde{\sigma}(\omega) = i\frac{nq^2}{m\omega}}$ ($\cancel{A}$)

is purely imaginary, and $\sim \frac{1}{\omega}$.

This is as if there were no collisions $\rightarrow$ response is inertial (ie from $F=ma$)

and $\tilde{\sigma} \rightarrow 0$ as $\omega \rightarrow \infty$ means

$$\frac{dv}{dt} = \frac{q}{m}\tilde{E}e^{-i\omega t} \rightarrow v = \frac{e^{-i\omega t}}{-i\omega}\frac{q\tilde{E}}{m} \rightarrow 0 \text{ as } \omega \rightarrow \infty$$

, ie electrons can't keep up/respond.

($\cancel{A}$) is purely from applied $\vec{E}$, so very general (independent of model of collisions of electrons). Will use this!

ASIDE

• If magnetic field is present, add to force a $\frac{q}{c}\vec{v}(t)\times\vec{B}(t)$ term. Then after multiplying by nq, $\frac{q}{c}\vec{j}(t)\times\vec{B}(t)$.

The FT of a product

$$\int dt \; a(t)b(t)e^{i\omega t} = \int\frac{d\omega_1}{2\pi}\int\frac{d\omega_2}{2\pi}\;\tilde{a}(\omega_1)\tilde{a}(\omega_2)\underbrace{\int dt\; e^{it(\omega-\omega_1-\omega_2)}}_{=2\pi\,\delta(\omega-\omega_1-\omega_2)}$$

$$= \int\frac{d\omega'}{2\pi}\;a(\omega')\,b(\omega-\omega')$$

So $$-i\omega\,\vec{j}(\omega) = -\frac{1}{\tau}\vec{j}(\omega) + \frac{nq^2}{m}\vec{E}(\omega) + \frac{q}{c}\int\frac{d\omega'}{2\pi}\;\tilde{\vec{j}}(\omega')\times\tilde{\vec{B}}(\omega-\omega')$$

Yikes! See Exercise 119.1 for static case.

General Properties of $\tilde{\sigma}(\omega)$. (Garg sec 121 – we'll go back to 120 later)

Let's write $\quad \tilde{\sigma} = \tilde{\sigma}_1 + i\tilde{\sigma}_2 \quad$ ($\tilde{\sigma}_{1,2}$ are real).

Then (much of this follows from the general susceptibility notes above).

1. $\tilde{\sigma}^*(\omega) = \tilde{\sigma}(-\omega)$

This is because $\vec{E}$ and $\vec{J}$ are real. For any real $f(t)$ its FT has $f^*(\omega) = f(-\omega)$ — we have seen this. And $\tilde{\vec{E}} = \tilde{\sigma}\tilde{\vec{J}}$.

2. $\sigma_1(\omega) > 0$. (This is slightly different than for χ above)!

Power dissipated $= P_{diss} = \vec{J}\cdot\vec{E} > 0$ by 2nd Law of Thermodyn's.

For

$$\int_{-\infty}^{\infty} dt\ \vec{J}(t)\cdot\vec{E}(t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\ \tilde{\vec{J}}(-\omega)\cdot\tilde{\vec{E}}(\omega) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\ \frac{1}{2}\left(\tilde{\vec{J}}^*(\omega)\cdot\vec{E}(\omega) + \tilde{\vec{J}}(\omega)\cdot\tilde{\vec{E}}^*(\omega)\right)$$

$$= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\ \mathrm{Re}\,\sigma(\omega)\ |\tilde{\vec{E}}(\omega)|^2$$

This must be positive for arbitrary $\vec{E} \Rightarrow \quad \sigma_1(\omega) > 0$.

3. $\sigma(\omega)$ is analytic for $\mathrm{Im}(\omega) > 0$

(Note that this depends on our definition of FT as $\int\frac{d\omega}{2\pi} f(t) e^{-i\omega t}$)

We have seen that causality $\Rightarrow \quad \sigma(t) = 0$ for $t < 0$.

So

$$\sigma(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\,\sigma(t) = \int_0^{\infty} dt\, e^{i(\mathrm{Re}\,\omega)t}\ e^{-\mathrm{Im}(\omega)t}\ \sigma(t)$$

For $\mathrm{Im}(\omega) > 0$ the integral converges provided $\sigma(t)$ does not grow exponentially.

We may safely assume $\sigma(t)$ does not grow exponentially. recall

$$\vec{j}(t) = \int_{-\infty}^{t} dt' \sigma(t-t') \vec{E}(t')$$

and we do not expect $\vec{j}(t)$ to depend on $\vec{E}(t')$ as $t' \to -\infty$ as $e^{-t'}$!

Moreover, we can safely take derivatives, as in

$$\frac{d^n \tilde{\sigma}}{d\omega^n} = (-i)^n \int_{-\infty}^{\infty} dt\, t^n e^{i\omega t} \sigma(t)$$

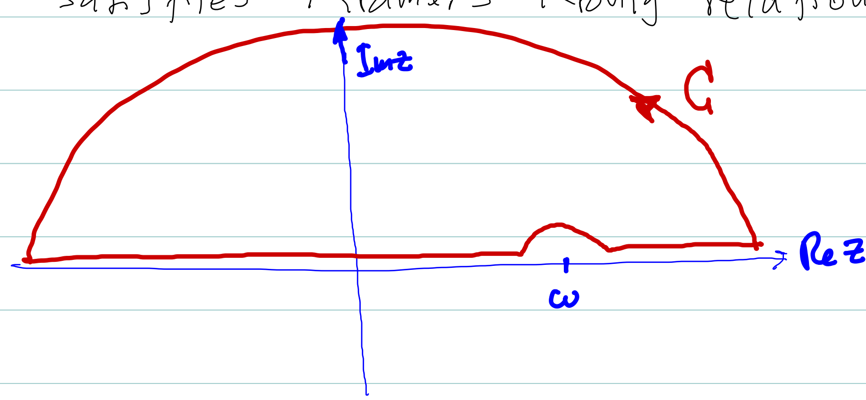
since this is still convergent for $\mathrm{Im}(\omega) > 0$.

$\Rightarrow$ $\tilde{\sigma}(\omega)$ is analytic in $\mathrm{Im}(\omega) > 0$.

Note also that $\sigma(\omega) \to 0$ as $\mathrm{Im}\,\omega \to \infty$.

This (analyticity in upper half plane plus vanishing at ∞) means:

4. $\tilde{\sigma}$ satisfies Kramers-Kronig relations



Consider $\oint_C \frac{dz\, \tilde{\sigma}(z)}{z-\omega} = 0$ (Cauchy: $\frac{\tilde{\sigma}(z)}{z-\omega}$ is analytic in region bounded by C).

On semicircle $|z| = R \to \infty$ the integral vanishes because $\sigma \to 0$, so $\left|\frac{\sigma}{z-\omega}\right| \to 0$ faster than $\frac{1}{|z|}$.

The small semicircle gives

(use $z = \omega + \epsilon e^{i\phi}$, $\epsilon \to 0$ and ϕ goes from π to 0):

$$-\lim_{\epsilon\to 0} \int_0^{\pi} \epsilon e^{i\phi}\, i\, d\phi \;\; \frac{\sigma(\omega+\epsilon e^{i\phi})}{\epsilon e^{i\phi}} = -i\pi\sigma(\omega)$$

The rest is the principal value of the integral on the real line, so

$$P\int_{-\infty}^{\infty} dx\, \frac{\sigma(x)}{x-\omega} - i\pi\sigma(\omega) = 0$$

Separating into real and imaginary parts, and using $x = \omega'$ so that the dummy variable reminds us it refers to frequency:

$$\sigma_2(\omega) = -\frac{1}{\pi} P\int_{-\infty}^{\infty} d\omega' \, \frac{\sigma_1(\omega')}{\omega-\omega'}$$

Kramers-Kronig relations

$$\sigma_1(\omega) = \frac{1}{\pi} P\int_{-\infty}^{\infty} d\omega' \, \frac{\sigma_2(\omega')}{\omega-\omega'}$$

If you know σ_1 (or σ_2) you can compute σ_2 (or σ_1)

5. $\tilde{\sigma}(\omega) \neq 0$ in upper half-plane.

This was done for $\tilde{\chi}(\omega)$ above, and won't repeat here.

6. f-sum rule

From Kramers-Kronig we have

$$\tilde{\sigma}_2(\omega) = -\frac{2\omega}{\pi} P \int_0^\infty d\omega' \frac{\tilde{\sigma}_1(\omega')}{\omega'^2 - \omega^2}$$

As $\omega \to \infty$ this is

$$\tilde{\sigma}_2(\omega) = \frac{2}{\pi\omega}\int_0^\infty d\omega' \, \tilde{\sigma}_1(\omega')$$

But from Drude's model, $\tilde{\sigma}(\omega) \simeq i\frac{nq^2}{m\omega}$ as $\omega \to \infty$, and we explained this is model independent. Comparing

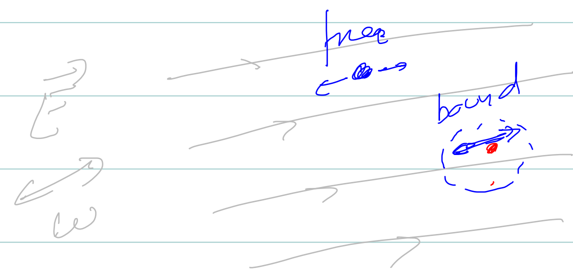
$$\frac{nq^2}{m\omega} = \frac{2}{\pi\omega}\int_0^\infty d\omega' \, \tilde{\sigma}_1(\omega)$$

or

$$\int_0^\infty d\omega' \, \tilde{\sigma}_1(\omega') = \frac{\pi n q^2}{2m} \qquad \text{"f-sum rule"}$$

Dielectric response function and Garg's "propensity"

The distinction between ρ_{free} and ρ_{bound}, and particularly $\vec{J}_{free}$ and $\vec{J}_{pol}$ (and we will see $\vec{J}_{mag}$ too) gets blurred with harmonic fields.



Both free and bound charges exhibit oscillatory motion. At very high frequency they are indistinguishable.

We will later consider a microscopic model of the response of bound electrons. But let's try to 1st capture the ambiguity in free vs bound described above, in a macroscopic description. From Ampere's law

(Here I deviate from Garg slightly: his breaking of $\vec{J} = \vec{J}^{ext} + \vec{J}^{int}$ – external and internal to the material – is non-sense, since these are local quantities, ie, $\vec{J} = \vec{J}(\vec{x}, t)$, $\tilde{\vec{J}} = \tilde{\vec{J}}(\vec{x}, \omega)$)

$$\vec{\nabla} \times \vec{H} - \frac{1}{c}\frac{\partial \vec{D}}{\partial t} = \frac{4\pi}{c}\vec{J}$$

In ω-domain:

$$\vec{\nabla} \times \tilde{\vec{H}} + \frac{i\omega}{c}\tilde{\vec{D}} = \frac{4\pi}{c}\tilde{\vec{J}} \implies \vec{\nabla} \times \tilde{\vec{H}} + \frac{i\omega}{c}\tilde{\epsilon}\tilde{\vec{E}} = \frac{4\pi}{c}\tilde{\sigma}\tilde{\vec{E}}$$

?

We can rewrite this as

$$\vec{\nabla} \times \vec{H} + \frac{i\omega}{c}\underbrace{\left(\tilde{\epsilon} + i\frac{4\pi\tilde{\sigma}}{\omega}\right)}\tilde{\vec{E}} = 0 \quad \text{or} \quad \vec{\nabla} \times \tilde{\vec{H}} = \frac{4\pi}{c}\underbrace{\left(\tilde{\sigma} - i\frac{\omega\tilde{\epsilon}}{4\pi}\right)}\tilde{\vec{E}}$$

sort of effective permitivity or kind of effective conductivity.

We also have $\vec{\nabla}\cdot\tilde{\vec{D}} = 4\pi\rho$. From continuity $\frac{\partial\rho}{\partial t} = -\vec{\nabla}\cdot\vec{j}$ we have $-i\omega\tilde{\rho} = -\vec{\nabla}\cdot\tilde{\vec{j}}$. Using $\tilde{D} = \epsilon\tilde{E}$ and $\tilde{j} = \tilde{\sigma}\tilde{E}$

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho \Rightarrow \vec{\nabla}\cdot(\tilde{\epsilon}\tilde{E}) = -\frac{4\pi i}{\omega}\vec{\nabla}\cdot(\sigma\tilde{E})$$

$$\Rightarrow \tilde{\nabla}\cdot\left[\left(\tilde{\epsilon} + i\frac{4\pi\tilde{\sigma}}{\omega}\right)\tilde{E}\right] = 0$$

Although you will find some textbooks that state that there is an ambiguity in whether we combine $\tilde{\epsilon}$ & $\tilde{\sigma}$ into permitivity or conductivity, the interpretation of Gauss's law suggests an effective permitivity is a better choice.

Garg invents the term (I have not seen it used elsewhere) "electric propensity" for

$$\varsigma(\omega) \equiv \tilde{\epsilon}(\omega) + \frac{4\pi i\,\tilde{\sigma}(\omega)}{\omega}$$

* the squiggle "ς" is intended as greek-zeta

or, to be fair, "full dielectric function"

Much of the literature calls it dielectric constant, or complex dielectric constant, or AC dielectric constant, ... None of these names capture the facts that (i) not a constant, (ii) not purely dielectric and (iii) not the same as $\tilde{\epsilon}(\omega)$ (even thoug this symbol is often used for $\varsigma(\omega)$).

We'll stick with Garg.

Aside: if there are additional currents $\vec{J}'$ not subject to Ohm's law (eg, superconducting current) then add to right hand side:

$$\vec{\nabla}\times\vec{\tilde{H}} + i\frac{\omega}{c}\tilde{\zeta}\vec{\tilde{E}} = \frac{4\pi}{c}\vec{\tilde{J}}'$$

$$\vec{\nabla}\cdot\tilde{\zeta}\vec{\tilde{E}} = 4\pi\tilde{\rho}'$$

Garg also defines $\vec{\tilde{Z}} = \tilde{\zeta}\vec{\tilde{E}}$ so that in t-domain

$$\vec{\nabla}\times\vec{H} - \frac{1}{c}\frac{\partial\vec{Z}}{\partial t} = \frac{4\pi}{c}\vec{J}' \quad , \quad \vec{\nabla}\cdot\vec{Z} = 4\pi\rho'$$

Beware than in most of the literature $\vec{\tilde{Z}} = \tilde{\zeta}\vec{\tilde{E}}$ is $\vec{\tilde{D}} = \tilde{\epsilon}\vec{\tilde{E}}$

Best is to understand what you are doing: then you don't get confused with symbols.

Electromagnetic energy in material media

We saw that $\tilde{\sigma} \neq 0$ in the upper half ω-plane, and in particular $\sigma_1 > 0$ on real axis. This was a result that followed from the 2^{nd} law, that energy is dissipated in the material body.

Now, the microscopic theory tells us exactly where the energy goes:

$$-\vec{\nabla}\cdot\vec{S} = \vec{j}\cdot\vec{e} + \frac{\partial u}{\partial t}$$

as was shown in 203A, where $\vec{S} = \frac{c}{4\pi}\vec{e}\times\vec{b}$ is the (microscopic version of) Poynting vector giving the energy flux and $u = \frac{1}{8\pi}(\vec{e}^2 + \vec{b}^2)$ the (microscopic energy density). The question is what replaces this that accounts for rate of heat dissipated in the presence of dielectrics.

The answer is

$$-\bar{\nabla}\cdot\overline{\vec{S}} = \overline{\vec{J}_{free}\cdot\vec{E}} + \frac{d\overline{U}}{dt} + \dot{Q}$$

where (i) Fields are assumed quasimonochromatic

(ii) $\overline{X(t)}$ means $X(t)$ is averaged over the period of the (quasi) monochromatic fields

(iii) $\bar{U}$ has the interpretation of internal energy and $\dot{Q}$ is d(heat)/dt, with

$$\bar{U} = \frac{1}{8\pi}\left[\frac{d}{d\omega}(\omega\epsilon_1(\omega))\overline{E^2} + \frac{d}{d\omega}(\omega\mu_1(\omega))\overline{H^2}\right]$$

and

$$\dot{Q} = \frac{1}{4\pi}\left[\omega\epsilon_2(\omega)\overline{E^2} + \omega\mu_2(\omega)\overline{H^2}\right]$$

The rest of this section is just computations deriving this result (plus a definition of terms, eg, "quasimonochromatic").

Consider quasimonochromatic field $\vec{E}(t)$. That is $\tilde{\vec{E}}(\omega)$ has frequencies centered on ω_0 with small dispersion.

We want to show that

$$\vec{E}(t) = \vec{a}(t)e^{-i\omega_0 t} + c.c.$$

To show this, consider

$$\vec{E}(t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\tilde{\vec{E}}(\omega)e^{-i\omega t} = \int_0^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}\tilde{\vec{E}}(\omega) + \int_0^{\infty}\frac{d\omega}{2\pi}e^{i\omega t}\tilde{\vec{E}}^*(\omega)$$

Now write $\tilde{\vec{E}}(\omega_0+\alpha) = \tilde{\vec{a}}(\alpha)$ so that ($\omega = \omega_0 + \alpha$ above)

$$\vec{E}(t) = e^{-i\omega_0 t}\int_{-\omega_0}^{\infty}\frac{d\alpha}{2\pi}\tilde{\vec{a}}(\alpha)e^{-i\alpha t} + c.c.$$

Now, assume that $\tilde{\vec{a}}(\alpha)$ is localized about some frequencies well above zero. Then we can approximately replace the lower limit by $-\infty$:

$$\vec{E}(t) = \vec{a}(t)e^{-i\omega_0 t} + c.c.$$

which is the desired result.

$\vec{a}(t)$ varies little over a period $\frac{2\pi}{\omega_0}$. So if we average $\vec{E}^2(t)$ over $t \gg \frac{2\pi}{\omega_0}$ the $e^{\pm 2i\omega_0 t}$ terms do not contribute

$$\overline{\vec{E}^2(t)} = 2\, \vec{a}(t)\cdot\vec{a}^*(t)$$

The average is over $t \gg \frac{2\pi}{\omega_0}$ but small over the typical time over which $\vec{a}(t)$ varies.

Likewise for other fields, like $\vec{H}(t)$.

Now, we have shown

$$\vec{\nabla}\cdot\left[-\frac{c}{4\pi}(\vec{E}\times\vec{H})\right] = \vec{J}\cdot\vec{E} + \frac{1}{4\pi}\left(\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right)$$

Integrate over volume, for some V. $\int_V d^3r\, \vec{J}\cdot\vec{E}$ = work/time done by $\vec{E}$ on free charges. $-\int d^3r\, \vec{\nabla}\cdot\left[\frac{c}{4\pi}(\vec{E}\times\vec{H})\right] = -\int_{\partial V} d^2r\, \hat{n}\cdot\left[\frac{c}{4\pi}(\vec{E}\times\vec{H})\right]$ is then energy/time flowing into V, so $\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{H}$ = energy flux/vol.

The last term must be the change in internal energy plus heat produced.

Consider $\vec{E}\cdot\frac{\partial\vec{D}}{\partial t}$. Write $\vec{E}$ and $\vec{D}$ in terms of $\tilde{\vec{E}}(\omega)$ and $\vec{D}(\omega) = \hat{\epsilon}(\omega)\,\tilde{\vec{E}}(\omega)$.

$$\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} = \int\frac{d\omega_1}{2\pi}\int\frac{d\omega_2}{2\pi}\, \tilde{\vec{E}}(\omega_1)^* e^{i\omega_1 t}\cdot(-i\omega_2)\,\epsilon(\omega_2)\,\tilde{\vec{E}}(\omega_2)\,e^{-i\omega_2 t}$$

$$= \int\frac{d\omega_1}{2\pi}\int\frac{d\omega_2}{2\pi}\, \tilde{\vec{E}}(\omega_2)\,e^{-i\omega_2 t}\cdot(i\omega_1)\,\epsilon^*(\omega_1)\,\tilde{\vec{E}}(\omega_1)^* e^{i\omega_1 t}$$

so adding these:

$$2\dot{E}\cdot\dot{D} = \int_{-\infty}^{\infty}\frac{d\omega_1}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}\, e^{i(\omega_1-\omega_2)t}\,\tilde{\vec{E}}(\omega_1)^*\cdot\tilde{\vec{E}}(\omega_2)\; i\left[\omega_1\epsilon^*(\omega_1) - \omega_2\epsilon(\omega_2)\right]$$

Write $\int_{-\infty}^{\infty}\frac{d\omega_1}{2\pi}\, e^{i\omega_1 t}\; \tilde{\vec{E}}(\omega_1)^* i\omega_1\epsilon^*(\omega_1)$

$$= \int_0^{\infty}\frac{d\omega_1}{2\pi}\left[e^{i\omega_1 t}\,\tilde{E}^*(\omega_1)\, i\omega_1\epsilon^*(\omega_1) + e^{-i\omega_1 t}\,\tilde{E}(\omega_1)(-i\omega_1)\,\epsilon(\omega_1)\right]$$

and so on, so that we only integrate over positive frequencies. Since we will want to average over quasimonochromatic fields, drop terms $\tilde{\vec{E}}(\omega_1)\cdot\tilde{\vec{E}}(\omega_2)$ or $\tilde{\vec{E}}(\omega_1)^*\cdot\tilde{\vec{E}}(\omega_2)^*$.

Next average over period $T=\frac{2\pi}{\omega_0}$. Use $\omega=\omega_0+\alpha$, and amp's $\vec{a}(t)$

$$\overline{2\vec{E}\cdot\dot{\vec{D}}} \approx 2\underbrace{\int_{-\omega_0}^{\infty}}_{\approx -\infty}\frac{d\alpha_1}{2\pi}\underbrace{\int_{-\omega_0}^{\infty}}_{\approx-\infty}\frac{d\alpha_2}{2\pi}\; i\left[\omega_1\epsilon^*(\omega_1) - \omega_2\epsilon(\omega_2)\right]\Big|_{\omega_i=\omega_0+\alpha_i}\tilde{\vec{a}}(\alpha_1)^*\cdot\tilde{\vec{a}}(\alpha_2)\, e^{i(\alpha_1-\alpha_2)t}$$

We separate $\epsilon(\omega)$ into real and imaginary parts since physically we expect $\epsilon_2=\mathrm{Im}\,\epsilon$ to be associated to heat (dissipation) while $\epsilon_1=\mathrm{Re}\,\epsilon$ ought to be related to internal energy. For each of this we expand in $\alpha=\omega-\omega_0$ and retain leading terms:

$$\mathrm{Re}\left[\omega_1\epsilon^*(\omega_1)-\omega_2\epsilon(\omega_2)\right] = (\omega_0+\alpha_1)\,\epsilon_1(\omega_0+\alpha_1) - (\omega_0+\alpha_2)\,\epsilon_1(\omega_0+\alpha_2)$$
$$= (\alpha_1-\alpha_2)\frac{d}{d\omega}(\omega\epsilon_1(\omega))\Big|_{\omega_0}$$

$$\mathrm{Im}[\;\text{idem}\;] = -(\omega_0+\alpha_1)\,\epsilon_2(\omega_0+\alpha_1) - (\omega_0+\alpha_2)\,\epsilon_2(\omega_0+\alpha_2)$$
$$= -2\omega_0\,\epsilon_2(\omega_0)$$

Write this in time domain (and drop "0" in ω_0):

$$\overline{\dot{\vec{E}}\cdot\vec{D}} = \underbrace{\frac{1}{2}\frac{d}{d\omega}(\omega\epsilon_1(\omega))\ \frac{d}{dt}\overline{\vec{E}^2(t)}} + \underbrace{\omega\epsilon_2(\omega)\ \overline{\vec{E}^2(t)}}$$

and this $\times \frac{1}{4\pi}$ is $\frac{d}{dt}\bar{U}$ as advertised; this $\times \frac{1}{4\pi}$ is $\dot{Q}$

What remains is justifying the interpretation of the two terms as above. Note that if you proceed slowly and adiabatically in polarizing the medium, the mechanical work done (which should go fully into internal energy) is, $\frac{1}{8\pi}\epsilon_1\vec{E}^2$. But $\frac{1}{8\pi}\frac{d}{d\omega}(\omega\epsilon_1(\omega))E^2$ gives this in the quasistatic approximation. So we interpret the first term as $\frac{d\bar{U}}{dt}$ and infer the 2^{nd} is heat that shows up when the process of polarizing the medium is <u>not</u> adiabatic.

Beware of the limits of applicability: we assumed linearity, quasi monochromatic fields, retained leading terms in Taylor expansion, ...

Electronic Response Model of Drude, Kramers, Lorentz.

Each atom/molecule a polarizable unit.

Model the 'atom' as a charge (electron) bound by a harmonic force, with dissipation and under and applied $\vec{E}$-field force:

$$m\ddot{\vec{r}} + m\gamma\dot{\vec{r}} + m\omega_0^2\vec{r} = q\vec{E}(t) \qquad (q = -e \text{ charge of } e^-).$$

If the "atom" is neutral and has no permanent dipole moment then we need a charge $-q$ (ie, $+e$) at the "center" $\vec{r}=0$. The dipole moment is then $\vec{d}(t) = q\vec{r}(t)$

In Fourier space (we have solved this Eq several times before)

$$-\omega^2\tilde{\vec{r}} - i\omega\gamma\tilde{\vec{r}} + \omega_0^2\tilde{\vec{r}} = \frac{q}{m}\tilde{\vec{E}}$$

or

$$\tilde{\vec{r}} = -\frac{q}{m}\tilde{\vec{E}}\frac{1}{\omega^2-\omega_0^2+i\omega\gamma}$$

With n = number density of bound electrons/volume

Polarization vector $\vec{P} = nq\tilde{\vec{r}} = -\frac{nq^2}{m}\tilde{\vec{E}}\frac{1}{\omega^2-\omega_0^2+i\omega\gamma}$

$$\Rightarrow \quad \tilde{\chi}_e = -\frac{nq^2}{m}\frac{1}{\omega^2-\omega_0^2+i\omega\gamma}$$

and $\tilde{\epsilon}(\omega) = 1 + 4\pi\tilde{\chi}_e(\omega) = 1 - \frac{4\pi n^2 q}{m} \frac{1}{\omega^2 - \omega_0^2 + i\omega\gamma}$

This is for rarefied media. For dense media a Clausius-Mossetti model treatment gives

$$\frac{\tilde{\epsilon}(\omega) - 1}{\tilde{\epsilon}(\omega) + 2} = \frac{4\pi}{3}\tilde{\chi}_e(\omega)$$

or $\tilde{\epsilon}\left(1 - \frac{4\pi}{3}\tilde{\chi}_e\right) = 1 + \frac{8\pi}{3}\tilde{\chi}$, $\tilde{\epsilon} = 1 + \frac{4\pi\tilde{\chi}_e}{1 - \frac{4\pi}{3}\tilde{\chi}_e}$

or $\tilde{\epsilon} = 1 + \frac{4\pi n^2 q}{m} \cdot \frac{(-1)}{\omega^2 - \omega_0^2 + i\omega\gamma} \cdot \frac{1}{1 - \frac{4\pi}{3}\frac{n^2 q}{m}\frac{(-1)}{\omega^2 - \omega_0^2 + i\omega\gamma}}$

$$= 1 - \frac{4\pi n^2 q}{m} \frac{1}{\omega^2 - \left(\omega_0^2 - \frac{4\pi}{3}\frac{n^2 q}{m}\right) + i\omega\gamma}$$

$$= 1 + \frac{4\pi n^2 q}{m} \frac{1}{\omega_1^2 - \omega^2 - i\omega\gamma}$$

where $\omega_1^2 = \omega_0^2 - \frac{4\pi}{3}\frac{n^2 q}{m}$

Improvement: many resonant frequencies of electrons in real atoms, given by

$$\hbar\omega_i = \underbrace{\mathcal{E}_i}_{i\text{th energy level}} - \underbrace{\mathcal{E}_0}_{\text{ground state energy}}$$

and introduce f_i = oscillator strength

= amplitude of dipole moment when oscillating between j-th state and ground state, with

$$\sum_i f_i = Z = \text{number of electrons in atom}$$

Then, the improved model is

$$\tilde{\epsilon}(\omega) = 1 + \frac{4\pi n q^2}{mZ} \sum_i \frac{f_i}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

(γ_i = damping of response at frequency ω_i).

Note:

This is a rough model. Do not attach too literal a meaning to constants like n & Z.

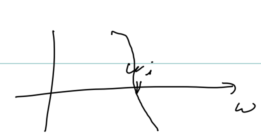
Let's plot $\tilde{\epsilon}_1(\omega)$ and $\hat{\epsilon}_2(\omega)$

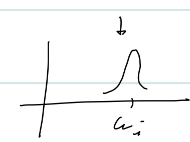
$$\tilde{\epsilon}_1(\omega) = Re(\tilde{\epsilon}(\omega)) = 1 + \frac{4\pi n q^2}{mZ} \sum_i \frac{f_i(\omega_i^2 - \omega^2)}{(\omega_i^2 - \omega^2)^2 + \omega^2\gamma_i^2}$$

$$\tilde{\epsilon}_2(\omega) = Im(\tilde{\epsilon}(\omega)) = \frac{4\pi n q^2}{mZ} \sum_i \frac{f_i \omega\gamma_i}{(\omega_i^2 - \omega^2)^2 + \omega^2\gamma_i^2}$$

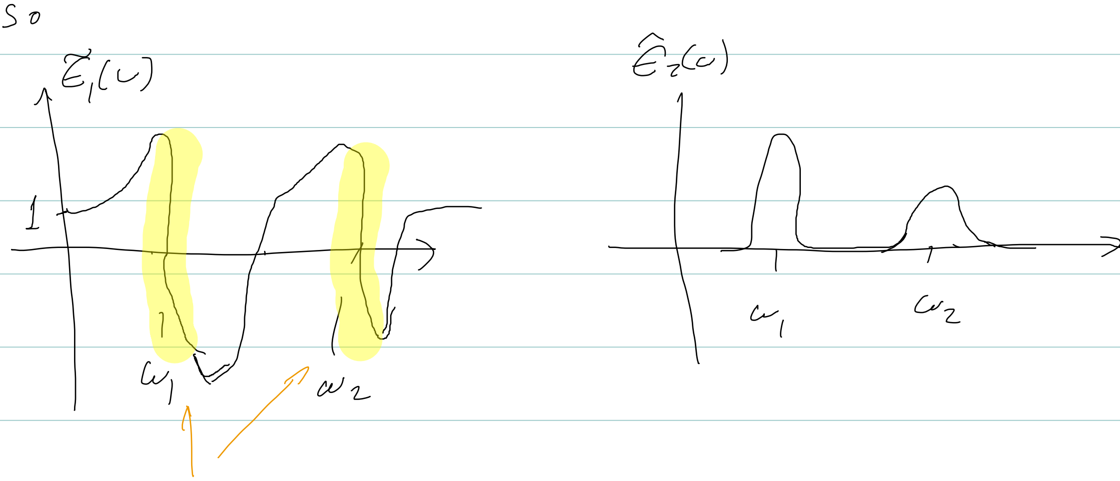
Near resonance i,

$$\tilde{\epsilon}_1(\omega) \simeq \frac{4\pi n q^2}{mZ} \frac{f_i(\omega_i^2 - \omega^2)}{\omega_i^2\gamma_i^2} \qquad \tilde{\epsilon}_2(\omega) \approx \frac{4\pi n q^2}{m} \frac{f_i \omega_i \gamma_i}{(\omega_i^2 - \omega^2)^2 + \omega_i^2\gamma_i^2}$$





So



$\frac{d\tilde{\epsilon}_1}{d\omega} < 0$ "anomalous dispersion" (see later, below)

Notes:

- If $\omega_1 = 0$ the contribution of this resonance to $\tilde{\epsilon} - 1$ is

$$-\frac{4\pi n q^2}{m Z} f_1 \frac{1}{\omega(\omega + i\gamma_1)}$$

which looks like conductivity in Drude's model: and it is! $\omega_1 = 0$ means no restoring force $\Rightarrow$ free electrons.

Recall the Drude model has

$$\tilde{\sigma}(\omega) = \frac{n_f q^2 \tau}{m} \frac{1}{1 - i\omega\tau} \qquad \text{with } n_f = n \text{ for free } e\text{'s}.$$

and "propensity" is

$$\tilde{\xi}(\omega) = \tilde{\epsilon}(\omega) + \frac{4\pi i \sigma(\omega)}{\omega}$$

So $\tilde{\xi}$ in Drude's model has $\tilde{\xi}(\omega) - \tilde{\epsilon}(\omega) = \frac{4\pi n_f q^2}{m} i\tau \frac{1}{\omega(1 - i\omega\tau)}$

$$= -\frac{4\pi n_f q^2}{m} \frac{1}{\omega(\omega + i\tau^{-1})}$$

which matches the above with

$$n_f = \frac{n f_1}{Z} \quad \text{and} \quad \tau = \gamma_1^{-1}$$

Nice to get a unified treatment in one simple model!

- The static case $\tilde{\epsilon}(0) = 1 + \frac{4\pi n q^2}{mZ} \sum_i \frac{f_i}{\omega_i^2}$

The large frequency limit

$$\tilde{\epsilon}(\omega) = 1 - \frac{4\pi n q^2}{m} \frac{1}{\omega^2}$$

(or $\tilde{\varsigma}(\omega)$ = this), where $\frac{1}{Z}\sum_i f_i = 1$ was used. Again, as in the case of $\tilde{\sigma}$, the $\omega \to \infty$ behavior is model independent (does not depend on f_i, γ_i, ω_i), since at high frequency electrons are "paralyzed" (in the words of Garg).

We may write $\tilde{\epsilon}(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$

where $\omega_p^2 \equiv \frac{4\pi n q^2}{m}$ is the "Plasma" frequency of the medium.

Addendum: Wave propagation in dispersive medium.

In PHYS 203A we discussed wave propagation in dispersive media briefly. We took

$$\omega(k) = vk = \frac{c}{\sqrt{\tilde{\epsilon}(\omega)\tilde{\mu}(\omega)}}k$$

and then found that $\frac{\omega}{k} = \frac{c}{\sqrt{\tilde{\epsilon}\tilde{\mu}}}$ is the phase velocity, while $v_g = \frac{d\omega}{dk}$ gives the group velocity.

with $\frac{v_g}{c} = \frac{1}{c}\left(\frac{dk}{d\omega}\right)^{-1} = \left[\frac{d}{d\omega}\sqrt{\tilde{\epsilon}\tilde{\mu}}\,\omega\right]^{-1}$

and taking $\tilde{\mu}=1$, we have $\frac{c}{v_g} = \sqrt{\tilde{\epsilon}} + \frac{1}{2}\frac{\omega}{\sqrt{\tilde{\epsilon}}}\frac{d\tilde{\epsilon}}{d\omega}$

Ignoring (for now) the imaginary part of $\tilde{\epsilon}$, we see that in the region of anomalous dispersion $\left(\frac{d\tilde{\epsilon}}{d\omega} < 0\right)$ $\frac{v_g}{c}$ increases. Worse $\tilde{\epsilon}_1 < 0$ so even if one neglects $\tilde{\epsilon}_2$, the index of refraction $\tilde{n}(\omega) = \sqrt{\tilde{\epsilon}}$ is purely imaginary so neither v_p nor v_g are well defined.

Sticking to the $\mu=1$ case, generally

$$\tilde{n} = \tilde{n}_1 + i\tilde{n}_2 = \sqrt{\tilde{\epsilon}_1 + i\tilde{\epsilon}_2} \Rightarrow \tilde{n}_1^2 - \tilde{n}_2^2 = \tilde{\epsilon}_1, \quad 2\tilde{n}_1\tilde{n}_2 = \tilde{\epsilon}_2$$

(Solve: $\tilde{n}_1^2 - \tilde{\epsilon}_1 - \frac{1}{4}\frac{\tilde{\epsilon}_2^2}{\tilde{n}_1^2} = 0 \Rightarrow \tilde{n}_1^2 = \frac{1}{2}\left(\tilde{\epsilon}_1 + \sqrt{\tilde{\epsilon}_1^2 + \tilde{\epsilon}_2^2}\right) \Rightarrow \tilde{n}_2^2 = \frac{1}{2}\left(-\tilde{\epsilon}_1 + \sqrt{\tilde{\epsilon}_1^2 + \tilde{\epsilon}_2^2}\right)$)

Recall wave equation is $\left(\nabla\times\vec{E} - i\frac{\omega}{c}\vec{B}=0,\ \ \vec{\nabla}\times\vec{B}+i\frac{\omega}{c}\tilde{\epsilon}(\omega)\vec{E}=0\right)$

$$\left[\vec{\nabla}^2 + \frac{\omega^2}{c^2}\tilde{\epsilon}(\omega)\right]\vec{E}=0$$

So in $e^{i(kz-\omega t)}$ for plane wave we have

$$k^2 = \frac{\omega^2}{c^2}\tilde{\epsilon}(\omega)$$

$$k = \frac{\omega}{c}\sqrt{\tilde{\epsilon}} = \frac{\omega}{c}(\tilde{n}_1 + i\tilde{n}_2)$$

So the region of anomalous dispersion, which coincides with non-negligible $\tilde{\epsilon}_2$ and therefore $\tilde{n}_2$, one has

$$E_\perp \cdot e^{i\omega(\tilde{n}_1\frac{z}{c}-t)}\ e^{-\tilde{n}_2\frac{\omega}{c}z}$$

and

$$\text{Intensity} \sim |E_\perp|^2 \propto e^{-z/\delta} \qquad \delta^{-1} = 2\frac{\tilde{n}_2\omega}{c}$$

where δ is the penetration length.

Sec 134: To make sense of velocity of propagation particularly in the region of anomalous dispersion one may define "energy velocity" $\vec{v}_E$:

$$\overline{\vec{S}} = \vec{v}_E \, \overline{U}$$

with Poynting and internal energy defined as previously. Since we are interested in velocity of propagation, but not on atenuation along the wave we ignore absorption (set $\tilde{n}_2 = 0$ so $\vec{k}$ is real). With this $\vec{S}$ is along $\vec{k}$. Moreover $\epsilon |\tilde{E}|^2 = \mu |\tilde{H}|^2$ so

$$\overline{\vec{S}} = \frac{c}{16\pi} \left(\tilde{E} \times \tilde{H}^* + c.c. \right) = \frac{c}{16\pi} \left[\sqrt{\frac{\epsilon}{\mu}} |\tilde{E}|^2 + \sqrt{\frac{\mu}{\epsilon}} |\tilde{H}|^2 \right]$$

Now $\overline{U}$ was determined earlier, $\overline{U} = \frac{1}{8\pi} \left\{ \frac{d}{d\omega} (\omega \tilde{\epsilon}_1) \overline{E^2} + \begin{matrix} \epsilon \to \mu \\ E \to H \end{matrix} \right\}$

So $\overline{S} = \frac{c}{8\pi} \sqrt{\frac{\epsilon}{\mu}} |\tilde{E}|^2$, $\overline{U} = \frac{1}{16\pi} \left[\frac{d(\omega \tilde{\epsilon}_1)}{d\omega} + \frac{\tilde{\epsilon}_1}{\tilde{\mu}_1} \frac{d(\omega \tilde{\mu}_1)}{d\omega} \right] |\tilde{E}|^2$ (the additional $\frac{1}{2}$ from $\overline{E(t)^2} = \frac{1}{2} |\tilde{E}(\omega)|^2$)

$$\Rightarrow \quad \frac{c}{v_E} = \frac{1}{2} \left[\sqrt{\frac{\tilde{\mu}_1}{\tilde{\epsilon}_1}} \frac{d(\omega \tilde{\epsilon}_1)}{d\omega} + \sqrt{\frac{\tilde{\epsilon}_1}{\tilde{\mu}_1}} \frac{d(\omega \tilde{\mu}_1)}{d\omega} \right]$$

$$\text{or} \quad \frac{c}{v_E} = \frac{d}{d\omega} \left(\omega \sqrt{\tilde{\epsilon}_1 \tilde{\mu}_1} \right) = \frac{d}{d\omega} \left(\tilde{n}_1 \omega \right) = \frac{c}{v_g}$$

the group velocity.

In the region of anomalous dispersion one cannot neglect absorption. The above treatment fails. But for the harmonicly-bound charges model one can compute explicitly.

See details in textbook. It shows $\frac{v_E}{c} \leq 1$.

(End Addendum)

Note on $\tilde{\mu}(\omega)$

For frequencies $\omega \gtrsim \omega_0$ where ω_0 is no higher than optical, but possibly lower, it makes no physical sense to distinguish between $\vec{H}$ & $\vec{B}$.

Recall

$$\vec{H} - \vec{B} = 4\pi \vec{M} \quad \text{and} \quad \vec{J}_{\text{bound}} = c \vec{\nabla} \times \vec{M} + \frac{\partial \vec{P}}{\partial t}$$

Under what condition is the 1st term bigger than 2nd?

Estimate

$$|c \vec{\nabla} \times \vec{M}| \sim c \frac{1}{\text{length of variation}} \cdot \chi_m B \sim \frac{\chi_m c B}{l}$$

Also for $\frac{\partial \vec{P}}{\partial t}$, use the induced $\vec{E}$ field (it is magnetic response we care about!) $\vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} \rightarrow E \sim l\omega B/c$

and $\vec{P} = \chi_e \vec{E} \sim \vec{E}$ $\quad \left|\frac{\partial \vec{P}}{\partial t}\right| \sim l\omega^2 B/c$

So for $\left|\frac{\partial \vec{P}}{\partial t}\right| << c|\vec{\nabla} \times \vec{M}|$ we need $l \frac{\omega^2 B}{c} << \chi_m \frac{cB}{l}$

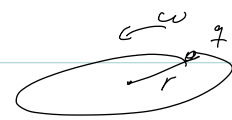
$$\Rightarrow \quad l^2 << \chi_m \frac{c^2}{\omega^2}$$

Moreover, the dimensions of the body over which variations are considered, l, should be much larger than atomic, $l >> a$.

Need to know some rough scaling of χ_m.

For a diamagnetic material (sec. 102)

model atom as bound electron in circular orbit



- $m_M = \frac{1}{2c} q \omega r^2$ ($\vec{m}_M$: magnetic dipole moment).

- $F = ma$: $\quad \underset{\substack{\uparrow \\ \text{mass (not } m_M)}}{m\omega^2 r} = \underset{\substack{\uparrow \\ \text{central}}}{\frac{Zq^2}{r^2}} + \underset{\substack{\uparrow \\ \text{applied } B}}{\frac{q\omega r B}{c}}$ $\quad \left(\vec{F} = \frac{q}{c}\vec{v}\times\vec{B}\right)$

- If ω_0 F_0, $B=0$ then $\omega_0^2 = \dfrac{Zq^2}{mr^3}$

and $\quad \omega^2 = \omega_0^2 + \omega_0\omega_L$

where $\quad \omega_0\omega_L = \dfrac{q\omega B}{mc}$

but we work at small B so set $\omega \simeq \omega_0$ on RHS

$$\Rightarrow \omega_L = \frac{qB}{mc} \quad \text{is the Larmor frequency.}$$

So the change in m_M due to the applied field is

$$\Delta m_M = \frac{q}{2c} r^2 \Delta\omega = \frac{q}{2c} r^2 \left(\frac{\omega_L}{2}\right) = \frac{q^2}{4mc^2} r^2 B$$

Magnetization: $M = n\,\Delta m_M = \dfrac{q^2 n}{4mc^2} r^2 B \Rightarrow \chi_m = \dfrac{q^2 n r^2}{4mc^2}$

Now $\dfrac{Zq^2}{m} = \omega_0^2 r^3$ and $r^3 n \lesssim 1$ so $\chi_m \lesssim \left(\dfrac{\omega_0 r}{c}\right)^2$

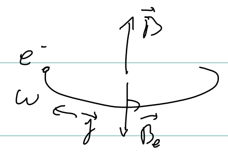
Added comment: there are two problems with the above (correct) argument

(i) we are taking r = constant, but this is not guaranteed.

(ii) $\vec{B}$ does no work but our higher $\omega = \omega_0 + \omega_L$ state has higher energy; $\vec{B}$ as taken ($\perp$ to plane of orbit) does no torque, so $\vec{L} = m\vec{r}\times\vec{v}$ = constant, so $mr^2\omega$ = constant, also not consistent with r = constant

The solution to this is that since $\vec{B}$ increases, $\vec{B} = \vec{B}(t)$ is not constant $\rightarrow$ $\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$ $\Rightarrow$ $\vec{E}$ is induced, does work and produces torque $\Rightarrow$ r remains constant.

Let's check: increase $\delta\vec{B} = \delta t \frac{\partial \vec{B}}{\partial t}$. Assume the current produced by circling electron produces a magnetic field B_e in direction opposite $\vec{B}$



Then $\Delta\vec{B}_e$ is along $\vec{B}_e$ (by Lenz's law): $\int \vec{E}\cdot d\vec{l} = -\frac{1}{c}\int d^2s\, \frac{\partial \vec{B}}{\partial t}$

$\Rightarrow E\, 2\pi r = \frac{1}{c}\pi r^2 \frac{\Delta B}{\Delta t}$. Note that $E\,2\pi r$ = work done on e^-.

and $N = qEr = \frac{q}{2c} r^2 \frac{\Delta B}{\Delta t}$ = torque on e^-. $\Rightarrow \Delta L = N\,\delta t$.

The initial trajectory has

$$\mathcal{E} = \frac{1}{2}m(\omega_0 r_0)^2 - \frac{Zq^2}{r_0} \quad \text{and} \quad L_0 = m\,\omega_0 r_0^2$$

The final one has

$$\mathcal{E}_f = \frac{1}{2} m (r\omega)^2 - \frac{Zq^2}{r} = \mathcal{E}_0 + \frac{1}{c}\pi r_0^2 \frac{\Delta B}{\Delta t} \qquad L = mr^2\omega$$

Now $L = m(r_0^2 + 2r_0\delta r_0)(\omega_0 + \delta\omega_0) = L_0 + 2mr_0\omega_0\delta r + m r_0^2 \,\delta\omega$

$$= L_0 + \frac{q}{2c} r^2 \frac{\Delta B}{\Delta t}\delta t \Rightarrow 2mr_0\omega_0\delta r + mr_0^2\,\delta\omega_0 = \frac{q}{2c} r_0^2 \Delta B$$

The EOM gives $(F = ma)$ $\quad m\omega$

$$m\dot{\omega} r = qE = r\frac{q}{2c}\dot{B} \rightarrow \omega = \omega_0 + \frac{qB}{2mc}$$

and

$$m\omega^2 r = \frac{Zq^2}{r^2} + \frac{q}{c}\omega r B$$

note, this already gives

$$\Delta\omega = \frac{1}{2}\omega_L = \frac{qB}{2mc}$$

that we used!

with $\frac{qB}{c} = m\omega_L$

$$m\omega(\omega - \omega_L) r = \frac{Zq^2}{r}$$

$$\Rightarrow m(\omega_0 - \tfrac{1}{2}\omega_L)(\omega_0 + \tfrac{1}{2}\omega_L) r = \frac{Zq^2}{r} \quad (= m\omega_0^2 r)$$

Something is wrong? Is the statement true only to linear order?

Also $\frac{dL}{dt} = N = qEr = \frac{qr^2}{2c}\frac{dB}{dt}$, $L = mr^2\omega$

$$\frac{dL}{dt} = 2mr\dot{r}\omega + mr^2\dot{\omega} = 2mr\dot{r}\omega + r^2\frac{q\dot{B}}{2c} = \frac{qr^2}{2c}\frac{dB}{dt} \Rightarrow \dot{r} = 0.$$

Clearly $r = r_0 + O(\Delta B^2)$ so good enough for our purpose (linear response); but would be nice to figure it out

END ADDENDUM

Returning to the question of when the frequency response becomes relevant we had

$$a^2 \ll l^2 \ll \chi_m \frac{c^2}{\omega^2} \quad \text{and now we know} \quad \chi_m \lesssim \left(\frac{\omega_0 a}{c}\right)^2$$

($a = r$ = atomic size = atomic separation ; the "$\lesssim$" because we used $na^3 = 1$, but for rare media $na^3 \ll 1$).

Hence

$$\boxed{\omega \ll \omega_0 \simeq \text{optical frequencies}}$$

is the condition for $\tilde{\mu}(\omega)$ to make sense physically. For optical frequencies and above (and possibly starting even below that, as the string of "$\gg$" and assumptions above shows) we may as well use $\mu = 1$ and keep track of $c\vec{\nabla}\times\vec{M} + \frac{\partial \vec{P}}{\partial t}$ (dominated by $\frac{\partial \vec{P}}{\partial t}$) through $\tilde{\epsilon}(\omega)$..

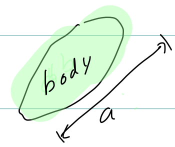
[Note added: why keep r fixed when turning $B \neq 0$? $\vec{D}$ does no work and no torque: $E_0 = E_{final}$, $E_0 = \frac{1}{2} m\omega^2 r^2 - \frac{Zq^2}{r} = \frac{1}{2}\frac{L^2}{mr^2} - \frac{Zq^2}{r}$, $L_0 = m\omega_0 r^2 = L_{final}$.

$\Rightarrow m\omega_f r_f^2 = m\omega_0 r_0^2$ and $\frac{L^2}{2mr_0^2} - \frac{Zq^2}{r_0} = \frac{L^2}{2mr^2} - \frac{Zq^2}{r} + \frac{q\omega r B}{c}$

$\Rightarrow$ 2 eqs, 2 unknowns $\Rightarrow$

Quasistatic phenomena in conductors

Quasistatic Fields



in time dependent field

study $a \ll \lambda$ = wavelength of $\vec{B}$ or $\vec{E}$ field

ie $\omega a \ll c$ eg $\omega < 1\,GHz$ for $a = 1\,cm$.

Recall (Drude's model) $\tilde{\sigma}(\omega) = \sigma_0 \frac{1}{1 - i\omega\tau}$ $\left(\sigma_0 = \frac{nq^2\tau}{m}\right)$

So for $\omega \ll \tau^{-1}$, $\tilde{\sigma}(\omega) \simeq \sigma_0$ = constant (independent of ω).

For good conductors $\tau^{-1} \gg \frac{a}{c}$ (unless a is tiny) so we will be in the regime where we can take $\tilde{\sigma}(\omega) = \sigma_0$.

Moreover, for good conductors $\sigma_0 \sim 10^{18}\,Hz$ so $\omega\sigma_0 \lll 1$.

The problem we want to solve is this: put a conductor in an external time dependent <u>magnetic</u> field, $\vec{H}_0(t)$. What are the fields (both magnetic and electric) inside the conductor? Is there a resulting electric field outside the conductor? How is $\vec{H}$ modified outside the conductor? What currents are produced in the conductor?

That $a \ll \lambda \Rightarrow$ working in "near zone", so there are no retardation effects to worry about.

Typical situation: conductor placed inside coil generating $\vec{H}_0(t)$. Also conductor moving into (possibly constant) field $\vec{H}_0$.

Simplification of Maxwell's macroscopic equations:

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \text{and} \quad \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0 \qquad \text{stay the same}$$

We want to use Faraday's law to give us $\vec{E}$ from $\vec{B}$. Since ω is small, we expect $|\vec{E}| \sim \omega|\vec{B}| \ll |\vec{B}|$.

Now $|\vec{D}| \sim |\vec{E}| \ll |\vec{B}| \sim |\vec{H}|$ so $\frac{\partial\vec{D}}{\partial t} \sim \omega\vec{D} \sim \omega^2 B$ can be neglected in

Ampere's law: $\vec{\nabla}\times\vec{H} - \frac{1}{c}\frac{\partial\vec{D}}{\partial t} = \frac{4\pi}{c}\vec{J} \quad \Rightarrow \quad \vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{J}$

Note also that

$$\frac{1}{c}\frac{\partial\vec{D}}{\partial t} \sim \frac{\omega}{c}\epsilon\vec{E} = \frac{\omega}{c}\sigma_0\epsilon\vec{J} \ll \frac{4\pi}{c}\vec{J}$$

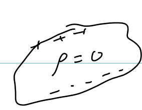
Using Ohm's law

$$\Rightarrow \quad \vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{J} \Rightarrow \vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\sigma_0\vec{E}$$

Now $\vec{\nabla}\cdot\left(\frac{4\pi}{c}\sigma_0\vec{E}\right) = \vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{H}\right) = 0 \quad \Rightarrow \vec{\nabla}\cdot\vec{E} = 0$

and with $\vec{\nabla}\cdot\vec{D} = 4\pi\rho \quad \Rightarrow \quad \rho = 0$

$\Rightarrow$ No free charges in bulk of conductor, just as in electrostatics.



(This is not a surprise: we are taking $\omega \approx 0$ in Maxwell equations for conductors).

Summary: $\vec{\nabla}\cdot\vec{B}=0 \qquad \vec{\nabla}\times\vec{E}+\frac{1}{c}\frac{\partial\vec{B}}{\partial t}=0$

$$\vec{\nabla}\times\vec{H}=\frac{4\pi}{c}\sigma_0\vec{E} \qquad \vec{\nabla}\cdot\vec{E}=0$$

Take $\vec{\nabla}\times(\vec{\nabla}\times\vec{H})=-\nabla^2\vec{H}$

$$\hookrightarrow = \vec{\nabla}\times\left(\frac{4\pi}{c}\sigma_0\vec{E}\right) = \frac{4\pi\sigma_0}{c}\left(-\frac{1}{c}\frac{\partial\vec{B}}{\partial t}\right) = -\frac{4\pi\sigma_0}{c^2}\mu\frac{\partial\vec{H}}{\partial t}$$

$$\Rightarrow \boxed{\nabla^2\vec{H}=\frac{4\pi\sigma_0\mu}{c^2}\frac{\partial\vec{H}}{\partial t}}$$

Alternatively, $\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=-\nabla^2\vec{E}$

$$\hookrightarrow \vec{\nabla}\times\left(-\frac{1}{c}\frac{\partial\vec{B}}{\partial t}\right) = -\frac{1}{c}\mu\frac{\partial}{\partial t}\vec{\nabla}\times\vec{H} = -\frac{4\pi\sigma_0\mu}{c^2}\frac{\partial\vec{E}}{\partial t}$$

$$\Rightarrow \boxed{\nabla^2\vec{E}=\frac{4\pi\sigma_0\mu}{c^2}\frac{\partial\vec{E}}{\partial t}}$$

Digression:

Each component of $\vec{H}$ and $\vec{E}$ satisfies the diffusion or heat conduction equation:

$$\nabla^2\psi=\kappa\frac{\partial\psi}{\partial t}$$

If $\psi(\vec{r},t)=\psi(z,t)$ only then $\frac{\partial^2\psi}{\partial z^2}=\kappa\frac{\partial\psi}{\partial t}$

To solve this let $\psi(z,t)=\int\frac{dk}{2\pi}\tilde{\psi}(k,t)\,e^{ikz} \Rightarrow -k^2\tilde{\psi}=\kappa\frac{\partial\tilde{\psi}}{\partial t} \Rightarrow \tilde{\psi}=\tilde{\psi}_0\,e^{-\frac{k^2}{\kappa}t}$

$\Rightarrow \psi(z,t)=\int\frac{dk}{2\pi}\tilde{\psi}_0\,e^{-\frac{k^2}{\kappa}t}e^{ikz}$; the exponent $-\frac{k^2}{\kappa}t+ikz=-\frac{t}{\kappa}\left(k-\frac{i\kappa z}{2t}\right)^2-\frac{\kappa z^2}{4t}$

$\Rightarrow \psi(z,t)=\tilde{\psi}_0\,e^{-\frac{\kappa}{4}z^2/t}\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{-\frac{t}{\kappa}k^2} = \psi_0\sqrt{\frac{\kappa}{t}}\,e^{-\frac{\kappa}{4}z^2/t}$ (I have absorbed a constant into ψ_0).

Check: $\frac{\partial^2}{\partial z^2}\psi=\psi_0\sqrt{\frac{\kappa}{t}}\frac{\partial}{\partial z}\left(-\frac{\kappa}{2}\frac{z}{t}e^{-\frac{\kappa}{4}z^2/t}\right)=\psi_0\sqrt{\frac{\kappa}{t}}\left(-\frac{\kappa}{2}\frac{1}{t}+\left(\frac{\kappa}{2}\frac{z}{t}\right)^2\right)e^{-\frac{\kappa}{4}z^2/t}$

$\kappa\frac{\partial}{\partial t}\psi=\kappa\psi_0\sqrt{\kappa}\left(-\frac{1}{2}t^{-3/2}+\frac{1}{\sqrt{t}}\frac{\kappa}{4}z^2/t^2\right)e^{-\frac{\kappa}{4}z^2/t}=\psi_0\sqrt{\frac{\kappa}{t}}\left(-\frac{\kappa}{2}\frac{1}{t}+\frac{\kappa^2}{4}\frac{z^2}{t^2}\right)e^{-\frac{\kappa z^2}{4t}}$ ✓✓

3D case: using $\psi(\vec{r},t) = X(x)\,Y(y)\,Z(z)$

$$\frac{1}{X}X'' + \frac{1}{Y}Y'' + \frac{1}{Z}Z'' = \kappa\left(\frac{1}{X}\dot{X} + \frac{1}{Y}\dot{Y} + \frac{1}{Z}\dot{Z}\right)$$

$$\Rightarrow \frac{1}{X}X'' = \kappa\frac{1}{X}\dot{X} + f_x(t) \quad \text{etc.} \qquad \text{with } f_x(t) + f_y(t) + f_z(t) = 0$$

For example, if $f_i(t) = 0$ we have three copies of the 1Dim case

$$\psi = \psi_0 \frac{1}{t^{3/2}} e^{-\frac{\kappa}{4} r^2/t}$$

These well known solutions are appropriate for diffusion: as $t \to 0+$ $\psi(z,t) \to \delta(z)$ and $\psi(\vec{r},t) \to \delta(\vec{r})$, with a clear interpretation: put a pointlike "drop" of fluid and it diffuses out, with distance $\sim \sqrt{t}$.

In the cases we study the problem is different. Imagine starting with a field $\psi_0(\vec{r})$ at $t=0$ (say an external field that is turned off). What happens next? To this end, solve the eigenvalue problem

$$\nabla^2 \psi_n(\vec{r}) = -\gamma_n \psi_n(\vec{r}) \qquad n = 1, 2, \ldots$$

Then $\psi(\vec{r},t) = \sum_n c_n e^{-(\gamma_n/\kappa)t} \psi_n(\vec{r})$ solves $\nabla^2\psi = \kappa \frac{\partial \psi}{\partial t}$

and the c_n's are chosen so that $\psi(\vec{r},0) = \psi_0(\vec{r}) = \sum_n c_n \psi_n(\vec{r})$

(As usual, with eigensystems, $(\psi_n, \psi_m) = 0$ if $\gamma_n \neq \gamma_m$ so one can orthonormalize the solutions so $c_n = (\psi_n, \psi_0)$).

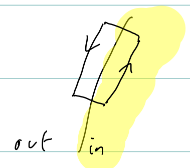
The important point is that ψ_n dies exponentially within a time $\tau \sim \kappa\gamma_1^{-1}$ (assuming $\gamma_1 < \gamma_2 < \ldots$).

Since we expect $\gamma_1 \sim \frac{O(1)}{a^2}$, the typical decay time is $\tau \sim a^2\kappa = \frac{4\pi\mu\sigma_0 a^2}{c^2}$

which for $a \sim 1\,\text{cm}$ and $\sigma_0 \sim 10^{17}\,\text{sec}$, $\mu \sim 1$, gives $\tau \sim 10^{-3}\,\text{sec}$.

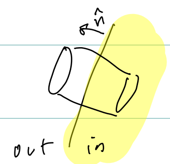
Boundary conditions: (to solve problem fully)

Assume boundaries are between conductor and vacuum.



$$\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0 \Rightarrow \vec{E}_{t\,in} = \vec{E}_{t\,out}$$

$$\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\sigma_0\vec{E} \longrightarrow \vec{H}_{t\,in} = \vec{H}_{t\,out}$$



$$\vec{\nabla}\cdot\vec{B} = 0 \longrightarrow B_{n\,out} = B_{n,in}$$

Note that with $\mu = 1$ this means $\vec{H}_{in} = \vec{H}_{out}$ (or $\vec{B}_{in} = \vec{B}_{out}$)

Left with E_n? L&L says: $\vec{\nabla}\cdot\vec{j} = 0$ and $\vec{j}_{out} = 0 \Rightarrow j_{n,in} = 0$

and since $\vec{E}_{in} = \sigma_0 \vec{j}_{in} \Rightarrow E_{n,in} = 0$.

Digression: Garg wants a more refined version. From the previous unit, we had

$$\vec{\nabla}\cdot\vec{\tilde{Z}} = 4\pi\tilde{\rho}' \qquad \text{where } \vec{\tilde{Z}} = \tilde{\varsigma}\vec{\tilde{E}} \quad \text{and } \tilde{\varsigma} = \tilde{\epsilon} + i\frac{4\pi\tilde{\sigma}}{\omega}$$

and $\tilde{\rho}'$ are charges from currents not subject to Ohm's law (ie, not included in $\vec{\tilde{E}} = \tilde{\sigma}\vec{\tilde{j}}$). From this $\tilde{E}_{n,out} = \tilde{\varsigma}\tilde{E}_{n,in}$ (for $\rho' = 0$).

From this we recover $E_{n,in} = 0$ (that is $E_{n,in} \sim \frac{\omega}{\sigma_0}E_{n,out} \to 0$ in the approx.).

If Σ = surface charge density (use Σ rather than σ, to avoid confusion with conductivity), then $Z_{n,out} - Z_{n,in} = 4\pi\Sigma$ (sign from $\hat{n}$ = outward pointing).

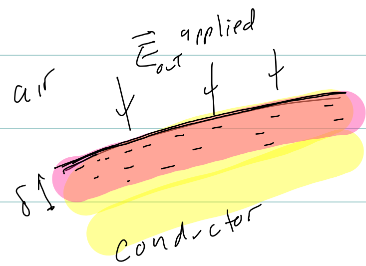
Then, using $E_{n,in} = -i\frac{\omega}{4\pi\sigma_0}E_{n,out} \Rightarrow E_{n,out}\left(1 - i\frac{\omega}{4\pi\sigma_0}\right) = 4\pi\Sigma \Rightarrow E_{n,out} \simeq \left(4\pi + i\frac{\omega}{\sigma_0}\right)\Sigma$

while $E_{n,in} \doteq -i\frac{\omega}{4\pi\sigma_0}E_{n,out} \simeq -i\frac{\omega}{\sigma_0}\Sigma$

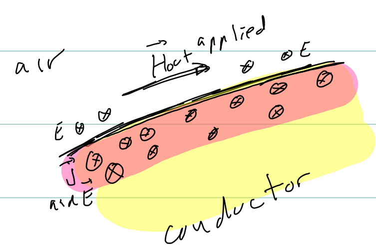
So where are we going with all this?

Put a conductor in an external quasistatic field ($\vec{E}$ or $\vec{H}$)

From the diffusion/heat transfer equation we expect the fields will not penetrate the conductor much. For $\vec{E}$ it is clear, much like in electrostatic case, charge at surface will screen. But now the charge is spread over some "skin depth" δ fixed by diffusion equation.



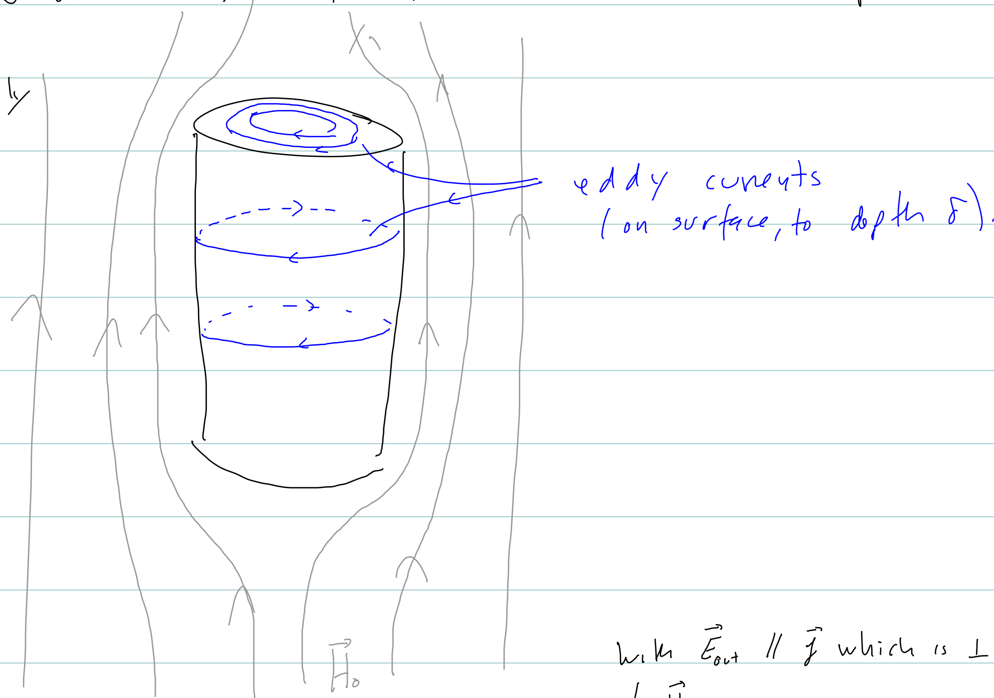
For magnetic field to be screened we need a current on the surface — down to depth δ. Since $\vec{J} = \sigma_0 \vec{E}$ and $J_n = 0$ at boundary, we will have an $\vec{E}_{t,in}$, but then $\vec{E}_{t,out} = \vec{E}_{t,in}$, so also outside



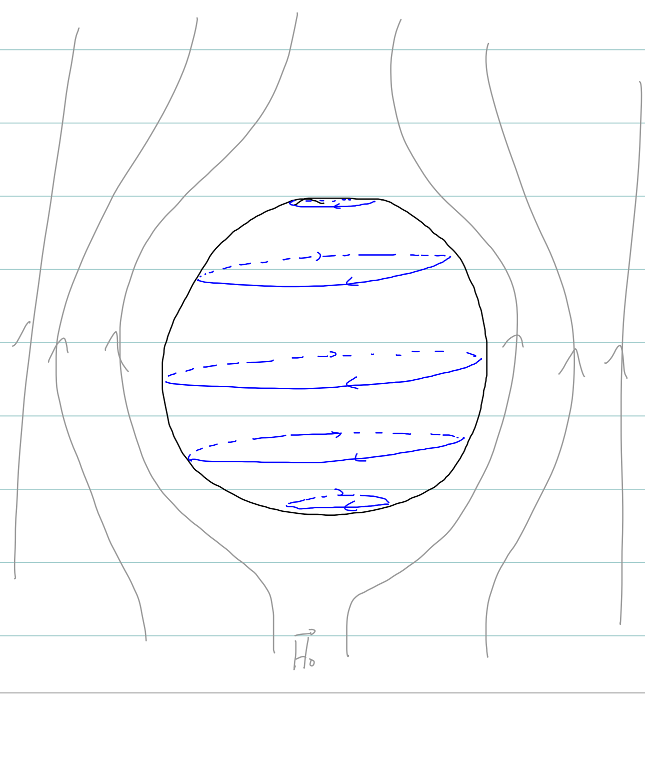
So we want to understand the skin depth and these currents called eddy currents.

Once we look at these in general terms, we can look at specific cases. Garg shows two geometries (of conductors) in time varying (harmonic) external $\vec{H}$: cylinder and sphere.

Qualitatively

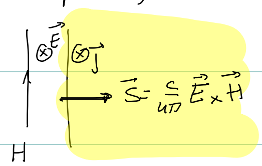


with $\vec{E}_{out} \parallel \vec{J}$ which is $\perp$ to $\vec{H}$



We also want to understand energy conservation:

We see (above pic's)



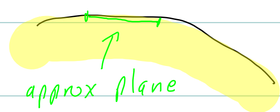
There is energy flow into conductor. Where does it go?

There is also energy dissipation, from $\vec{J}\cdot\vec{E} = \sigma E^2$ in the conductor

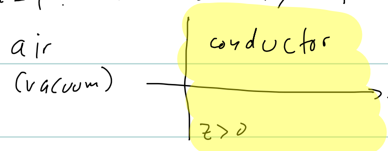
The energy that flows in = energy dissipated.

Plane conductor

While $a \to \infty$ is outside the regime we are studying, we can look at a plane conductor as a local approximation of a large but finite size conductor



Take $\mu = 1$. Set boundary of conductor on xy plane, and conductor on $z > 0$.



Assume $\vec{B} \, (= \vec{H}) = B_0 \hat{x}$ (along plane; $e^{-i\omega t}$ dependence implicit) for $z = 0-$.

$\Rightarrow$ $\vec{B} = B_0 \hat{x}$ for $z = 0+$ because $\vec{B}_{\text{in}} = \vec{B}_{\text{out}}$.

The difussion equation is

$$\nabla^2 \vec{B} = -i \frac{4\pi \sigma_0 \omega}{c^2} \vec{B}$$

For $B_{y,z}$ with b.c. $B_{y,z} = 0$ at $z = 0$ gives $B_{y,z} = 0$.

For B_x, we look for a solution that depends on z only, $B_x = B_0 \hat{x} \, e^{ikz}$

$$\Rightarrow \quad k^2 = i \frac{4\pi \sigma_0 \omega}{c^2} \quad \Rightarrow \quad k = \pm \sqrt{i} \sqrt{\frac{4\pi \sigma_0 \omega}{c^2}}$$

With $\sqrt{i} = \left(e^{i\pi/2}\right)^{1/2} = e^{i\pi/4} = \frac{1}{\sqrt{2}}(1+i)$ this is $k = \pm(1+i)\sqrt{\frac{2\pi\sigma_0\omega}{c^2}}$

This gives $\sim e^{\pm(i-1)\sqrt{\frac{2\pi\sigma_0\omega}{c^2}}\, z}$. The $-$ sign solution gives B_x <u>increasing</u> with z, which is unphysical. So keep only $+$ sign:

$$B_x = B_0 e^{-(1-i)\sqrt{\frac{2\pi\sigma_0\omega}{c^2}}\, z} = B_0 \, e^{-(1-i)z/\delta}$$

where $\delta = \frac{c}{\sqrt{2\pi\sigma_0\omega}}$ is the "skin depth".

For Cu @ 300°K, $\delta(60\,\text{Hz}) \simeq 8.5\,\text{mm}$, $\delta(100\,\text{MHz}) = 7\,\mu\text{m}$

From $\vec{\nabla}\times\vec{H} = \frac{4\pi\sigma_0}{c}\vec{E}$ we can compute $\vec{E}$:

$$\epsilon_{ijk}\,\partial_j B_k = \epsilon_{izx}\frac{\partial B_x}{\partial z} = \delta_{iy}\frac{\partial}{\partial z}\left(B_0 e^{-(1-i)z/\delta}\right) = \delta_{iy}\left[-(1-i)\frac{1}{\delta}B_0 e^{-(1-i)z/\delta}\right]$$

$$\Rightarrow\quad \vec{E} = -\frac{cB_0}{4\pi\sigma_0\delta}(1-i)\,\hat{y}\;e^{-(1-i)z/\delta} = E_0\hat{y}\,e^{-(1-i)z/\delta}$$

where $E_0 = -\frac{cB_0}{4\pi\sigma_0\delta}(1-i)$

Writing the fields as 'real part of' and restoring ω-dependence we discover phase shift:

$$\vec{B} = B_0\hat{x}\,e^{-z/\delta}\cos\left(\frac{z}{\delta}-\omega t\right)$$

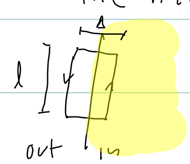
$$\vec{E} = -|E_0|\hat{y}\,e^{-z/\delta}\cos\left(\frac{z}{\delta}-\omega t-\frac{\pi}{4}\right)$$

(The phase shift is from $1-i = \sqrt{2}e^{-i\pi/4}$).

In addition $\vec{j} = \sigma_0\vec{E}$ is now determined. Note that for $\delta \ll a$ the current is confined to the "surface" of the conductor, and can be modeled by a surface current density $\vec{K} = \int_0^\infty dz\,\vec{j} = \sigma_0 E_0\,\hat{y}\,\frac{\delta}{1-i} = -\frac{cB_0}{4\pi}\hat{y}$

In $\vec{K} = -\frac{cB_0}{4\pi}\hat{y}$ there is no σ_0 ! $\vec{K}$ is there to shield $\vec{B}_{in}$:

In the naive approach one has, from Ampere's law



$$\int d^2a\,\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\int d^2a\,\vec{j}$$
$$\parallel$$
$$\oint \vec{B}\cdot d\vec{l} = (B_{in\,t} - B_{out\,t})\,l = \frac{4\pi}{c}\,l\,K_\perp$$

ie $B_{in} - B_{out} = \frac{4\pi}{c}K_\perp$.

We have $K_\perp = -\frac{c}{4\pi}B_{out}$ so it must be that $B_{in} = 0$. In this naive approximation B_{in} corresponds to the our B_{in} at $z \gg \delta$, hence vanishingly small.

Define "surface impedance" Z_s : $\vec{E} = Z_s \vec{K}$

So that in the present case $Z_s = \frac{(1-i)}{\sigma_0 \delta}$

Note also that, as expected $|\vec{E}|/|\vec{B}| \ll 1$:

$$\frac{|\vec{E}|}{|\vec{B}|} = \frac{c}{4\pi\sigma_0}\frac{\sqrt{2}}{\delta} = \frac{c}{4\pi\sigma_0}\sqrt{2}\,\frac{\sqrt{2\pi\sigma_0\omega}}{c} = \sqrt{\frac{\omega}{4\pi\sigma_0}} \ll 1$$

Energetics! Compute $\vec{S}_{out}$; $\vec{E}_{t\,in} = \vec{E}_{t\,out}$ gives $\vec{E}$ outside conductor ($E_{n,out} = \tilde{\varepsilon} E_{n,in} = 0$). Then

$$\vec{S} = \frac{c}{4\pi}(\vec{E}\times\vec{B}) = \frac{c}{4\pi}\left(E_0 B_0\, \hat{y}\times\hat{x}\right) = \frac{c}{4\pi} E_0 \left(-\frac{4\pi\sigma_0\delta}{c(1-i)} E_0\right)(-\hat{z})$$

$$= \frac{\sigma_0\delta}{1-i} E_0^2\,\hat{z}$$

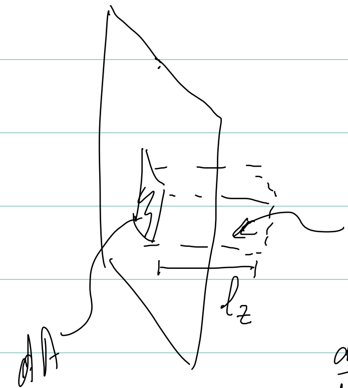
Brief review of averaging over time: complex fields $a e^{-i\omega t}$ are really $\frac{1}{2}(a e^{-i\omega t} + a^* e^{i\omega t})$. Then $\overline{ab} = \frac{1}{T}\int_0^T dt\, \frac{1}{4}(a e^{-i\omega t} + c.c)(b e^{-i\omega t} + c.c)$

$$= \frac{1}{4}(ab^* + a^*b) = \frac{1}{2}\mathrm{Re}(ab^*)$$

Time average $\vec{S}$:

$$\overline{\vec{S}} = \frac{1}{2}\mathrm{Re}\left(\frac{\sigma_0\delta}{1-i} E_0 E_0^*\,\hat{z}\right) = \frac{1}{4}\sigma_0\delta|E_0|^2\,\hat{z}$$

Let's compare with the energy dissipated. Work done per unit volume per unit time: $\vec{j}\cdot\vec{E}$. Time averaged: $\frac{1}{2}\,\mathrm{Re}(\vec{j}\cdot\vec{E}^*)$.



work done in volume: $\frac{1}{2}\mathrm{Re}(\vec{j}\cdot\vec{E}^*)\, l_z\, dA$

$\Rightarrow$ work done / unit surface area

$$\frac{dQ}{dt\,dA} = \int_0^\infty dz\, \frac{1}{2}\mathrm{Re}(\vec{j}\cdot\vec{E}^*)$$

$$= \int_0^\infty dz\, \frac{1}{2}\mathrm{Re}(\sigma_0\, \vec{E}\cdot\vec{E}^*)$$

Use $\vec{E} = E_0\,\hat{y}\, e^{-(1-i)z/\delta}$ $\longrightarrow$

$$= \frac{1}{2}\sigma_0 |\vec{E}_0|^2 \int_0^\infty dz\, e^{-2z/\delta}$$

$$= \frac{1}{4}\sigma_0\,\delta\, |E_0|^2$$

Same as $|\vec{S}|$! Energy flows in = energy dissipated. ☺

Note one can also write $Z_s^{-1} = \frac{\sigma\delta}{1-i} = \frac{\sigma\delta}{2}(1+i)$ so

$$\frac{dQ}{dt\,dA} = \frac{1}{2}\mathrm{Re}\left(\frac{1}{Z_s}\right)|E_0|^2 = \frac{1}{2}\mathrm{Re}\left(\frac{1}{Z_s}\vec{E}_0\cdot\vec{E}_0^*\right) = \frac{1}{2}\mathrm{Re}(\vec{K}\cdot\vec{E}_0)$$

Or since $\left|\frac{\vec{E}}{\vec{B}}\right|^2 = \frac{\sqrt{2}\,c}{4\pi}\frac{1}{\sigma_0\delta}$, $\frac{dQ}{dt\,dA} = \frac{1}{4}\,2\left(\frac{c}{4\pi}\right)^2\frac{1}{\sigma_0\delta}|\vec{B}_0|^2 = \frac{1}{2}\left(\frac{c}{4\pi}\right)^2 \mathrm{Re}\, Z_s\, |\vec{B}_0|^2$

$\vec{E}$ outside conductor? (ie for $z<0$).

As we said in the introduction, it is given by Faraday's law

$$\vec{\nabla}\times\vec{E} - i\frac{\omega}{c}\vec{B} = 0$$

with b.c. $\vec{E}(z=0-) = E_0\,\hat{y}$

By symmetry $\vec{E} = \hat{y}\, E_y(z)$ only, and recall $\vec{B} = \hat{x} B_0$

so $(\vec{\nabla}\times\vec{E})_x = -\frac{\partial E_y}{\partial z} = i\frac{\omega}{c}B_0$

$$\Rightarrow \quad E_y = E_0 - i\frac{\omega}{c}B_0 z \qquad \text{(disagree with sign in Garg).}$$

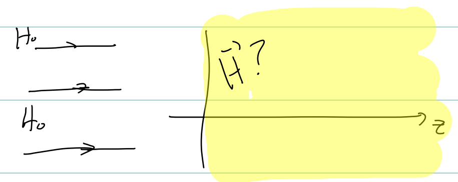
$\lambda\frac{\omega}{2\pi} = c$ $$= E_0 - 2\pi i\left(\frac{z}{\lambda}\right)B_0$$

So $E_y(z)$ seems to increase without bounds as $z\to-\infty$. But this is not so: we are assuming distance scales are $\ll \lambda$ so the solution is a good approximation only close to the conductor. For example if conductor is inside a solenoid, $z\to-\infty$ will take us outside this region. Moreover, as $|\vec{E}|$ increases, it cannot be neglected in Ampere's law.

$\delta(\omega)$: Note $\delta(\omega) = \frac{c}{\sqrt{2\pi\sigma_0\omega}} \sim \frac{1}{\sqrt{\omega}} \to \infty$ as $\omega\to 0$! It would appear that in static case $\vec{E}$ penetrates the whole conductor! But wait $|\vec{E}|/|\vec{B}| = \sqrt{\frac{\omega}{4\pi\sigma_0}} \to 0$ as $\omega\to 0$. So there is no field.

$\vec{H} = H_0 \hat{z}$?

Now suppose the external applied field is perpendicular to the surface $z=0$, $\vec{H}_0 = H_0 \hat{z}$



We want to find $\vec{H}$ inside. But importantly, $\vec{H}$ outside is not $\vec{H}_0$. Still as a 1[st] guess we can assume $\vec{H} \simeq \vec{H}_0$ at $z=0-$ (just outside) $\Rightarrow$ $\vec{H}_{in} = \vec{H}_{out}$ means $\vec{H}(z=0+) = H_0 \hat{z}$. Let's use the diffusion equation to find $\vec{H}$ $\left(= \frac{1}{\mu}\vec{B} = \vec{B}\right.$ since we are using $\left.\mu = 1\right)$: but this is the same as before, except for $H_z = B_z$ instead of B_x:

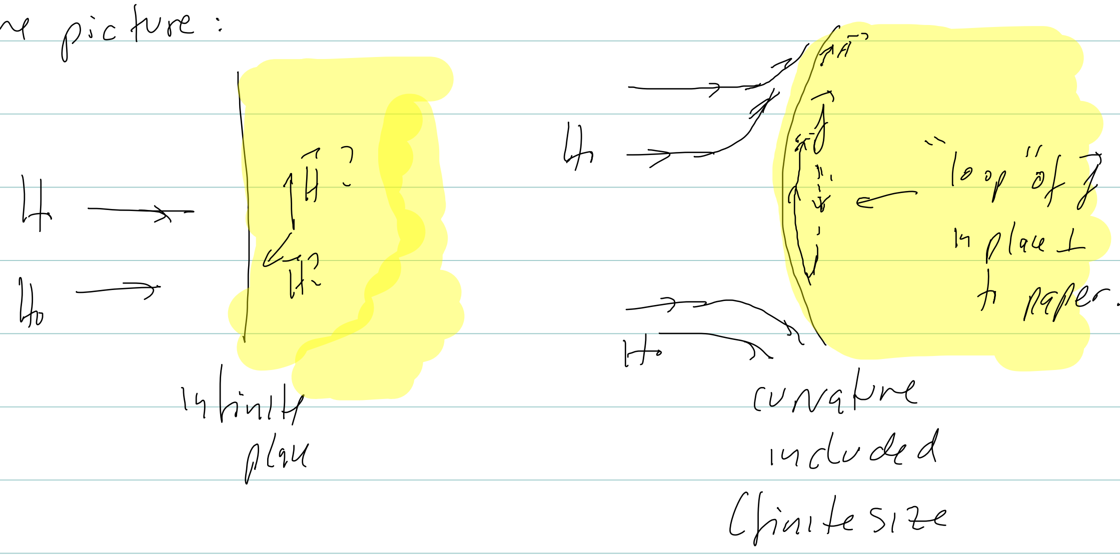
$$H_z = H_0 \hat{z}\, e^{-(1-i) z/\delta}$$

The problem here is $\vec{\nabla}\cdot\vec{B} = \overset{1}{\mu}\, \vec{\nabla}\cdot\vec{H} = \partial_z H_z \neq 0$ in violation of $\vec{\nabla}\cdot\vec{B} = 0$.

To understand what happens we cannot continue to take an infinite plane approximation to the finite size body, of size $a \ll \lambda$. A field $\vec{H}_{\parallel}$ must be produced that is parallel to the surface (Garg calls it $B_\perp$ because it is perpendicular to $\vec{H}_0$ — I find his nomenclature confusing). With this one can satisfy $\vec{\nabla}\cdot\vec{B} = 0$ since $\vec{\nabla}\cdot\vec{B} = \frac{\partial B_{\parallel}}{\partial x_{\parallel}} + \frac{\partial B_z}{\partial z}$

Note that $\vec{B}$ is then confined to a region of depth δ in the conductor.

Let's assume $\delta \ll a$ (the opposite limit $\delta \gg a$ is basically that of $\omega = 0$, ie, magnetostatics). Then, in order to shield the bulk of the conductor from $\vec{B}$ we need a current $\vec{J}$ in the skin. What breaks the symmetry in the xy plane if $\vec{H}_0 = H_0 \hat{z}$, ie, is $H_\parallel$ along $\hat{x}$ or $\hat{y}$? The answer is the finite size, as is easily seen from the picture:



And, of course that means there is an $\vec{E}$ field $(\vec{E} = \frac{1}{\sigma}\vec{J})$. Note that $H_\parallel$ changes on scale of curvature, which itself is the scale of the size a of the body, while H_z changes over scale δ. Since $\vec{\nabla}\cdot\vec{B} = 0$ we have $\frac{H_\parallel}{a} \sim \frac{H_0}{\delta}$ or $H_\parallel = \frac{a}{\delta} H_0 \gg H_0$: The $\parallel$-field dominates and again $\vec{H}_{in} = \vec{H}_{out}$ means now that close to the body $\vec{H}$ is nothing like the uniform applied $\vec{H}_0$ — on surfaces that are not parallel to $\vec{H}_0$.

One can see this explicitly in analytic solutions of the cylinder and sphere problems (those shown in p. 7 of these notes), but we will not go through those calculations. The general principle is what we are after, and that is enough to figure out generally what happens in other geometries, eg: