Chapter 4: Fields of Moving Charges

Lienard-Wiechert potential $\mathbf{A}(x)$

To determine the field due to a point charge in arbitrary motion given by a specified trajectory $y(t)$ (we use $y^0$ for the $x^0$ so as to not get confused with the argument of $\phi(y)$), we use the retarded Green's function $G_{ret}(x,x')$. Recall

$$G_{ret}(x,x') = G_{ret}(0,x-x')$$

with $\delta^4(x) = \delta^4(x_0 - x_0')$ trajectory $y(t)$

and

$$A^\mu_r(x) = A^\mu_{ret}(x) + \int d^4y G_{ret}(x,x') \frac{\partial}{\partial x^\mu} \phi(y)$$

(we call $\partial A^\mu_r = \frac{\partial}{\partial x^\mu} \phi$ in Lorentz gauge).

Notice that the $\delta$-function in $G_{ret}$ means that the field at $(x,x')$ is determined by the charge at the retarded time $x^0_r = x^0 - \frac{1}{c} |x-x'|$. That is, $t_r = t - \frac{1}{c} |$ at time $t_r$.

$A^\mu_r(x)$ has $\delta^4 A^\mu_r = 0$ and has a simple interpretation: if the charge is infinitely far away as $t \to -\infty$, then the only contribution to $A^\mu_r(x)$ is from $A^\mu_0(x) \equiv 0$, the "initial" value of $A^\mu_r(x)$, specified at $t = -\infty$. We set it to zero (can add it back at no cost).

The 4-current of the point charge is $J^\mu(x) = eq \delta^4 \delta(x-y(x))$.

Trick: multiply by $\int d^4y \, \delta(x-y(y))$ to obtain a covariant expression and, more usefully, a $\delta(y):$ $\int d^4y \, \delta(x-y(y))$ where $u^\mu = \frac{dy^\mu}{dy}$.

So we have

$$A^\mu_r(x) = \int d^4y' G_{ret}(x,x') \frac{\partial}{\partial x^\mu} \left[ \delta^4 \left( x-y' \right) \right] = \int d^4y' G_{ret}(x,y') \frac{\partial}{\partial x^\mu} \left[ \delta^4 \left( x-y' \right) \right]$$

Then we can use an integral. The elegant way involves expressing $G_{ret}$ in a Lorentz-invariant way: since $x^0 = y^0$ is on the future light cone, consider

$$\delta(x^0-y^0) = \delta(x^0-y^0) \theta(y^0) = \frac{\delta(x^0-y^0)}{2y^0}$$

so $G_{ret} = \frac{1}{2y^0} \delta((x-y)^2) \theta(x^0-y^0)$

$$\Rightarrow A^\mu_r(x) = \int d^3y' \delta((x-y'^0) \theta(x^0-y'^0)) = \int d^3y' \frac{1}{(x-y'^0)y^0} \frac{\partial}{\partial x^\mu} \left[ \frac{1}{(x-y'^0)y^0} \right]$$

\[\text{Lienard-Wiechert potentials}\]
Here, evaluated at $\lambda_0$ means, as anticipated, at retarded time: $\lambda_0$ is the solution to 

$$(x-y(\lambda_0))^2 = 0 \quad \text{with} \quad x^+ = y^+(\lambda_0)$$

which, of course, is just 

$$(y^+(\lambda)) = \lambda_0 - |x-y(\lambda)|$$

The alternative way is to use $G_{\mu\nu}(x) = \frac{1}{16\pi} \delta(x-R)$ directly:

$$A^\mu(x) = \frac{1}{4\pi} \int \frac{u^{-1} \sigma^\mu \frac{1}{i\sigma^\nu} \delta(x-y^\nu(x))}{i\sigma^\nu} \left[ \frac{1}{R \sqrt{1-R^2}} \right]$$

where $\frac{d}{dx} (y^\nu + i\sigma^\nu) = \frac{dy^\nu}{dx} + \frac{(x-y^\nu + i\sigma^\nu)}{i\sigma^\nu}$

This equals the previous expression, since

$$\int \frac{1}{i\sigma^\nu} \left[ \frac{(x-y^\nu + i\sigma^\nu)}{i\sigma^\nu} \right] du^{-1} = \frac{1}{R} \left( \frac{x-y^\nu}{R} \right)$$

so

$$A^\nu(x) = \frac{qU^\nu}{|x-y^\nu - i\sigma^\nu|} = \frac{qU^\nu}{R \sqrt{1-R^2}}$$

Let's write the potentials in terms of the velocity $\vec{v}$ of the charge and the distance $R$.

Now, retarded change to $x$. Using $x-y^\nu = (1,\vec{v})$. As above,

$$A^\nu(x) = \frac{qU^\nu}{16\pi R} \frac{1}{\frac{d}{dx} (x-y^\nu + i\sigma^\nu) - \frac{x-y^\nu}{R \sqrt{1-R^2}}} = \frac{qU^\nu (1,\vec{v})}{R} \cdot \frac{1}{1-R^2}$$

where $\vec{v}$ is at $\ell$ (retarded).

We can also compute $\vec{E}$ and $\vec{B}$. The application is that in this $\frac{d\mu}{dx}$ we are changing not just $x$ but also $\lambda_0$ (think of $\lambda_0 = \lambda_0(x)$ determined by $(x-y(\lambda_0))^2 = 0$).

$$(x-y(\lambda_0))^2 = 0 \quad \text{and} \quad (x+\delta x - y(\lambda_0 + \delta\lambda))^2 = 0 \quad \Rightarrow \quad (x+\delta x - y(\lambda_0 + \delta\lambda))^2 = 0$$

For $E_{\mu\nu}$ we need $\frac{d}{dx} A^\nu$, and for its $\frac{d}{dx} (x-y^\nu) = \frac{\partial}{\partial x} (x-y^\nu) = \frac{\partial}{\partial x} \left( x^+ - y^+(\lambda_0) \right)$

$$= \frac{\partial}{\partial x} x^+ = \frac{(x-y)^x}{x-y-x}$$

where $\frac{d}{dx} A^\nu = \frac{d}{dx} \sigma^\nu = \frac{\partial}{\partial x} \frac{\partial}{\partial x} u^\nu = \frac{\partial}{\partial x} \frac{\partial}{\partial x} (x-y)^\nu \frac{\partial}{\partial x} \frac{\partial}{\partial x}$
Then \( E_\nu = \beta \left( \frac{\alpha}{(x-y) \cdot u} \right) - \mu \gamma \nu \)

\[
- \frac{\beta}{[(x-y) \cdot u]^3} \left[ (x-y)_\mu \gamma_{\lambda} - (x-y)_\lambda \gamma_{\mu} \right] \gamma_{\nu} - (x-y)_\lambda \frac{(x-y)_\mu}{[(x-y) \cdot u]^2} - \mu \gamma \nu
\]

This gives \( E \) and \( B \) in terms of retarded \( y^0(\mathbf{y}) \), \( u^0(\mathbf{y}) \) and \( \alpha^0(\mathbf{y}) \). Note that it is reparametrization invariant. It is useful to write this more explicitly in terms of 
\( \beta, \mathcal{R} \) and \( \nu \). So here \( \nu = \gamma^0 \).

Let's separate this into the \( u \)-dependent \( \beta \cdot E^\text{acc} \) and \( u \)-independent \( E^\text{v} \):

\[
E^\text{acc}_\nu = \frac{\alpha}{[(x-y) \cdot u]^3} \left( \frac{(x-y)_\mu}{[(x-y) \cdot u]} - \mu \gamma \nu \right)
\]

\[
E^\text{v}_\mu = \frac{\alpha}{[(x-y) \cdot u]^3} \left[ (x-y)_\lambda u^\lambda - (x-y)_\lambda u^\lambda \right]
\]

Then \( u^- = (1, \beta) \), \( (x-y) = u^\nu (x-y)_\nu \), \( u = [x \cdot y] y \), \( R = \frac{1}{1 - \beta^2} \).

\[
E^\text{v}_\mu = \frac{\alpha}{\delta \beta} \left( \frac{(x-y)_\mu}{(1 - \beta^2)^{3/2}} \right) \Rightarrow E^{\text{v}_\mu} = - \frac{\alpha}{\delta \beta} \left( \frac{1}{(1 - \beta^2)^{3/2}} \right) R \beta^2 (\mathbf{R} - \mathbf{R}^\prime)
\]

and \( R^{\text{v}_\mu} = \frac{\delta \beta}{\delta \beta} \cdot R^{\text{v}_\mu} = - \frac{\delta \beta}{\delta \beta} \cdot \frac{1}{(1 - \beta^2)^{3/2}} = - \frac{\delta \beta}{\delta \beta} \cdot \frac{R \beta^2}{(1 - \beta^2)^3} \)

which we recognize as the \( E \) and \( B \) fields of a moving charge with \( \beta \) constant.

But for \( \beta^2 \to 0 \) we have additional terms. We need

\[
\alpha^0 = \frac{\partial \alpha}{\partial y^0} = \frac{\partial \alpha(1, \beta)}{\partial y^0} = (0, \hat{z}) \quad \text{with} \quad \alpha = \frac{\partial \alpha}{\partial y^0} = \frac{1}{\hat{z}} \alpha
\]

\[
E^\text{acc} = - \frac{\alpha}{\delta \beta (1 - \beta^2)^{3/2}} \left[ R \left( \alpha + \frac{\beta}{1 - \beta^2} \right) - \frac{\beta}{1 - \beta^2} \right] = \frac{\alpha}{\delta \beta} \left( \frac{1}{1 - \beta^2} \right) \left[ \frac{\beta}{1 - \beta^2} \right]
\]

\[
\beta^{\text{acc}} = \frac{\alpha}{\delta \beta (1 - \beta^2)^{3/2}} \left( \hat{x} + \frac{\hat{z}}{1 - \beta^2} \right)
\]

\[
\text{Note that} \quad B^{\text{acc}} = \hat{R} \times E^{\text{acc}} \quad \text{and} \quad |\mathbf{E}^{\text{acc}}| \sim \frac{1}{\beta}. \quad \text{So} \quad \mathbf{E} \sim \frac{1}{\beta} \hat{R} \times \mathbf{E} \text{ "radiation field"}
\]

\[
\text{Note that} \quad \hat{R} \times \left[ (\hat{R} - \beta) \times \hat{z} \right] = (\hat{R} - \beta) \hat{R} \times \hat{z} - \hat{R} \cdot \hat{z} \hat{R} (\hat{R} - \beta) = - (\hat{R} - \beta) \left( \hat{x} \cdot \hat{z} \frac{\hat{z}}{1 - \beta^2} \right)
\]

\[
\text{so one may write} \quad E^{\text{acc}} = \frac{\alpha}{\delta \beta (1 - \beta^2)^{3/2}} \hat{R} \times \left[ (\hat{R} - \beta) \times \hat{z} \right]
\]
At long distances only the radiation field is significant. It has $\vec{E}$ & $\vec{B}$ perpendicular to $\vec{R} = \text{the retarded position vector}$, and are perpendicular to each other.

**Non-relativistic limit:** $E^{acc} = \frac{q}{R} \hat{R} \times (\hat{R} \times \vec{a}) = \frac{q}{\epsilon^2 R^2} \hat{R} \times (\hat{R} \times \vec{a})$

For linear motion, $\vec{p} = \beta \vec{a} = E^{acc} = \frac{q}{\epsilon^2 R^2} \frac{1}{(1 - \beta^2)^{3/2}} \hat{R} \times (\hat{R} \times \vec{a})$

**Power Radiated**

Power vector: energy flux

$\vec{S} = \frac{c}{4\pi} E \times B = \frac{c}{4\pi} E_0^2 \hat{R}$

$= cu \hat{v}$ with $u = \frac{1}{\sqrt{1 - \beta^2}} (E_0^2 / \beta^2)$.

Consider at $t$ a sphere centered at $x$ of retarded charge position $\vec{x}(t)$. The energy

$\vec{R}$

$= dS \times \vec{v} = R^2 \sin \theta d\theta d\phi$

$\frac{d\rho}{dt} = \frac{\text{energy through area}}{\text{time}} = \frac{c}{4\pi} (R^2 d\theta d\phi)$

$= \frac{c}{4\pi} R^2 \hat{R} \cdot \hat{v} = \frac{c}{4\pi} \left( R \left( E_0^2 \right)^{1/2} - \frac{q}{\epsilon^2 R^3} \frac{1}{(1 - \beta^2)^{3/2}} \hat{R} \times (\hat{R} \times \vec{a}) \right)^2$

Note that as $R \to \infty$ the contribution of $E^{acc}$ vanishes, which we have neglected it here.

This expression gives the energy per unit time in the inertial frame, which is measured by a far-away observer. Sometimes we are interested in a time interval measured at the particle, $d\rho = d\rho/\gamma_0^2$. Now, recall

$\frac{\partial y_1}{\partial x^2} = \frac{(x-x')U}{(x-x')\dot{v}}$ so that $\frac{\partial y_1}{\partial x^2} = \frac{1}{1 - \beta^2}$

and we have then

$\frac{d\rho}{d\theta} = \frac{c \cdot y_1}{4\pi} \left( \frac{1}{(1 - \beta^2)^{3/2}} \right) \hat{R} \times (\hat{R} \times \vec{a})^2$

(The reason for using time as seen by particle is (i) unit time), power radiated between particle at $y_1$ and $y_2$, and (ii) $d\rho$ is then a Lorentz invariant; see below.)
The relativistic expression presents some challenges when integrated:

\[
\frac{d\rho}{d\theta} = \frac{c^3 q^2}{4\pi} \frac{1}{(1-\beta^2)^{3/2}} \left[ \dot{x}^2 (1-\beta^2) + (\vec{\beta} \cdot \vec{a})^2 \right]^{1/2}
\]

\[
= \frac{c^3 q^2}{4\pi} \frac{1}{(1-\beta^2)^{3/2}} \left[ \dot{x}^2 (1-\beta^2) + 2(\vec{\beta} \cdot \vec{a})^2 \dot{x} \cdot (1-\beta^2) \right]
\]

\[
= \frac{c^3 q^2}{4\pi} \frac{1}{(1-\beta^2)^{3/2}} \left[ \dot{x}^2 (1-\beta^2) - (\vec{\beta} \times \vec{a})^2 (1-\beta^2) + 2\dot{x} \cdot \vec{a} \dot{x} \cdot (1-\beta^2) \right]
\]

To write this in terms of angles, we pick a frame (\( \vec{a} \equiv \hat{z} \) in x-y plane)

\[
\begin{align*}
\vec{a} &= \vec{a}(\sin \theta, \cos \theta) \\
\vec{b} &= \vec{b}(0, 0, 1) \\
\vec{r} &= \vec{r}(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)
\end{align*}
\]

\[
\Rightarrow \quad \vec{a} \cdot \vec{r} = \rho \sin \theta \cos \phi \\
\vec{b} \cdot \vec{r} = \rho \sin \theta \sin \phi \quad \text{(but these are independent of } \theta \text{.)}
\]

Now \( \int_0^{2\pi} \) is trivial, with \( \theta = 0 \).

\[
P = \frac{c^3 q^2}{2} \int_0^{2\pi} \frac{1}{(1-\beta^2)^{3/2}} \left[ \dot{x}^2 (1-\beta^2) - (\vec{\beta} \times \vec{a})^2 (1-\beta^2) + 2\dot{x} \cdot \vec{a} \dot{x} \cdot (1-\beta^2) \right]
\]

(We used \( \int_0^{2\pi} \cos \theta \sin \phi = 2\pi \int_0^1 \cos \phi \).

The remaining integral is straightforward but tedious (and we have Mathematica):

\[
P = \frac{c^3 q^2}{2} \left[ \frac{1}{3} \dot{x}^2 (1-\beta^2) \right] \text{ where } s^2 = \sin^2 \theta = 1 - \beta^2
\]

or

\[
P = \frac{c^3 q^2}{3} \left[ \dot{a}^2 - (\vec{\beta} \times \vec{a})^2 \right]
\]

Lionard (1939)

In the non-relativistic limit, \( \beta \ll 1 \), we obtain "Larmor's formula" (NR limit, "Larmor's formula")

\[
P = \frac{c^3 q^2 m}{3}
\]
**Comment:** It is easy to obtain Larmor's formula from the non-relativistic limit of

\[
\frac{d\mathbf{p}}{dt} = \frac{e}{m} \left( \mathbf{E} + \mathbf{v} \times \mathbf{B} \right).
\]

Then \( \mathbf{p'} = \mathbf{p} \) and integrating over \( d\Omega \) (recall \( \mathbf{a} = \frac{e}{m} \mathbf{E} \))

\[
\mathbf{p} = \frac{e}{m} \frac{q_a^2}{c^3}.
\]

Many textbooks use Lienard's formula as follows: argue that \( I \), the frame invariant, then find a Lorentz scalar limit and reduces to Larmor's formula in the NR limit.

The 1st part of the argument is this: with \( d\mathbf{p} = \frac{d\mathbf{p}}{dt} dt = m \frac{d\mathbf{v}}{dt} dt \), this is everywhere as measured by comoving observer.

2nd part: substitute \( \mathbf{a} = \frac{1}{m} \frac{d\mathbf{p}}{dt} \) in Larmor's:

\[
\mathbf{a}^2 = \frac{1}{m^2} \frac{d\mathbf{p}}{dt} \cdot \frac{d\mathbf{p}}{dt} = \frac{1}{m^2} \left[ -c^2 \frac{d\mathbf{p}'}{dt} \cdot \frac{d\mathbf{p}'}{dt} \right]
\]

where the last step is valid in the NR limit (we'll verify below).

Now

\[
-\frac{c^2}{m^2} \frac{dt}{ds} \frac{d\mathbf{p}'}{dt} = \frac{c^2}{m^2} \left( \frac{dt}{ds} \right)^2 \left[ \left( \frac{d(c_\mathbf{p})}{dt} \right)^2 - \left( \frac{d(c_\mathbf{p})}{dt} \right)^2 \right]
\]

\[
= -c^2 \gamma^2 \left[ \left( \mathbf{p}^{\prime 2} - \mathbf{p}^{\prime 2} \right) - \mathbf{p}^{\prime 2} \mathbf{p}^{\prime 2} \right]
\]

\[
= -c^2 \gamma^2 \left[ \mathbf{p}^{\prime 2} \mathbf{p}^{\prime 2} \mathbf{p}^{\prime 2} \right] - 2c^2 \gamma^2 \mathbf{p}^{\prime 2} \mathbf{p}^{\prime 2} - 2c^2 \gamma^2 \mathbf{p}^{\prime 2} \mathbf{p}^{\prime 2}
\]

\[
= c^2 \gamma^2 \left[ \mathbf{p}^{\prime 2} \mathbf{p}^{\prime 2} \mathbf{p}^{\prime 2} \right] - c^2 \gamma^2 \mathbf{p}^{\prime 2} \mathbf{p}^{\prime 2} - 2c^2 \gamma^2 \mathbf{p}^{\prime 2} \mathbf{p}^{\prime 2}
\]

\[
= c^2 \gamma^2 \left[ \mathbf{p}^{\prime 2} - \mathbf{p}^{\prime 2} \mathbf{p}^{\prime 2} \right] = c^2 \gamma^2 \left[ \mathbf{p}^{\prime 2} - \mathbf{p}^{\prime 2} \mathbf{p}^{\prime 2} \right]
\]

\[
\mathbf{p} = \frac{e}{m} \frac{q_a^2}{c^3} \left[ \mathbf{a}^2 - \mathbf{p}^{\prime 2} \mathbf{p}^{\prime 2} \right] \quad \text{as before.}
\]
Angular Distribution

We already have the basic equation for this in

\[ \frac{\partial P}{\partial \Omega} \]

given in terms of \( \theta \times \phi \) angles in \( p \leq \) above.

We look at special cases:

1. Linear motion: \( \vec{p} \parallel \vec{\alpha} \).

\[ \frac{\partial P}{\partial \Omega} \] is independent of \( \phi \) (symmetric under rotations about axis defined by \( \vec{p} \)).

Explicitly

\[
\frac{\partial P}{\partial \Omega} = \frac{c q^2}{4 \pi^2 (1 - \beta^2)^5} \left| \hat{\mathbf{r}} \times (\hat{\mathbf{p}} - \hat{\mathbf{e}}) \times \hat{\mathbf{\alpha}} \right|^2
\]

\[
= \frac{c q^2}{4 \pi^2} \frac{\alpha^2 \sin^2 \theta}{(1 - \beta \cos \theta)^5}
\]

The direction of maximum radiation can be found analytically, but can also be quickly approximated by noting that for \( p \ll 1 \) it is at small \( \theta \) and the denominator is small for

\[ 1 - \beta \cos \theta \approx 1 - \beta (1 + \theta^2) \approx 1 - \beta \Rightarrow \frac{1}{\beta^2} \sin^2 \theta = \text{small}, \quad \text{or} \quad \frac{1}{\beta} = (1 - \beta)(1 + \beta) \approx 2(1 - \beta)
\]

\[ \Rightarrow \theta^2 \ll \frac{1}{\beta^2} \Rightarrow \frac{\partial P}{\partial \Omega} \] is peaked at \( \theta = 0 \)

\[ \frac{\partial P}{\partial \Omega} \approx \frac{c q^2}{4 \pi^2} \frac{\alpha^2 (1 + \theta^2)^5}{(1 + \theta^2)^5} \]

or

\[ \frac{\partial P}{\partial \Omega} \approx \frac{c q^2}{4 \pi^2} \frac{\alpha^2 (1 + \theta^2)}{(1 + \theta^2)^5} \]

In the opposite limit, \( p \ll 1 \)

\[ \frac{\partial P}{\partial \Omega} \approx \frac{c q^2}{4 \pi^2} \sin^3 \theta \] (Larmor), so \( \theta_{\text{max}} = \frac{\pi}{2} \)

Radiation pattern

\[ \theta_{\text{max}} \]

(\( \text{The distance from origin represents} \ \frac{\partial P}{\partial \Omega} \))

Note: Figures of revolution (i.e., the letter i is a forward cone).
Another case of interest is when \( \beta \perp \mathbf{a} \), as in circular motion. Then, from previous lecture

\[
\frac{dP}{d\Omega} = \frac{e^2}{4\pi} \frac{\alpha^4}{(1-\alpha^2)^5} \left[ \alpha^2 \left(1 - (\mathbf{n} \cdot \mathbf{\beta})^2\right) - \alpha (\mathbf{n} \cdot \mathbf{\beta}) (1-\beta) + 2 \alpha^2 \mathbf{\beta} \cdot \mathbf{\hat{a}} (1-\mathbf{n} \cdot \mathbf{\beta}) \right]
\]

\[
= \frac{e^2}{4\pi} \frac{\alpha^4}{(1-\alpha^2)^5} \left[ (1-\alpha^2)^3 - (\mathbf{n} \cdot \mathbf{\beta})^2 (1-\beta) \right]
\]

In the coordinate system we used earlier (\( \mathbf{\beta} = \hat{z}, \mathbf{\alpha} \) in xz plane),

we have now \( \mathbf{\alpha} = \hat{z} \) and \( \mathbf{\hat{r}} \cdot \mathbf{\beta} = \omega \), \( \mathbf{\hat{r}} \cdot \mathbf{\alpha} = \sin \theta \cos \phi \)

(\( \mathbf{\hat{r}} = \mathbf{\hat{\beta}} = \mathbf{\hat{z}} \) and \( \mathbf{\hat{a}} = \mathbf{\hat{r}} \))

\[
\frac{dP}{d\Omega} = \frac{e^2}{4\pi} \frac{\alpha^4}{(1-\alpha^2)^5} \left[ \frac{1}{(1-\alpha^2)^3} - (1-\beta) \frac{\sin \theta \cos \phi}{(1-\alpha^2)^5} \right]
\]

Using \( 1 - \alpha^2 \approx \frac{1}{2} \left( 1 + \alpha^2 \right)^{-1} \) for \( \beta \approx 1 \) we have

\[
\frac{dP}{d\Omega} \approx \frac{2e^2}{n \pi} \frac{\alpha^4}{(1 + \alpha^2)} \left[ 1 - 4 \frac{\beta \alpha^2 \cos \phi}{(1 + \alpha^2)} \right] \frac{d\Omega}{d\Omega}
\]

\[\theta = \frac{\pi}{2} \]

\[\phi = \frac{\pi}{2} \]

\[\beta = 0 \]

\[\theta = \frac{1}{2} \]

\[\phi = 0 \]

\[\beta = \frac{1}{2} \]

\[\theta = 0 \]

\[\phi = \pi \]

The radiation is emitted preferentially in the direction of \( \mathbf{\beta} \), to within a cone of angular size \( \theta = \frac{\pi}{2} \); there is slightly more power radiated off the \( 2 - \beta \) plane (ie \( \theta = \frac{\pi}{2} \)) than on plane (\( \theta = 0 \)), but the difference is order \( 1/\beta \).
At small $\beta$, retaining lowest order in $\beta$:

$$d\Phi = \frac{e^2 q d\xi}{4\pi} \left[ 1 - \sin^2 \theta \cos^2 \beta + \beta \cos \theta \left( 3 - 5 \sin^2 \theta \cos^2 \beta \right) \right]$$

$$= \frac{e^2 q^2 d\xi}{4\pi} \left\{ \begin{array} \{rcl\} & & \cos^2 \theta - \beta \cos \theta (2 - 5 \sin^2 \theta) \\ & & \cos^2 \theta \end{array} \right\}$$

for $\beta = 0$

$\phi = \frac{\pi}{2}$

Drawings:

![Diagram](image)

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How about polarization? Recall we first derived (Lienard-Weichert):

$$E_{rad} = -\frac{q}{R} \frac{1}{(1-\beta)^2} \left[ 2 - \beta^2 \frac{\hat{r} - \hat{p}}{1 - \beta^2} \right]$$

For $u > 1$, radiation is mostly along $\hat{\beta} = \hat{p}$, so focusing on net direction.

and noting $\hat{r} \cdot \hat{p} = 0$ for circular motion, $E_{rad} \approx -\frac{q}{R} \frac{1}{(1-\beta)^2} \hat{r}$.

That is, fully in the plane of the circular motion to good approximation.
Chapter 4: Fields of moving charges

**Heuristic discussion of radiation flow**

*Arbitrary motion with $r >> 1$ and into to spectral decomposition*

**Arbitrary relativistic motion ($r >> 1$):**

Radiation is forward cone $\Omega \sim \frac{1}{r}$

For a fixed observer as the particle transits a small section of the curved path, there is radiation within the cone $\Omega \sim \frac{1}{r}$ at observer.

\[
\rho = \frac{1}{r} \quad \text{so burst of emission over time} \quad \Delta t = \frac{\rho}{\sqrt{v^2 - \rho^2}} = \frac{\rho}{v}
\]

**Front edge of radiation moves a distance**

\[
\Delta l = c\Delta t = \frac{c}{v}
\]

**Back edge of radiation "pulse"** is emitted from particle. Not moved distance $d$.

\[
\text{Width of pulse} \quad \Delta x_c = \Delta l - d = \frac{c}{v} - \frac{c}{v} \frac{1}{1 - \rho} = \frac{c}{v} \frac{\rho^2}{v^2}
\]

"Width" of pulse after $\Delta t$

**Observed intensity**

\[
\frac{\Delta \omega}{\Delta \omega} \approx \frac{1}{c^2 \rho^2}
\]

**Fourier**

\[
\frac{\Delta \omega}{\Delta \omega} = \frac{1}{c^2} - \frac{\rho^3 c}{v}
\]

For circular motion $\frac{\Delta \omega}{\Delta \omega} \approx \Delta \omega \sim \rho^3 \Delta \omega$

*Ref: Jackson 14.4*
The amplification factor $Y^3$ is important.

For example, for $\omega = 10^4 \text{ MHz}$, one can produce $10 \text{ keV}$ X-rays ($\omega \sim 10^{19} \text{s}^{-1}$) with $\gamma \sim (10^9/10)^{1/3} \sim 10^5$. For electrons with $m c^2 = 1 \text{ MeV}$ this requires $E \sim 10 \text{ GeV}$ — see energies of synchrotrons used by X-ray sources!

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Spectral Analysis

Clearly of interest is to have a more quantitative analysis.

We would like to have an expression for

$$\frac{dI}{d\Omega \, d\omega} = \frac{\text{observed intensity of radiation}}{\text{solid angle} \times \text{frequency}}$$

Now $\frac{dI}{d\Omega}$ is just $\frac{dp'}{dt}$ integrated over time. We use $p'$ which refers to per time of lab frame, i.e., observer's time as is appropriate for this question. The expression is in these notes (p.4),

$$\frac{dI}{d\Omega} = \int_0^\infty dt \frac{dp}{dt}$$
Poisson's theorem
\[ f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{f}(k) \quad \Rightarrow \quad \int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} \frac{dk}{2\pi} |\tilde{f}(k)|^2 \]

**Physical proof:**
\[ \int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{dk'}{2\pi} e^{ikx} \tilde{f}(k) e^{-ik'x} \tilde{f}^*(k') \]
\[ = \int \frac{dk}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k) \tilde{f}^*(k') \int_{-\infty}^{\infty} e^{i(k-k')x} dx \]
\[ = 2\pi \delta(k-k') \int \frac{dk}{2\pi} |\tilde{f}(k)|^2 \]

Since we had
\[ \frac{dP'}{ds^2} = \mathbf{\hat{s}} \cdot \mathbf{\hat{P}} = \frac{c}{4\pi} \left| \mathbf{r} \times (\mathbf{\hat{P}} \times \mathbf{\hat{r}}) \right|^2 = \frac{c}{4\pi} \left| \frac{\mathbf{\hat{r}}}{(\mathbf{\hat{r}} \cdot \mathbf{\hat{P}}) \times \mathbf{\hat{r}}} \right|^2 \]

we write this as
\[ \frac{dP'}{ds^2} = |\tilde{\mathbf{P}}(k)|^2 \quad \tilde{\mathbf{P}}(k) = \frac{c}{4\pi} \frac{\mathbf{\hat{r}}}{(\mathbf{\hat{r}} \cdot \mathbf{\hat{P}}) \times \mathbf{\hat{r}}} \]

Then
\[ \frac{d\tilde{\mathbf{P}}}{d\Omega} = \int \frac{d\omega}{2\pi} \frac{\tilde{\mathbf{P}}(\omega)}{2\pi} \] (when \( \tilde{\mathbf{P}}(\omega) \) is the Fourier transform of \( \mathbf{\hat{P}}(k) \))
\[ = \int_{0}^{\infty} d\omega \left[ |\tilde{\mathbf{P}}(\omega)|^2 + |\tilde{\mathbf{P}}(-\omega)|^2 \right] \]

so
\[ \frac{d\tilde{\mathbf{P}}}{d\Omega} \frac{d\omega}{2\pi} \]

is simply
\[ \frac{1}{2\pi} \left[ |\tilde{\mathbf{P}}(\omega)|^2 + |\tilde{\mathbf{P}}(-\omega)|^2 \right] = \frac{1}{2\pi} |\tilde{\mathbf{P}}(\omega)|^2 \]
Useful approximations:

\[ \int_0^\infty dt \ e^{it} \left( \frac{\text{func}}{\epsilon_{\text{radi}} \ \epsilon_{\text{radi}}} \right) e^{-i\omega t} = \int_0^\infty dt' \ (1 - \hat{r} \cdot \hat{p}) e^{i\omega (t' + R \hat{v} / c)} \left( \text{func}(t') \right) \]

Now, consider the vector $\hat{p}(t')$ which is little different from $\hat{p}(t)$.

(i) When the origin of the $a + \hat{r}$ coordinate system is chosen somewhere in the blob (i.e., particle trajectory)

\[ p(t') = \left| \hat{r} - \hat{y}(t') \right| \approx \left| \hat{r} - \hat{\gamma}(t') \right| \]

(ii) If observer is at $a + \hat{r}$

\[ R(t') = \left| \hat{r} - \hat{y}(t') \right| = r - \hat{r} \cdot \hat{y}(t') \]

So (drops primes and $e^{i\omega t} = \text{constant}$) recall trajectory of $a$.

\[ \tilde{\rho}(\omega) = \frac{e^{i\omega r}}{4\pi} \int_0^\infty dt' \ e^{i\omega (t' + \hat{r} \cdot \hat{y}(t'))} \frac{\hat{r} \times [\hat{r} - \hat{\gamma}(t') \times \hat{\omega}]}{(1 - \hat{r} \cdot \hat{p})^2} \]

And

\[ \frac{dI}{d\Omega \ dw} = \frac{e^{i\omega r}}{4\pi^2} \int_0^\infty dt' \ e^{i\omega \cdot \hat{r} \cdot \hat{y}(t')} \left( \frac{\hat{r} \times [\hat{r} - \hat{\gamma}(t') \times \hat{\omega}]}{(1 - \hat{r} \cdot \hat{p})^2} \right)^2 \]
This integral is hard. There is a trick that simplifies significantly. Note that

\[
\frac{\hat{r} \times \left[(\hat{r} \cdot \vec{p}) \vec{x}\right]}{(1 - \hat{r} \cdot \vec{p})^2} = \frac{1}{c} \frac{d}{dt} \left[ \frac{\hat{r} \times \left[(\hat{r} \cdot \vec{p}) \vec{x}\right]}{1 - \hat{r} \cdot \vec{p}} \right]
\]

\[
\hat{p} \text{mak:} \frac{1}{c} \frac{d}{dt} \left[ \frac{\hat{r} \times \left[(\hat{r} \cdot \vec{p}) \vec{x}\right]}{1 - \hat{r} \cdot \vec{p}} \right] = \frac{\hat{r} \cdot \vec{p}}{(1 - \hat{r} \cdot \vec{p})^2} \hat{r} \times \left[(\hat{r} \cdot \vec{p}) \vec{x}\right] + \frac{\hat{r} \times (\hat{r} \times \vec{p})}{1 - \hat{r} \cdot \vec{p}}
\]

\[
= \frac{1}{(1 - \hat{r} \cdot \vec{p})^2} \left[ \hat{r} \times (\hat{r} \cdot \vec{p} - \vec{p}) - \hat{r} \times (\hat{r} \cdot \vec{x}) - \hat{r} \cdot \vec{p} (\hat{r} \times \vec{x} - \vec{x}) \right]
\]

\[
= \frac{1}{(1 - \hat{r} \cdot \vec{p})^2} \left[ (\hat{r} \cdot \vec{p}) \hat{r} - (\hat{r} \cdot \vec{x}) \vec{p} + \hat{r} \times (\hat{r} \times \vec{x}) \right]
\]

\[
= \frac{1}{(1 - \hat{r} \cdot \vec{p})^2} \left[ -\hat{r} \times (\hat{r} \times \vec{x}) + \hat{r} \times (\hat{r} \times \vec{x}) \right]
\]

Now, we willy-nilly integrate by parts

\[
\bar{\vec{P}}(\omega) = \sqrt{\frac{c}{\pi}} \frac{q}{\varepsilon_0} \int_0^\infty \frac{dt}{c} e^{i \omega (t - \hat{r} \cdot \vec{x})} \frac{1}{c} \frac{d}{dt} \left[ \frac{\hat{r} \times \left[(\hat{r} \cdot \vec{p}) \vec{x}\right]}{1 - \hat{r} \cdot \vec{p}} \right]
\]

\[
= \sqrt{\frac{c}{\pi}} \frac{q}{\varepsilon_0} \int_0^\infty \frac{dt}{c} e^{i \omega (t - \hat{r} \cdot \vec{x})} \hat{r} \times \left[(\hat{r} \cdot \vec{p}) \vec{x}\right]
\]

and noting that \( \frac{d}{dt} (t - \hat{r} \cdot \vec{x}) = 1 - \hat{r} \cdot \vec{p} \) we have

\[
\bar{\vec{P}}(\omega) = -i \omega \sqrt{\frac{c}{\pi}} \frac{q}{\varepsilon_0} \int_0^\infty dt e^{i \omega (t - \hat{r} \cdot \vec{x})} \hat{r} \times \left[(\hat{r} \cdot \vec{p}) \vec{x}\right]
\]

This is much simpler. Now

\[
\frac{d}{d\omega} = \frac{2i \omega^2}{4 \pi^2 c} \left| \int_0^\infty dt e^{i \omega (t - \hat{r} \cdot \vec{x})} \hat{r} \times \left[(\hat{r} \cdot \vec{p}) \vec{x}\right] \right|^2
\]
Spectrum of circular motion  
(Synchrotron radiation)

Note: I use θ for angle with x-axis (not \( \phi \)).

\[ \chi(t) = \rho \left( \sin \left( \frac{vt}{p} \right), \cos \left( \frac{vt}{p} \right), 0 \right) \quad \text{(putting } \chi(0) = \rho \left( 0, 1, 0 \right) \text{)} \]
and no argument of \( \exp \) in \( \beta \) i,

\[ \omega \left( t - \frac{\chi(t)}{c^2} \right) = \omega \left( t - \frac{\rho \sin \left( \frac{vt}{p} \right) \cos \theta}{c^2} \right) \]

\[ = \frac{\omega}{c^2} \left( \left( \frac{1}{\beta^2} + \theta^2 \right) t + \frac{c^2}{\beta^2} \right) \]

where we used our previous heuristic discussion. For |\( \beta | > 1 \), radiation is in a cone \( 0 \sim \frac{\pi}{2} \) around \( \beta \), and since the observer is in the xz plane the burst of radiation that reaches him/her is from when \( \beta = \pm \hat{x} \), i.e. from a small time interval around

\[ \chi(t) \approx \rho \left( 1, 0, 0 \right) \Rightarrow \text{expand about } \sin \left( \frac{vt}{p} \right) = 0 \] and of course, expand about \( \theta = 0 \).

The higher order terms are suppressed by either \( \theta^2 \), \( \frac{1}{\beta^2} \) or
by \( \left( \frac{vt}{p} \right)^2 \sim \left( \frac{v \Delta t}{p} \right)^2 \) with \( \Delta t \sim \frac{\lambda}{p} \) from our previous arguments

So \( \left( \frac{vt}{p} \right)^2 \sim \frac{1}{\beta^2} \).

Note that this makes sense if we only integrate over one cycle of the circular motion. If we really integrate over all
times, we get infinitely many equal contributions \( \to \infty \).

The reason is clear: it is periodic motion, with fixed frequency = angular frequency = \( \omega_q = \frac{\nu}{\beta} \).

The proper mathematical analysis is then to do a Fourier series rather than an integral:

\[
\hat{\vec{\rho}}(t) = \sum_{n=-\infty}^{\infty} e^{i\omega_n t} \hat{\vec{\rho}}_n
\]

with

\[
\hat{\vec{\rho}}_n = \frac{1}{2\pi} \int_0^{2\pi} \omega_n dt \ e^{-i\omega_n t} \hat{\vec{\rho}}(t)
\]

Physically we expect \( n \omega_n \approx \tilde{\omega} \), the number of modes.

Just an infinite sum of discrete contributions is huge, the approximation above of just retaining one pulse replaces the large (but exact) number of discrete modes by a continuum:

\[
\sum_{n=1}^{\infty} \to \int
\]

and

\[
\frac{1}{2\pi} \sum_{n=1}^{\infty} |\hat{\vec{\rho}}_n|^2 \to \frac{1}{2\pi} \int d\omega \ |\hat{\vec{\rho}}(\omega)|^2
\]
The approximation to $\vec{E}(t) \propto \hat{r} \times (\hat{r} \times \vec{v})$

is $\perp$ to $\hat{r}$: physically, the EM wave at the observer has $\vec{E}$ (and $\vec{B}$) $\perp$ to line of sight. We can decompose $\vec{E}$ into 2 polarizations in the plane $\perp$ to $\hat{r}$:

$E_{\parallel}$: in the xy plane
$E_{\perp}$: in a direction, $E_{\perp} = \hat{\phi} \times E_{\parallel}$

Since radiation is from $\hat{\beta} = \hat{x}$, ($x\hat{x} + \hat{y}\hat{y} + 0$)

$\Rightarrow \vec{E}_p = (0, 1, 0)$

(And $\vec{p} = p(\cos(\phi), \sin(\phi), 0)$).

Then $E_\perp = \hat{r} \times E_{\parallel} = (0, p, 0) \times E_{\parallel} = (-\sin\phi, 0, \cos\phi)$

Decompose

$\hat{r} \times (\hat{r} \times \vec{p}) = \rho_\parallel E_{\parallel} + \rho_\perp E_{\perp}$

$\rho_\parallel = \vec{E}_{\parallel} \cdot [\hat{r} \times (\hat{r} \times \vec{p})] = (\vec{E}_{\parallel} \times \hat{r}) \cdot (\hat{r} \times \vec{p}) = -\vec{E}_{\parallel} \cdot (\hat{r} \times \vec{p})$

$= - (\vec{E}_{\parallel} \times \hat{r}) \cdot \vec{p} = -\vec{E}_{\parallel} \cdot \vec{p} = -\vec{E}_{\parallel} \cdot (\rho \cos(\phi), \rho \sin(\phi), 0) = -\vec{E}_{\parallel}$

$\rho_\perp = \vec{E}_{\perp} \cdot [\hat{r} \times (\hat{r} \times \vec{p})] = (\vec{E}_{\perp} \times \hat{r}) \cdot (\hat{r} \times \vec{p}) = \vec{E}_{\perp} \cdot (\hat{r} \times \vec{p})$

$= (\vec{E}_{\perp} \times \hat{r}) \cdot \vec{p} = -\vec{E}_{\perp} \cdot \vec{p} = \rho \cos(\phi) \sin(\phi) = \Theta$

$S_0 \frac{dI}{d(2\omega)} = \frac{q^2 \omega^2}{4\pi^2 c^2} \int_{-\infty}^{\infty} dt \ e^{i \omega t} \left[ \frac{1}{4 \rho^2} \frac{\vec{E}_\parallel \cdot \vec{E}_\perp}{\rho} \right] \frac{1}{\left( \frac{c}{\rho} E_{\parallel} + \Theta \vec{E}_{\perp} \right)^2}$
The integral can be expressed in terms of the Airy function

\[ Ai(x) = \frac{1}{2\pi} \int_0^\infty dt \, e^{i(xt + \frac{1}{3} t^3)} = \frac{1}{\pi} \int_0^\infty dt \, \omega_s \left( x t + \frac{1}{3} t^3 \right) \]

and its derivative

\[ Ai'(x) = \frac{2}{2\pi} \int_{-\infty}^\infty dt \, t \, e^{i(xt + \frac{1}{3} t^3)} \]

Our integral is of the form

\[ \int_{-\infty}^{\infty} dt \, e^{i(xt + \frac{1}{3} t^3)} \]

So rescale \( t = \frac{1}{b^{1/3}} \), \( \frac{1}{b^{1/3}} \int_{-\infty}^{\infty} dt \, e^{i \left( \frac{a}{b^{1/3}} t + \frac{1}{3} t^3 \right)} = \frac{2\pi}{b^{1/3}} \frac{a}{b^{1/3}} \]

and \( \int_{-\infty}^{\infty} dt \, t \, e^{i(xt + \frac{1}{3} t^3)} = \frac{2\pi}{b^{1/3}} \frac{a}{b^{1/3}} \)

In our case, \( a = \frac{\omega_s}{2} (1 + \Theta^2) \) and \( b = \frac{\omega_s}{2} c^3 \). Let \( \xi = \frac{\omega_s}{2} (1 + \Theta^2) \)

or \( \Omega = \left( \frac{\omega_s}{2} \right)^{1/2} (1 + \Theta^2) \)

With this,

\[ \frac{\partial^2 I}{\partial \Omega^2} = \frac{\omega_s^2}{4\pi \rho c} \left. \frac{1}{\rho \Omega^2 \left( \frac{\omega_s}{2} c^3 \right)^{1/2}} Ai' \left( \frac{\omega_s}{2} c \right) \xi + \Theta \frac{2\pi}{\left( \frac{\omega_s}{2} c^3 \right)^{1/2}} \xi \right|_{\xi} \]

\[ = \frac{\omega_s^2}{4\pi \rho c} \left( \frac{\omega_s}{2} c \right)^{3/2} \left[ Ai'(\xi)^2 + \Theta^2 \left( \frac{\omega_s}{2} c \right)^{3/2} \left[ Ai'(\xi)^2 \right] \right] \]

Note that the angular frequency of the circular motion is \( \omega_0 = \frac{\omega_s}{\rho} \)

so the result is given in terms of the ratio \( \frac{\omega_s}{c} = \frac{\omega_s}{\omega_0} \).
Now we can compute

\[ A_1(t) = \frac{i}{\pi} \Re \int_0^\infty e^{it \sqrt{x^2 + t^2}} \, dt. \]

Change variable \( u = \frac{1}{2} t^2 \)

\[ (du = t \, dt \Rightarrow dt = \frac{du}{(3u)^{3/2}}) \]

Then

\[ \int_0^\infty e^{iu} \, du = \int_0^\infty e^{iu} \, du = \frac{e^{i\pi/2}}{3^{3/2}} \Gamma(1/3) \]

Then change \( u = 2v \) (formally consider \( \oint dt e^{it} = 0 \)).

\[ \frac{1}{3} \int_0^\infty e^{-v} \frac{i \, dv}{\sqrt{2} \sqrt{3v} \sqrt{2v}} = \frac{i^{1/2}}{3^{3/2}} \int_0^\infty e^{-v} \sqrt{3}^{-1} \, dv = \frac{e^{i\pi/2}}{3^{3/2}} \Gamma(1/3) \]

So, \( A_1(t) = \frac{1}{\pi} \frac{\cos(\pi/2)}{2^{3/2}} \Gamma(1/3) \), or \( v \sim \sin(\pi/2) \Gamma(1/3) \Gamma(1/3) = \frac{\pi}{\sin(\pi/2)} \)

\[ A_1(t) = \frac{1}{3^{3/2} \Gamma(1/3)} \approx 0.655 \]

Similarly, we find \( A_2(t) = -\frac{1}{3^{3/2} \Gamma(1/3)} \approx -0.259 \)

The large-\( x \) behavior can be obtained by stationary phase:

\[ \frac{d}{dt} \left( tx + \frac{1}{2} t^2 \right) = 0 \Rightarrow t^2 = -x ; \quad \frac{d^2}{dt^2} (tx + \frac{1}{2} t^2) = ct \]

So

\[ x \frac{1}{2} t^2 = x \sqrt{x} + \frac{1}{2} (-x)^{3/2} + \frac{1}{2} \sqrt{x} (t - \sqrt{x})^2 + \ldots \]

So we get

\[ \int_0^\infty dt e^{i(\frac{3}{2} x + \sqrt{x} (t - \sqrt{x}))} \approx \int_0^\infty dt e^{i(\frac{3}{2} x + \sqrt{x} (t - \sqrt{x}))} \]
Use \( F_x = i x \) (the other solution blows up (skeerat "ascent").

\[
S_x = e^{-\frac{2}{3} x^{3/2}} \int_0^\infty dt \ e^{-\sqrt{x} (t - i\Delta t)}
\]

\[
= e^{-\frac{2}{3} x^{3/2}} \int_0^\infty dv \ e^{\sqrt{x} v^2} = e^{-\frac{2}{3} x^{3/2}} \frac{2}{\sqrt{x}} \int_0^\infty du \ e^{-u^2}
\]

\[
S_x = \int_0^\infty v^2 \frac{dv}{\sqrt{x}} = 2e^{-\frac{2}{3} x^{3/2}} \int_0^\infty \frac{dJ}{2 \sqrt{x}} e^{J^2} = e^{-\frac{2}{3} x^{3/2}} \frac{\Gamma(1/4)}{\sqrt{x}^{1/4}} = \sqrt{\pi} e^{-\frac{3}{2} x^{3/2}}
\]

And \( A_1(x) \sim \frac{e^{-\frac{2}{3} x^{3/2}}}{2 \sqrt{\pi} x^{1/4}} \) as \( x \to +\infty \).

For \( A_1'(x) \), \( x \mapsto x^{1/4} \).

\[
A_1'(x) \sim -\frac{x^{1/4}}{2 \sqrt{\pi}} e^{\frac{3}{2} x^{3/2}}
\]

For \( x \to -\infty \) and relation to \( K_v \) see Garg.
We can now analyze the behavior of $\frac{d^2 I}{d \omega^2}$.

Recall

$$\frac{d^2 I}{d \omega^2} = -\frac{q}{c} \left( \frac{\omega}{\omega_0} \right)^{7/3} \left[ \frac{A_1'(\omega)}{\omega_0} \right]^2 + \theta^2 \left( \frac{\omega}{\omega_0} \right)^{7/3} \left[ A_1(\omega) \right]^2$$

where $z = \left( \frac{\omega}{\omega_0} \right)^{1/3} \left( 1 + \theta \right)$

For fixed $\theta'$, we have at small $\omega$

$$\frac{d^2 I}{d \omega^2} = \frac{q}{c} \left[ \left( \frac{\omega}{\omega_0} \right)^{7/3} \left[ A_1'(\omega) \right]^2 + \theta^2 \left( \frac{\omega}{\omega_0} \right)^{7/3} \left[ A_1(\omega) \right]^2 \right] = \frac{q}{c} \left[ A_1'(\omega) \right]^2 \left( \frac{\omega}{\omega_0} \right)^{7/3}$$

and for large $\omega$

$$\frac{d^2 I}{d \omega^2} = \frac{q}{c} e^{-\frac{\omega_0^2}{2 \left( \frac{\omega}{\omega_0} \right)^{7/3} \left( 1 + \theta \right)^{1/3}}} \left( \frac{\omega}{\omega_0} \right)^{7/3} \left[ \frac{A_1'(\omega)}{\omega_0} \right]^2$$

$$= \frac{q}{c} e^{\frac{\omega_0^2}{2 \left( \frac{\omega}{\omega_0} \right)^{7/3} \left( 1 + \theta \right)^{1/3}}} \left[ \frac{A_1'(\omega)}{\omega_0} \right]^2$$

The exponential gives a rapid drop $e^{-2 \left( 1 + \theta \right)^{1/3} \frac{\omega}{\omega_0}}$

where the critical frequency $\omega_c = 3 \frac{\omega_0}{c}$ characterizes the frequency beyond which $(\omega > \omega_c)$ radiation is negligible even for $\theta = 0$.

Note also that for small $\omega$, the polarization is largely $\parallel$. Like wise, at $\theta = 0$, only $\parallel$ contributes.
Jackson has a nice plot:

\[ \frac{d^2 T}{d^2 \omega^2} \]

where \( \omega_c = 3 \frac{g^3}{\omega_0} \) is defined by \( z = 1 \) at \( \theta = 0 \), the critical frequency beyond which there is negligible radiation for any angle.

Added: At \( \theta = 0 \) we have

\[ z = \left( \frac{\omega}{\omega_0} \right)^{\frac{4}{3}} = \left( \frac{3g^3}{2g^2 \omega_0} \right)^{\frac{4}{3}} = \frac{3}{2} \omega_c^{\frac{2}{3}} \]

and

\[ \frac{d^2 T}{d^2 \omega^2} \left|_{\theta=0} \right. = \frac{4g^2 \omega_c^2}{3} \left( \frac{3g^3}{2g^2 \omega_0} \right)^{\frac{2}{3}} \left[ A_1 \left( \frac{3g^3}{2g^2 \omega_0} \right) \right]^2 \]

This explains \( \theta = 0 \) behavior of curve above.
Units and Dimensions

We have postponed discussion of units. Most of our studies have been formal. But at some point we would like to plug in numbers, compare with measurements and so on.

Two most common systems:

1. Gaussian: CGS
2. SI - MKS

CGS = cm, gram, second MKS = m, kg, sec denote the units used for mechanical quantities.

Gaussian is more natural:

1. $\vec{E}$ and $\vec{B}$ have same dimensions
2. All units are derived from CGS.

SI is more common (volts, amperes, coulombs)

1. $\vec{E}$ and $\vec{B}$ have different dimensions (and units)
2. Introduces one new basic unit: Ampere for current

Will not discuss how units are defined. See

https://www.nist.gov/si-redefinition/definitions-si-base-units

That $\vec{E}$ and $\vec{B}$ don’t have same dimensions in CGS vs SI implies that formule containing them change from one system to the other.

More detail:
For mechanics, formulae have the same form in MKS, CGS, etc.
\[ F = \frac{dP}{dt}, \quad L = \vec{r} \times \vec{P}, \quad \rho = m \vec{v}, \text{ etc.} \]

For EM, formulae depend on system of units:

**Gaussian**
\[ \bar{F} = \frac{q}{\sqrt{\mu_0}} \bar{A} \]
\[ U = \frac{1}{2} \left( \varepsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) \]
\[ F = q \left( \vec{E} + \vec{v} \times \vec{B} \right) \]

\[ \Rightarrow \text{ Translating between systems requires changes in formulas.} \]

Denote by \( [\text{J}] \) dimension (as usual) with \([\text{J}]=L, \ [\text{M}]=M, \ [\text{T}]=T\)

(with units that are measured in cm-g-s in CGS or m-kg-s in MKS).

We can see what the dimensions are of each quantity in each system. In particular the new (non-mechanical) quantities \( q \) (charge), \( \vec{E} \) \& \( \vec{B} \).

**Gaussian**

From \( F = \frac{q}{\sqrt{\mu_0}} \), \( [q] = \left[ \left( F = m \vec{a} \right) \left[ \text{m} \right] \right]^{1/2} = [M]^{1/2} [L]^{3/2} [T]^{-1} \) is statcoulomb

\[ U = \frac{1}{2} \left( \varepsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) \]
\[ [E] = [\text{V}] = \left[ \left( m \vec{a} \right) \left[ \text{m} \right] \right]^{1/2} = [M]^{1/2} [L]^{1/2} [T]^{-1} \] is statvolt cm\(^{-1}\)

Sanity check: \( F = q \varepsilon_0 \left[ \left( [M][L][T]^{-2} \right)^{3/2} \left[ [M]^{1/2} [L]^{1/2} [T]^{-1} \right] \right] \)

Other quantities trivially follow, e.g. \( \vec{E} = -\nabla \phi \) \( ( \phi = \text{V}) \)

\[ \Rightarrow [\phi] = [E] [L] = [M]^{1/2} [L]^{1/2} [T]^{-1} \left( \left[ \text{m} \text{kg} \text{V} \text{s} \right] \right) = \text{statvolt} \]
$\Sigma$

\[ [I] \text{ is a new basic dimension} \rightarrow \text{ Ampere} \]

\[ I = \frac{dq}{dt} \rightarrow [Q] = [I][T] \rightarrow \text{ Coulomb} \]

\[ F = \frac{1}{\mu_0} \frac{q^2}{|x|^3} \rightarrow [M][L][T]^{-2} = [e_0]^{-1}[L]'[T]'[L]'^{-2} \rightarrow [e_0] = [L]'[T]'[L]^{-2}[M]' \]

\[ u = \frac{1}{\mu_0} (e_0 \vec{E} - \mu_0 \vec{B}) \rightarrow \text{ two relations} \]

\[ [\vec{E}] = ([M][L]'[T]'[L]'^{-2} ([L]'[T]'[L]'^{-3} [M]'^{-1})^{-1/2} = [M][L]'[T]'[L]'^{-2} [D]^{-1} \]

\[ [\vec{B}] = [M][L]'[T]'[L]'^{-2} = [M][L]'[T]'[L]'^{-2} [D]^{-1} \]

\[ F = q \left( \vec{E} + \vec{v} \times \vec{B} \right) \rightarrow \text{ two relations} \]

\[ [F] = [E][V][B] \rightarrow \text{ some as above} \]

From \( U \): \[ [U] = [B]^2 [M][L]'[T]'[L]'^{-2} = [M][L]'[T]'[L]'^{-2} [E] \]

\[ \text{Check:} \ [e_0][e_0] = [L]'^{-2} [T]'^{-1} \text{ consistent with} \ c^2 = \frac{1}{e_0/\mu_0} \]

\[ \text{Exercise: Check that Maxwell's Equations in SI are dimensionally consistent.} \]

\[ \text{Translating between systems} \]

\[ \text{We can go from an expression (say any of Maxwell's)} \]

\[ \text{in one system to another if we develop a dictionary.} \]

\[ \text{For example} \]

\[ F = \frac{q^2}{x^4} \quad (\text{cgs}) \quad F = \frac{1}{\mu_0} \frac{q^2}{x^3} \quad \text{(SI)} \]

\[ \text{Since} \ F \times \vec{x} \text{ are mechanical quantities, they are the same in both systems. We} \]

\[ \text{inert} \quad \frac{q^2}{4\pi \varepsilon_0} = \frac{q^2}{4\pi \varepsilon_0} \quad \text{or} \quad q_c = \frac{q_{54}}{4\pi \varepsilon_0} \]
And from \( u = \frac{1}{\sqrt{c^2 + B_0^2}} = \frac{1}{2} \left( e_0 \frac{E_0}{\mu_0} \right) \)

we get 
\[
\vec{E}_0 = \sqrt{\frac{c^2}{\mu_0}} \vec{E}_{0j} \quad \text{and} \quad \vec{B}_0 = \sqrt{\frac{c^2}{\mu_0}} \vec{B}_{0j}
\]

As above, \( \vec{E}, \vec{B} \) are sufficient to obtain the rest of the dictionary.

The Lorentz force gives nothing new — except if we do not know a priori that \( c^2 = \frac{1}{\epsilon_0 \mu_0} \cdot \vec{F} = q_0 (\vec{E}_0 + \frac{v}{c} \times \vec{B}_0) = q_{5j} (\vec{E}_{5j} + \vec{v} \times \vec{B}_{5j}) \)

\[
\Rightarrow q_0 \vec{E}_0 = q_{5j} \vec{E}_{5j} \quad \text{consistent with the above} \quad q_0 \vec{E}_0 = \left( \frac{q_{5j}}{\sqrt{\epsilon_0 \mu_0}} \right) \left( \sqrt{\frac{c^2}{\epsilon_0 \mu_0}} \vec{E}_{5j} \right)
\]
and \( q_0 \vec{B}_0 = q_{5j} \vec{B}_{5j} \rightarrow \frac{1}{c} \left( \frac{q_{5j}}{\sqrt{\epsilon_0 \mu_0}} \right) \left( \sqrt{\frac{c^2}{\epsilon_0 \mu_0}} \vec{B}_{5j} \right) = q_{5j} \vec{B}_{5j} \rightarrow \frac{1}{c} \frac{q_{5j}}{\sqrt{\epsilon_0 \mu_0}} = 1 \rightarrow c^2 = \frac{1}{\epsilon_0 \mu_0}.

We can correct any formula Gaussian \( \rightarrow \) SI. For example, we derive Larmor’s formula in Gaussian, so
\[
P = \frac{e}{2} \frac{q^2 \alpha^2}{c^2} \quad \text{where mechanical} \quad P = \frac{e}{2} \frac{q^2 \alpha^2}{\sqrt{\epsilon_0 \mu_0}} \cdot \frac{1}{\sqrt{\epsilon_0 \mu_0}} = \alpha \cdot \frac{q^2 \alpha^2}{\epsilon_0 \mu_0}
\]

Or Maxwell’s equations, es
\[
\nabla \times \vec{E}_0 + \frac{1}{c^2} \frac{\partial \vec{B}_0}{\partial t} = 0 \quad \Rightarrow \nabla \times \left( \mu_0 \vec{E}_{0j} \right) + \nabla \left( \frac{c^2}{\mu_0} \frac{\partial \vec{B}_{0j}}{\partial t} \right) = 0
\]
\[
\Rightarrow \nabla \times \vec{E}_{5j} + \frac{1}{c^2} \frac{\partial \vec{B}_{5j}}{\partial t} = 0
\]

Student can derive the rest in SI from Gaussian.

Exercise: Derive the Poynting vector in SI by translating from Gaussian.

Numerics. How do convert quantities from one system of units to the other? And what about \( m_0, e_0 \)? As physicist I like Gaussian, but the “voltmeter does not give statvolts, nor statamperes, etc.”
1. Since the only new dimension is [I] is SI, we should be able to translate any amount of anything between systems once we know how to translate current \( \rightarrow \) or, equivalently, charge.

Of course we also need \( 1 \text{m} = 10^2 \text{cm} \), \( 1 \text{kg} = 10^3 \text{g} \), \( 1 \text{s} = 1 \text{hr} \).

Consider Coulomb's law. In CGS

\[
F = \frac{q_1^2}{4\pi \varepsilon_0 r^2}
\]

means two charges of \( q_{\text{esu}} = 1 \text{ statcoulomb} \) 1 cm apart experience a force of 1 dyne.

ESU is derived very much like dyne or erg.

(esu and statcoulomb are used interchangeably; franklin (fr) is also sometimes used - less common)

The same two charges (tesu) at the same distance (km) experience the same force (f) in other systems. In SI the force is \( 1 \text{dyn} = 10^{-5} \text{N} \), distance 1 cm = 10-2 m and charge is \( 1 \text{ esu} = \times C \) (C = coulomb). So we have

\[
10^{-5} = \frac{1}{4\pi \varepsilon_0 (10^{-2})^2} \quad \text{or} \quad x = \sqrt{\frac{4\pi \varepsilon_0}{10^{-5}}}
\]

With \( \varepsilon_0 = 8.854 \times 10^{-12} \text{ F/m} \)

\[
x = \sqrt{\frac{4\pi \times 8.854 \times 10^{-12}}{10^{-5}}} = 7.336 \times 10^{-10} = \frac{1}{(2.998 \times 10^9)}
\]

Usually written \( 1 \text{ esu} = 10^{-3} \text{ C} \) or \( 1 \text{C} = 3 \times 10^9 \text{ esu} \)

but "3" is 2.998, suspiciously the same digits don't appear in \( c = \text{speed of light} \)
2. $E_0 \times \mu_0 \propto \gamma$

Because $[\gamma]$ is a new dimension in SI, formulae like

$$F \propto \frac{q_1 q_2}{r^2} \quad \text{and} \quad B \propto \frac{1}{r} \quad \text{or} \quad F/\ell \propto \frac{1}{\gamma^*}$$

require introduction of dimensionful constants. For example, we have

$$F = \frac{1}{4\pi \varepsilon_0} \frac{q_1 q_2}{r^2} \quad \text{with} \quad [\varepsilon_0] = [T]^{-2} [L]^{-3} [M]^{-1}$$

And, say, $B = \frac{\mu_0 \mathcal{E}}{2\pi r}$. But also $[\mu_0][\varepsilon_0] = [T] [L]^{-2} [M]^{-1}$, there is no need for two different conversion factors. This is clear just from dimensional analysis, has nothing to do with speed of light. (Had we introduced $E_0 \times \mu_0$ with different coefficients in Coulomb and Ampère's laws, we would still get $\mu_0 E_0 \propto \gamma^2$, but with some proportionality constant.) Since only one is needed, one may define one and measure the other, or define one, measure the other, and infer the other from $\mu_0 E_0 = \gamma^2$.

You probably know $\mu_0 = 4\pi \times 10^{-7}$ H/m by definition. This fixes

$$\varepsilon_0 = \frac{1}{\mu_0 \gamma^2} = 8.854 \times 10^{-12} \ \text{F/m}. \ \text{But there is a better way to write this:}$$

$$4\pi \varepsilon_0 = (\frac{1}{2\mu_0}) \gamma^2 \quad \text{so} \quad 4\pi \varepsilon_0 = 10^9 \gamma^2 \quad \text{(in F/m if c in m/s).}$$

As we saw above, $\varepsilon_{\text{su}} = \gamma \varepsilon_0$ with $\gamma = \sqrt{4\pi \varepsilon_0 \cdot 10^9}$

$$\Rightarrow x = \frac{10^9}{10^6} = \frac{1}{100} = \frac{1}{2.998 \times 10^8} \quad \text{as before.}$$

Exercise: Derive the conversion factors for 5 other quantities in the conversion tables 1.1-1.2 of Garg or Table 4 of Appendices of Jackson.
RADIATION REACTION

A charge is the source of EM field. A field produces a force on a charge. What are the effects of a field produced by a charge on the charge itself?

Start with a simple question: what is the energy in the field of a point charge? (Say the electron, which, as best we know, is a point particle).

If \( q \) is at \( \vec{x}' \), then \( \vec{E}(\vec{x}) = \frac{q}{4\pi \varepsilon_0 |\vec{x} - \vec{x}'|} \)

(For continuous distribution \( \rho \), \( \vec{E}(\vec{x}) = \int d\vec{x}' \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} \))

Then \( u = \frac{1}{8\pi} \varepsilon_0 \vec{E}^2 \) \( (+ \beta^2 \), but \( \beta = 0 \) for stationary charge, assumed here for simplicity). Then

\[ \mathcal{E}(\text{energy}) = \int d\vec{x} \frac{1}{8\pi} \varepsilon_0 \vec{E}^2 = \frac{q^2}{8\pi} \int d\vec{x} \frac{1}{|\vec{x} - \vec{x}'|} \rightarrow \infty \]

The divergence arising from the region \( \vec{x} = \vec{x}' \).

Is this a problem? An additive but constant (infinite) energy does not affect dynamics — which depends on energy differences. BUT an accelerated charge looses (radiates) energy, so it does matter.

If instead we replace point charge by smeared distribution of charge, we get a finite self-energy:

For a continuum \( \mathcal{E} = \frac{1}{8\pi} \int d\vec{x} \int d\vec{x}' d\vec{u} \frac{\rho(\vec{x}) \rho(\vec{x}')}{|\vec{x} - \vec{x}'|} \frac{(\vec{x} - \vec{u}')}{|\vec{x} - \vec{u}'|} \)

\[ = \frac{1}{2} \int d\vec{x} \int d\vec{x}' \frac{\rho(\vec{x}) \rho(\vec{x}')}{|\vec{x} - \vec{x}'|} \text{ is finite (except for } \rho(x) \sim \delta^n(x)) \]
Mathematical digression: (not for class)

We have used our knowledge from electromagnetics that

\[ E = \frac{1}{2} \int d^3x \frac{\rho(x) \rho(y)}{|x-y|^2} \]

which is usually derived from \( E = \frac{9\mu_0}{2} \) for two charges, so

\[ E = \frac{9\mu_0}{2} \sum_{j=1}^{n} \frac{\rho_j}{r_j^3} = \frac{3\mu_0}{2} \sum_{j=1}^{n} \frac{\rho_j}{r_j^3} \quad \text{(introducing infinite self-energy)} \]

Now replacing \( \sum_{j=1}^{n} \frac{\rho_j}{r_j^3} \rightarrow \int d^3x \rho(x) \).

Question: can we obtain this directly from the integral \( \frac{1}{2\pi^2} \int E^2 \)?

Answer: yes! We need to show

\[ I = \int d^3x \frac{(\hat{x} - \hat{r}) \cdot (\hat{x} - \hat{r})}{|x-r|^2} = 4\pi \frac{1}{|x-r|^2} \]

Writing \( \hat{x'} = \hat{r} + \hat{r} \), \( \hat{x} = \hat{r} - \hat{r} \) so \( r = \frac{1}{2} (\hat{x} - \hat{x'}) \), we have (after shifting \( \hat{x} - \hat{x'} \))

\[ I = \int d^3x \frac{\hat{x} \cdot \hat{r}}{(\hat{x} + \hat{r})^2 - 4\hat{r}^2 \cos^2 \theta} \]

Now \( \frac{d}{ds} \frac{1}{\sqrt{a-bs^2}} = \frac{1}{(a-bs^2)^{3/2}} \) so

\[ I = 4\pi \int_0^{\infty} dx \frac{x^3}{(x^2 + r^2)^{5/2}} = \frac{4\pi}{r} \left[ \int_0^{\infty} x^{1/2} \frac{x^3}{(x^2 + r^2)^{5/2}} - \int_0^{r} x^{1/2} \frac{x^3}{(x^2 + r^2)^{5/2}} \right] = \frac{2\pi}{r} \frac{4\pi}{r} \]

End digression.
Generally a difficult integral. But we can gain insight by dimensional analysis. If \( p \) is the total charge (e.g., electron) and \( r \) is for a spherical distribution with radius \( r_e \), then we expect \( E \sim \frac{e^2}{r_e} \) and this should be no larger than the rest energy \( m_e c^2 \). The lower bound on \( r_e \) is the classical electron radius (I use the same symbol).

\[
\frac{e^2}{r_e} = m_e c^2 \implies r_e \approx \frac{e^2}{m_e c^2} \approx 2.8 \times 10^{-13} \text{ cm}
\]

The time it takes light to traverse \( r_e \) is \( \frac{e^2}{r_e c} = \frac{e^2}{m_e c^3} \approx 10^{-23} \text{ s} \).

Therefore, we should not trust classical E&M at length scales shorter than \( r_e \) — or time scales shorter than \( r_e/c \). Note that this is just from the internal consistency of Maxwell’s theory, regardless of quantum limitations: we should not trust it either for distance scales shorter than the Compton wavelength \( \lambda_e = \frac{h}{m_e c} \). Note that \( \lambda_e \frac{r_e}{c} = \frac{h}{m_e c^2} = 137 \), the breakdown of classical E&M occurs well before the self-inconsistency at \( 137 \sim r_e/c \) kicks in!

If no point charges are allowed in classical E&M, what is the electron? We need a mathematical model of a charge distribution that consistently accounts for it. (See digression next page)

Alternatively redefine “bare” mass to also be infinite and precisely cancel the infinity from self-energy. This can be done consistently — the result is independent of the precise manner in which the calculation is regulated (meaning, modified to make intermediate steps finite — discussed later).
Historical digression:

Abraham Loewicz proposed the electron’s structure is purely electromagnetic.

In particular that its mass/energy and momentum are completely due to the EM field. There is a charge density \( \rho(x) \) localized to \( \pm e \), and we know from above this should give an energy \( mc^2 \) for the electron.

To explore the relation between \( E \times B \), consider electron in motion, with vel \( \vec{\beta} \).

In the rest frame fields are \( \vec{E}_0 \) and \( \vec{B}=0 \). Boosting to frame with vel \( -\vec{B} \), we have \( \vec{E} \approx \vec{B} \) related by \( \vec{B}=\vec{\beta} \times \vec{E} \)

Exercise: check this.

\[
\vec{B}=\gamma \vec{\beta} \times \vec{E}_0 \approx \vec{E}_0 - \frac{\gamma \vec{\beta}}{c} \vec{E} \rightarrow \vec{B}=\vec{\beta} \times (\vec{E}_0 - \vec{E})
\]

Now \( E = \frac{1}{\gamma^2} \int dx \left( E^2 + B^2 \right) \approx \frac{1}{\gamma^2} \int dx \left[ E^2 + (\gamma E^2 - (\gamma \vec{E})^2) \right] \)

The momentum is \( \vec{p} = \int dx \vec{p} = \frac{1}{\gamma mc} \int dx \vec{E} \times \vec{B} = \frac{1}{\gamma mc} \int dx \vec{E} \times (\gamma \vec{E}) = \frac{1}{\gamma mc} \int dx \left[ \vec{E} \times (\gamma \vec{E}) \right] \)

In the NR limit \( (\beta<<1) \) \( \vec{E} = \vec{E}_0 + \theta(\beta) \) is spherically symmetric so

\[
\int d^3 x \ E^2 = \frac{1}{2} \delta^{ij} \int d^3 x \ E^2.
\]

Hence \( \vec{p} = \frac{\vec{E}_0}{\gamma mc} \int d^3 x \ E^2 (1 - \frac{1}{2}) = \frac{2}{3} \gamma \vec{E}_0 \cdot (\theta \vec{E}) = \frac{4}{3} \gamma \vec{v} (\%)
\]

Oops! Is the \( 4/3 \) (which should be \( 1 \) ) an algebra mistake?

No! It is an inherent failure of the model.
Poincaré proposed a solution with two important ingredients:

1. There must be additional non-EM forces holding the charge together.
   Then \( T^{\mu\nu} = T_{EM}^{\mu\nu} + T_{X}^{\mu\nu} \) where \( X \) is another mysterious force field.

2. Proper treatment of Lorentz covariance:
   \[ \rho^\mu = \int dx^\mu T^\mu \]

   is a 4-vector \( \frac{\partial}{\partial} \rho^\mu = 0 \)

He showed \( T_{X}^{\mu\nu} \) annihilates \(-\frac{1}{3}\) to \( E_0 \) so that \( \vec{p} = \vec{E_0} \), and gives fully covariant results (this came later: Fermi, Kikel, Rohrlich, Wilson, ...).

None of this addresses successfully other issues with radiation-reaction that we turn to next.
Digression:

If $\sum_{\mu} T^{\mu} = 0$ then $p^\mu = \int d^3x \ T^{\mu}$ is a 4-vector.

Proof: $T^\mu(x)$ is a tensor: $T^{\mu\nu}(x) = \Lambda^\mu_{\nu} \Lambda^\rho_{\sigma} T^{\rho\sigma}(\Lambda^{-1}x)$.

We aim at showing that $p^\mu = \sum_{\nu} p^\nu$ where $p^\mu$ is constructed as above by an observer in frame $K'$, and $p^\nu$ in frame $K$.

$$p^\mu = \int d^3x \ T^{\mu}(x) \quad p^\nu = \int d^3x \ T^{\nu}(x)$$

Now let $A^\mu = \Lambda^\mu_{\nu} x^\nu$ does not imply $d^3x = d^3\bar{x}$.

We do have $d^3x = d^3\bar{x} \delta(x^0) = d^3x \delta(x^0, x^\nu)$.

So $p^\mu = \int d^3x \delta(x^0, x^\nu) \Lambda^\mu_{\nu} \Lambda^\rho_{\sigma} T^{\rho\sigma}(x)$

We have two issues to contend with: (i) the integral is not over $d^3x$ at $x^0 = 0$ (or any constant), and (ii) we have $T^{\rho\sigma}$ rather than $T^{\rho\gamma}$.

Now $\Lambda^\mu_{\nu} x^\nu = \gamma (x^0 - \bar{x}^0)$. So we have

$$p^\mu = \Lambda^\mu_{\nu} \int d^3x \frac{1}{\gamma} \gamma [T^{\mu\nu} - p^\nu T^{\mu\nu}] = \int d^3x \ T^{\mu\nu}(x) = \int d^3x \ T^{\mu\nu}(x^0 - \bar{x}^0, x)$$

We'd like to show that this does not depend on the choice of spacelike hypersurface. Consider

$$0 = \int d^2x \ T^{\mu\nu} = \int d^2x \ T^{\mu\nu}$$

where $n_\nu$ is the normal to the surface $\partial V$ at element $d^2x$. Now, we want $V$ to be

We assume $T^{\mu\nu}$ is real, so the real content of the surface does not contribute.

The other two (spacelike) surfaces have $n_\nu = (0, 0, 0)$ and $n_\nu = (1, -\beta^\nu)$.

$$\int d^2x \frac{1}{\gamma} T^{\mu\nu} = \int d^2x \ T^{\mu\nu} \bigg|_{x^0 = 0}$$

End Digression
Going back to question on trusting classical E.M., we may ask more pointedly: under what conditions can we neglect the reactive effects of radiation?

Qualitatively: simple criterion $E_{\text{radiated}} \ll E$ (typical energy relevant to problem)

By the same token $E_{\text{rad}} \sim E$ should give us an estimate of when radiation reaction cannot be neglected.

Now $P = \frac{2}{3} \frac{q^3}{c^3} a^2 \quad \Rightarrow \quad E_{\text{rad}} \sim \frac{q^2 a^2}{c^3} \Delta t$

where $\Delta t$ is time over which particle is accelerated.

For $E$ we have to look case by case, and be judicious (after all $E$ is defined up to additive constant, e.g., we can swamp all energies in the NR case if we write $E = mc^2 + \text{kinetic}$).

Two typical cases:

(i) Accelerate particle from rest $\rightarrow E = \frac{1}{2} m v_{\text{final}}^2 \sim ma^2 \Delta t$

RR non-negligible: $\frac{q^2 a^2 \Delta t - ma^2 (\Delta t)^2}{c^3} = \Delta t \approx \frac{q^2}{mc^2}$

For $q = e$ and $m = me$ this is $\Delta t \approx T_e$. And RR can be neglected for $\Delta t > T_e$

A bit of a surprise: $T_e \approx c \epsilon_0$ was introduced above as the minimum size of a charge distribution so that mass does not exceed $mc^2$. This was only from static fields. If we get around that somehow (by subtraction?) the scales $T_e, \epsilon_0$ reappear!
(VI) Circular motion:

acceleration: \( \omega^2 p \)
energy: \( \frac{1}{2} m v^2 = \frac{1}{2} m (\omega p)^2 \)

Circular:
\[ \frac{q^2 (\omega p)^2}{e^2} \Delta t \approx m (\omega p)^2 \left( \frac{q^2 \omega^2}{mc^2} \right) \Delta t \approx 1 \]

For electron, \( \Delta t \approx 1 \)

We cannot use such long \( \Delta t \) that \( p \) changes appreciably (then no longer circular and formulae are incorrect). So we need \( \Delta t \) to work for at least one period, or \( \Delta t \approx \frac{1}{\omega} \). This condition is then \( T_e > \omega \approx 1 \) For angular frequencies \( \omega > \frac{1}{T_e} \approx 10^{13} \) Hz

RR cannot be neglected.

Note that this argument applies equally to any (quasi-)periodic motion with angular frequency \( \omega \), (and, irrelevant, amplitude \( p \)).

In both cases we see \( T_e \) sets the relevant scale. And it is so small (\( \frac{1}{T_e} \) so large) that for most (but not all) practical purposes RR can be safely neglected (justifying the success of all you’ve learned in EM that neglects RR).
Quantifying RR: baby version.

If a body loses energy by radiation it must decelerate, but $F=ma$ says there must be a force acting on it. Of course in order to radiate it has to accelerate, so there already was an external force ($\vec{F}_{\text{ext}}$) acting on it:

$$m \ddot{\vec{x}} = \vec{F}_{\text{ext}} + \vec{F}_{\text{RR}}$$

$\text{RR} = \text{"radiative reaction"}$

How to determine $\vec{F}_{\text{RR}}$? We'll see below a derivation, fully covariant, using retarded Green's function. Abraham and Lorentz's method is similar but (i) NR and (ii) Te (small but non-zero) used as cut-off.

For now, cheat a little. Set

$$\left( \text{energy radiated in } \Delta t \right) = \left( \text{work done by } \vec{F}_{\text{RR}} \text{ in } \Delta t \right)$$

$$\int_0^t \frac{1}{2} \rho \dot{\vec{x}}^2 \, dt = \frac{2}{3} \frac{\rho}{C^3} \vec{x}^3 - \int_0^t \vec{F}_{\text{RR}} \cdot \dot{\vec{x}} \, dt$$

Now the LHS has

$$\int_0^t \vec{v} \cdot \dot{\vec{x}} \, dt = \vec{v}_f \cdot \vec{x}_f - \vec{v}_0 \cdot \vec{x}_0$$

(because $\vec{x} = 0$ at $t = 0$ or periodic).

Then we identify

$$\vec{F}_{\text{RR}} = \frac{2}{3} \frac{\rho}{C^3} \vec{x}$$

Since it is $T$-odd ($t \rightarrow -t \Rightarrow \vec{x} \rightarrow -\vec{x}$) we expect dissipation (much like for $\vec{F} = \frac{1}{2} m \vec{a}$, air drag). Of course dissipation is precisely what we expect.
We can now write
\[ m \left( \dot{x} - \frac{\dot{v}}{c^3} \right) = -\frac{1}{c} \frac{\partial F_{\text{ext}}}{\partial x} \]

where \( \tau = \frac{\hbar \omega}{mc^2} \) is just as before, but now including the factor of \( \frac{\tau}{3} \)
so that we do not have to carry it around.

**WEIRD!**

For \( F_{\text{ext}} = 0 \) \Rightarrow \frac{d}{dt} \frac{\dot{x}}{c} = \dot{\dot{x}} = \ddot{x} = \frac{a_x}{c} \; e^{t/c}

Clearly only \( \dot{a_x} = 0 \) is physical. But useful lesson: when we turn on \( F_{\text{ext}} \), there will also be unphysical as well as physical solutions.

In preparation for next consider the case of a general time dependent \( F_{\text{ext}}(t) \). We use the method of Green functions:

\[ \left( \frac{d^2}{dt^2} - \frac{\dot{v}}{c} \frac{d}{dt} \right) G(t) = \delta(t) \Rightarrow X(t) = \int_{-\infty}^{\infty} dt' G(t-t') F_{\text{ext}}(t') \]

To find \( G \) consider its Fourier transform \( \hat{G}(\omega) \):

\[ G(t) = \int_{-\infty}^{\infty} d\omega \; e^{i\omega t} \hat{G}(\omega) \]

\[ = \left( \frac{d^2}{dt^2} - \frac{\dot{v}}{c} \frac{d}{dt} \right) \hat{G}(\omega) = \int_{-\infty}^{\infty} d\omega \; e^{i\omega t} \left( 1 - \omega^2 - i\omega \tau \right) \hat{G}(\omega) \]

and set \( \omega \tau \), equals \( \delta(t) = \int_{-\infty}^{\infty} d\omega \; e^{-i\omega t} \)

\[ \Rightarrow \hat{G}(\omega) = -\frac{1}{\omega^3 (1+i\omega \tau)} \; \text{ and } \; G(t) = \int_{-\infty}^{\infty} d\omega \; \frac{e^{-i\omega t}}{\omega^3 (1+i\omega \tau)} \]
As with other Green functions we encounter poles and we need to choose a contour of integration about them. We should expect our choice will determine behavior appropriate for various boundary conditions, to which we now turn.

![Diagram](https://via.placeholder.com/150)

Poles of $\tilde{G}(\omega)$ in complex $\omega$-plane (and possible contours of integration)

Consider each possibility:

- **Top (green):** Can close contour on upper half and get $G(t)=0$ provided the argument of exp(-i\omega t) has negative real part $<\omega$.

\[ \text{Re}[-i(\omega t+i\omega)^{-1}] = \text{Im}(\omega)t < 0 \] for $t < 0$. This is the analogue of a retarded Green function: it is causal. For $t > 0$ we get contribution from $\omega = i\omega$ and $\omega = 0$.

\[
\int \frac{d\omega}{\omega} e^{-i\omega t} = \frac{1}{2\pi} \int e^{i\omega t} - i e^{-i\omega t} d\omega = G(t) = -te^{\frac{v}{c}t} + (t+c)
\]

This is unphysical: an impulsive force at time $t=0$ sets the charge in accelerated motion for all $t > 0$ times.
\[ F(t) = \begin{cases} 
\frac{g^2}{\lambda^2} & t < 0 \\
 t + 2 & t > 0 
\end{cases} \]

In this case, the charge moves at constant velocity after hit by a hammer. But it starts moving \( T = \frac{1}{3} \frac{g^2}{\lambda^2} \) before contact. This is acausal behavior. This is not a concern because on those time (and distance) scales, quantum mechanical effects take over. As best we know QED (quantum electrodynamics) is fully causal - in the quantum mechanical sense.
Line Brethren and shift of oscillator.
Consider a harmonic oscillator: $F_{\text{ext}} = -ma^2 x$

\[ \ddot{x} + 2\dot{x} + \omega_0^2 x = 0 \]

For simplicity, do 1-dim only.

Look for solutions $A e^{i\omega t}$: $-\dot{x}^2 - \dot{x} \ddot{x} + \omega_0^2 x = 0$

There are 3 solutions. Two of them survive even if we take $\epsilon \to 0$ and correspond to the solutions in the absence of RR. The 3rd solution gives the unphysical exponential growth and we must discard it.

Since $\epsilon$ is small, solve perturbatively (in powers of $\epsilon \omega_0^2$).

$\alpha = \alpha_0^{(0)} + \alpha_0^{(1)} + \alpha_0^{(2)} + \cdots$ where $\alpha_0^{(n)} = (\omega_0)^{2n}$ and $i^{n/2}$ labels the sols.

Clearly $\alpha_0^{(0)} = \omega_0 \Rightarrow \alpha_0^{(1)} = \omega_0$ and $\alpha_0^{(2)} = -\omega_0$.

Then $-(\pm \omega_0 + \alpha_0^{(1)}) - i(\pm \omega_0)^2 + \omega_0^2 = 0 \Rightarrow c_1^{(1)} + i\omega_0^2 = 0 \Rightarrow \alpha_0^{(2)} = -\frac{i}{2} \omega_0^2$

and we can write the solution up to 1st order

\[ x = A_1 e^{i(\omega_0 - \frac{i}{2} \omega_0^2) t} + A_2 e^{-i(\omega_0 - \frac{i}{2} \omega_0^2) t} \]

This is a damped oscillator, with $x \propto e^{-\gamma t}$ and $\gamma = \omega_0^2 \epsilon$

exactly as expected from energetic considerations early on.

This charge is radiating. The field oscillates with the same frequency.

More precisely, the spectral analysis of the radiation is

\[ \frac{d}{d\omega} \propto \text{Fourier transform of } |x(\omega)|^2 \]
Now \[
\int_0^\infty dt \, e^{i\omega t} \left[ e^{-i(\omega - \omega_0) t} \right] = \frac{1}{\omega - \omega_0 + \frac{i\nu}{2}}
\]

where we have assumed the oscillator is excited from \( t = 0 \) onward and neglect the process of turning on. (One can clearly do a better, cleaner job).

Then \[
\frac{dI}{d\omega} \propto \left| \frac{1}{\omega - \omega_0 + \frac{i\nu}{2}} \right|^2
\]

\[
= \frac{1}{(\omega - \omega_0)^2 + (\nu/2)^2}
\]

\[
\int_0^\infty d\omega \frac{1}{(\omega - \omega_0)^2 + (\nu/2)^2} = \frac{\pi \nu}{4} \quad \text{or} \quad \text{done by contour integration}
\]

or \[
\frac{dI}{d\omega} = \frac{1}{\pi} I_0 \left[ \frac{1/2}{(\omega - \omega_0)^2 + (\nu/2)^2} \right]
\]

where \( \int_0^\infty \frac{d\omega}{\omega} = I_0 \)

If the spectrum is given as a function of wavelength, \( \omega \lambda = 2\pi c \)

then a small interval in \( \lambda \) is \( \delta \lambda = \frac{2\pi c}{\omega} \delta \omega \). Since the half-width is small, \( \lambda/\lambda_0 = \omega/\omega_0 \ll 1 \) (by assumption), the half-width in \( \lambda \)-space is \( \Delta \lambda = c \frac{\delta \omega}{\omega_0} = 2\pi c \omega_0 \) for the classical electron radius (up to \( \lambda \approx \hbar/2\pi \)), independent of \( \lambda \).
Final comments:

- Quantum mechanics: for atomic transitions photons are not monochromatic. Same phenomenon. AM line widths depend on strength of transition, and are related to partial lifetimes of levels $T_i^{-1} = \sum_j \Gamma_j$.

- Had we retained $\sigma (\omega \xi)^2$ in our solution for $\chi (\propto \omega \xi)$ we would've obtained

$$\chi = (\omega_0 + \Delta \omega) - \frac{i}{\hbar} \Gamma$$

with $\Gamma$ as before and $\Delta \omega = -\frac{5}{8} \omega_0^3 \xi^2$.

This is the line shift. It moves the center of the curve $\chi(\omega)$.

But very little $\frac{\Delta \omega}{\omega} = O(\omega \xi) \ll 1$. AM effects have $\Delta \omega \sim \Gamma$.

In atoms this is called a Lamb shift (Lamb 1st observed).

$$\frac{\Delta \omega}{\omega_0} = \frac{\omega_0}{c^2} \frac{\hbar}{\xi^2} \text{ vs classical } \frac{\Delta \omega}{\omega_0} \sim (\omega \xi)^2$$
Self-Field of Electron & the Force of Radiation Reaction

I believe this amounts to a modern (and relativistic) version of Abraham-Lorentz.

Rather than inferring the self-force on the electron (due to its radiation field) by matching its rate of kinetic energy loss to the power radiated, which only gives us a time average so the inference of \( \mathbf{F}_{\text{re}} \) ("RR" = radiation reaction) is not completely justified, can we obtain \( \mathbf{F}_{\text{re}} \) directly?

The program should be clear:

1. Compute \( \mathbf{A}_e \rightarrow \mathbf{E}_e \) due to electron.
2. Compute \( \mathbf{F}_{\text{re}} \) due to \( \mathbf{E}_e \) → give motion of electrons.

Of course, there is no ordering here (which is first, the chicken or the egg? Ans: both/neither). These are simultaneous equations. As often done with simultaneous equations, solve for one variable in terms of the other, then plug into second equation.

To keep the aim clear, let's list expectations:

* The static component of self-force should give an infinite self-energy (i.e., mass).
  
  We should regulate this (i.e., cut-off the integral near \( x = \text{electron} \)) by subtracting it using a "bare" mass (i.e., a contribution to the energy which is not of electromagnetic origin). Call this \( M_0 \).

* The radiation field should give rise to a force responsible for energy loss; it should be \( T \)-odd (dissipative! think air drag \( F_\infty \cdot \mathbf{v} \)) and we expect \( \mathbf{F}_{\text{re}} \propto \mathbf{E}_e \).
The two equations are:

- **Field due to point charge (electron):**
  \[
  A_\mu(x) = 4\pi\varepsilon \int d^3y \, \epsilon_{\mu\nu\rho} G_{\nu\rho}(x-y) \quad (\text{Notes from PHYS 203A, p1 of chapter 4})
  \]
  and

- **Equation of motion**
  \[
  \frac{d\rho_{\mu}}{d\lambda} = \frac{q}{c} F_{\mu\nu} U^\nu \quad (\text{PHYS 203A, chap 2, p.6})
  \]

And we take \( \rho^\mu = m_0 U^\mu \) with \( m_0 \) as explained above.

The integral giving \( A_\mu \) will diverge at the origin. To get around this problem we introduce a cut-off. How it's introduced should not matter provided (i) It only affects \( |k-R_0| < r_c \) and (ii) It has a parameter that removes the cutting-off in some limit. For example,

\[
\begin{align*}
\int d^3k \ e^{ikx} & \rightarrow \int d^3k \ e^{ikx} \underbrace{A^\text{regulated}}_{R_0} \underbrace{A^\text{unregulated}}_{\lambda} \\
& = \left( \frac{4\pi}{\lambda^3} \right)^{\frac{3}{2}} \frac{1}{\lambda^2} \int d^3k \ e^{ikx}
\end{align*}
\]

Remove cut-off by \( \lambda \rightarrow 0 \).

Our choice of cut-off is in wave-number space: recall

\[
G(x) = -\int \frac{d^3k}{(2\pi)^3} \frac{e^{-ikx}}{k^2} \quad (\text{PHYS 203A, chap 2, p.13 of revised notes}).
\]

The \( x \rightarrow 0 \) region corresponds to \( k \rightarrow \infty \). So we take

\[
G(x) = -\int \frac{d^3k}{(2\pi)^3} \frac{e^{ikx}}{k^2} \left[ \frac{1}{k^2} - \frac{1}{k^2 - \lambda^2} \right] \quad (\text{cut-off removed by } \lambda \rightarrow \infty).
\]

It is \( k^2 - \lambda^2 \) rather than \( k^2 + \lambda^2 \) so that poles are at \( k = \pm \sqrt{\lambda^2 + \lambda^2} \), real.
We are ready to compute. We need $E_\nu$ so take $A, A_\nu$ above:

$$E_\nu = 2 A_\nu - A_\nu = 4 \pi \varphi \int d\lambda \, U_\nu \partial \nabla^+ G^{\text{ret}}(x-y, \lambda) - c (\rho, \omega, \nu)$$

Here $U_\nu = \frac{d\rho_\nu}{d\lambda}$. The integral runs over $-\infty < \lambda < \lambda_0$, where $\lambda_0$ solves the retarded condition $x-y, \lambda_0 = 0$. We choose as parameter

$$\lambda = \zeta^2 = (x-y)^2$$

This is useful because $G^{\text{ret}}(x)$ is a scalar function with dimensions of $L^{-2}$, i.e., wave-vector, so it depends on $x$ and $\zeta$ only through the combination $\zeta^2 x^2$ which is dimensionless, and to get dimensions right we write

$$G^{\text{ret}}(x-y) = \zeta^2 f(\zeta^2)$$

for some function $f$. This function can be computed by performing the integral explicitly; this is difficult and not illuminating, and not necessary.

Use $\partial^\nu G^{\text{ret}}(x-y) = \partial^\zeta \frac{\partial^\lambda}{\partial \lambda} \partial^\beta \frac{\partial^\kappa}{\partial \kappa} \partial^\mu \frac{\partial^\nu}{\partial \nu}$ and $\int_0^\infty \frac{\partial^\nu}{\partial \nu} = 2(x-y) \int_0^\infty \frac{\partial^\nu}{\partial \nu}$

Recall

$$\frac{\partial^\nu}{\partial \nu} = \frac{(x-y) \partial^\zeta}{(x-y) \partial \zeta}$$

so $\frac{\partial^\nu}{\partial \nu} = (x-y)^2 \left[ \frac{(x-y) \partial^\zeta}{(x-y) \partial \zeta} \right] = (x-y)^2 \partial^\beta \frac{\partial^\gamma}{\partial \gamma}$ so we have

$$\partial^\nu G^{\text{ret}} = \frac{\partial^\zeta}{\partial \zeta} \frac{(x-y)^2}{2} \partial^\beta \frac{\partial^\gamma}{\partial \gamma} \partial^\mu \frac{\partial^\nu}{\partial \nu}$$

and $E_{\mu \nu} = 4 \pi \varphi \zeta^2 \int_0^\infty \frac{\partial^\nu}{\partial \nu} \frac{\partial^\nu}{\partial \nu} \frac{\partial^\nu}{\partial \nu} - (\rho, \omega, \mu, \nu)$
Integrate by parts

\[ F_{\nu}^{\infty}(x) = -4\pi q A^2 \int_0^\infty d\alpha f(\alpha) \frac{d}{d\alpha} \left[ \left( \frac{\alpha^2}{2} \right) \frac{dy}{dx} \right] - f \nu \cdot c \nu \]

\[ = 4\pi q A^2 \int_0^\infty d\alpha f(\alpha) \left[ \frac{(\nu \cdot \alpha)}{\alpha} \frac{dy}{dx} - \frac{(\nu \cdot \alpha)}{\alpha^2} \frac{d^2y}{dx^2} \right] - f \nu \cdot c \nu \]

\[ \text{Now, we are interested in } F_{\nu}^{\infty}(x) \text{ for } x = \text{location of charge } q. \]

So at some time \( x^0 \), we want \( X = Y(\lambda) \) with \( \lambda_x \) determined by \( x^0 = Y^0(\lambda_x) \). In terms of \( \varepsilon \), \( \lambda_x \) is \( \varepsilon = 0 \). So \( x \cdot y = y(0) - y(x) \).

Since the divergences are associated with the field at \( x = x_{\text{electron}} = Y(0) \), we expand the integrand in powers of \( \varepsilon \).

Note that

\[ \int_0^\infty d\varepsilon f(\lambda \varepsilon) \varepsilon^n = \frac{1}{n+1} \int_0^\infty d\varepsilon f(\lambda \varepsilon) \varepsilon^n \geq \text{ca } \frac{1}{n+1} \text{ for some pure number} \]

So only a finitely number of terms need be retained beyond some power the expansion terms vanish as \( \Lambda \to \infty \). This will leave us with some divergent terms (expected, like self-energy), and some \( \Lambda \)-independent terms, the big pay-off of this long computation.

In fact, since there is a \( A^2 \) in front we need include only \( n=1 \) above.
So we have

$$F_{\mu} (x) = 4 \pi g \Lambda^2 \int_0^\infty d\tau \, f(\xi) \left[ \frac{(x^\gamma \gamma^\nu - (x^\gamma) \gamma^\nu)}{2} \frac{d^2 \phi}{d\tau^2} \right] - (\text{secr})$$

Use $y^\mu(x) = y^\mu(0) + \frac{d}{dz} \int_0^z \frac{d^2 \phi}{dz^2}$ and $\frac{d^2 \phi}{dz^2} = \frac{d^2 \phi}{dz^2}$, etc.

and let d\tau denote derivatives at current time, i.e., $\dot{y}^\mu = \frac{d y^\mu}{d\tau}$, etc.

$$F_{\mu} (\gamma^\mu) = -4 \pi g \Lambda^2 \int_0^\infty d\tau \, f(\xi) \left[ (\dot{y}^\nu + \dot{\gamma}^\nu \gamma^\nu) \gamma^\mu - \gamma^\mu \right] + O \left( \frac{1}{\Lambda^2} \right)

= -4 \pi g \left[ c_0 \Lambda (\dot{y}^\nu \dot{\gamma}^\nu - \dot{\gamma}^\nu \gamma^\nu) + c_1 (\dot{y}^\nu \dot{\gamma}^\nu - \dot{\gamma}^\nu \gamma^\nu) \right]

Postpone determination of $c_0$ and $c_1$. Instead, we are ready to compute $F^{\nu}_{\mu}$

$$\frac{d}{dt} m_\nu = \frac{q}{c} F_{\nu} \gamma^\mu \quad \text{or}

m_\nu \dot{y}^\mu = \frac{q}{c} F_{\nu} \gamma^\mu = -\frac{4 \pi g \Lambda}{c} \left[ c_0 \Lambda (\dot{y}^\nu \dot{\gamma}^\nu - \dot{\gamma}^\nu \gamma^\nu) + c_1 (\dot{y}^\nu \dot{\gamma}^\nu - \dot{\gamma}^\nu \gamma^\nu) \right]

Now, as $\tau \to 0$, $\frac{d\tau}{dt} \to 1$, so we can interpret the derivatives as w.r.t. $\tau$

so $\dot{y}^\nu = 1$ and $\dot{\gamma}^\nu \gamma^\nu = 0$ (and $\dot{y}^\nu \gamma^\nu = 0$). So

$$\left( m_\nu - \frac{4 \pi g \Lambda}{c} \right) \dot{y}^\nu = \frac{4 \pi g \Lambda}{c} \left( \dot{y}^\nu \dot{\gamma}^\nu - \dot{\gamma}^\nu \gamma^\nu \right)

The divergent self-energy can be combined with a divergent bare mass $m_0(x)$ to leave a finite mass, the physical electron mass $M_e = m_0 - \frac{4 \pi g \Lambda}{c}$. (so we don't much care what $c_0$ is). So we have

$$m_\nu \dot{y}^\nu = \frac{4 \pi g \Lambda}{c} c_1 (\dot{y}^\nu \dot{\gamma}^\nu - \dot{\gamma}^\nu \gamma^\nu)$$
In the non-relativistic limit, $\dot{\gamma} \ll 1$ and we recognize the FR version of $F_{\text{FR}}$, proportional to $\dot{\gamma}^3$. Comparing with our simplistic energy conservation-on-average argument we can read off the constant $c_i$:

$$F_{\text{FR}} = \frac{2}{3} \frac{q^2}{c^2} \frac{d^2 \gamma}{d t^2} \Rightarrow 4 \pi c_i = \frac{2}{3} \left( c_i = \frac{1}{6} \right).$$

So finally

$$m \ddot{\gamma}^n = \frac{2}{3} \frac{q^2}{c} (\ddot{\gamma} + \dot{\gamma} \ddot{\gamma}^2).$$
Self-Field of Electron & Its Force of Radiation Reaction

I believe this amounts to a modern (and relativistic) version of Abraham-Lorentz.

Rather than inferring the self-force on the electron (due to its radiation field) by matching its rate of kinetic energy loss to the power radiated, which only gives us a time average so the inference of \( F_{ee} \) ("RR" = radiation reaction) is not completely justified, can we obtain \( F_{ee} \) directly?

The program should be clear:

1. Compute \( A_e \rightarrow F_{ee} \) due to electron.
2. Compute \( \frac{F_{ee}}{m} \) due to \( F_{ee} \rightarrow \) give motion of electrons.

Of course, there is no ordering here (which is first, the chicken or the egg? Ans: both/neither). These are simultaneous equations. As often done with simultaneous equations, solve for one variable in terms of the other, then plug into second equation.

To keep the aim clear, let's list expectations:

* The static component of self-force should give an infinite self-energy (i.e., mass).
  We should regulate this (i.e., cut-off the integral near \( x = \) electron), then subtract it using a “bare” mass (i.e., a contribution to the energy which is not of electromagnetic origin). Call this \( m_0 \).

* The radiation field should give rise to a force responsible for energy loss; it should be \( T \)-odd (dissipative!), think air drag \( F \propto \dot{x} \).
  And we expect \( F_{ee} \propto \dot{x} \).
The two equations are

- **Field due to point charge (electron):**
  \[ A_\mu(x) = \frac{1}{4\pi^2} \int d^4y \, G_{\mu
u}(x-y) \Phi(x-y) \]
  Notes from PHYS 203A, p.1 of chapter 4
  "Fields of Moving charges"

and

- **Equation of motion**
  \[ \frac{d\rho^a}{dx^a} = -\frac{1}{c} F_{\mu
u} U^\nu \]
  (PHYS 203A, chap. 2, p.6)

And we take \( \rho^a = m_0 U^a \) with \( m_0 \) as explained above.

The integral giving \( A_\mu \) will diverge at the electron. To get around this problem we introduce a cut-off. How it's introduced should not matter provided (i) it only affects \( 1/R^2 \sim 1/\rho \), and (ii) \( \Lambda \) has a parameter that removes the cutting-off in some limit. For example

\[ \int \rightarrow \int \Lambda^{-1} \int \phi(x) A_\mu(x) \text{ unregulated} \]

Remove cut-off by \( R \rightarrow 0 \).

Our choice of cut-off is in wave-number space: recall

\[ G(x) = -\int_{\text{cut-off}} d^4k \, e^{-ikx} \frac{1}{k^2} \]


The \( x \rightarrow 0 \) region corresponds to \( k \rightarrow 0 \). So we take

\[ G(x) = -\int_{\text{cut-off}} d^4k \, e^{-ikx} \left[ \frac{1}{k^2} - \frac{1}{k^2 - \Lambda^2} \right] \]

(cut-off removed by \( \Lambda \rightarrow \infty \)).

It is \( k^2 - \Lambda^2 \) rather than \( k^2 + \Lambda^2 \) so that poles are at \( k^2 = \pm \sqrt{k^2 + \Lambda^2} \), real.
We are ready to compute. We need $F^\nu$ so take $\partial_\nu A_\nu$ above:

$$F^\nu = \partial^\nu A_\nu - \partial_\nu A^\nu = 4\pi g \int d\lambda \, U^\nu \, \partial^- G_\Lambda^\nu (x-y(\lambda)) - (\partial \rho / \partial \nu)$$

Here $U^\nu = dx^\nu / d\lambda$. The integral runs over $-\infty < \lambda < \lambda_0$, where $\lambda_0$ solves the retarded condition $x-y(\lambda_0) = 0$. We choose as parameter

$$\lambda = z^2 = (x-y)^2$$

This is useful because $G_\Lambda^\nu (x)$ is a scalar function with dimensions of $L^{-2}$, i.e., of wave-vector, so it depends on $x$ and $\Lambda$ only through the combination $\Lambda^2 x^2$ which is dimensionless, and to get dimensions right we write

$$G_\Lambda^\nu (x-y) = \Lambda^2 \, f(\Lambda^2 x^2)$$

for some function $f$. This function can be computed by performing the integral explicitly; this is difficult and not illuminating, and not necessary.

Use $\partial_\nu G_\Lambda^\nu (x-y) = \Lambda^2 \, \partial^2 / \partial z \partial \tau$ and $\partial_\nu \partial^\nu = 2(\partial (x-y))^2 \, \partial (x-y)$. Recall

$$\partial_\nu y_\tau = (x-y) \, U^\nu / (x-y) \, U^\nu$$

(Phys 509A, chm 4, p.2)

So $\partial_\nu \partial^\nu = (x-y)^2 \, \left[ (x-y) \, U^\nu / (x-y) \, U^\nu \right] = (x-y)$ so we have

$$\partial_\nu G_\Lambda^\nu (x-y) = \Lambda^2 - \partial / \partial \nu$$

and $F^\nu = 4\pi g \Lambda^2 \int_0^\infty d\tau \, \partial / \partial \tau (x-y) \, \partial / \partial \tau - \partial (\rho \, \partial / \partial \nu)$
Integrate by parts

$$F_{\nu}^{\alpha}(t) = - \frac{\hbar q}{2} \int_{0}^{\infty} dt' \int_{0}^{\infty} dz' \left[ \frac{\partial}{\partial z'} \frac{\partial}{\partial z} \right] \left[ \frac{\partial}{\partial z} f (z, t) \frac{\partial}{\partial z} \right] - \frac{\hbar q}{2} \frac{\partial}{\partial z} f (z, t)$$

$$= \frac{\hbar q}{2} \int_{0}^{\infty} dz' \int_{0}^{\infty} dz'' \left[ \frac{\partial}{\partial z'} f (z, t) \frac{\partial}{\partial z} \left( z'' - z' \right) - \frac{(z'' - z')^2}{2} \frac{\partial}{\partial z} \right] - \frac{\hbar q}{2} \frac{\partial}{\partial z} f (z, t)$$

Now, we are interested in $F_{\nu}^{\alpha}(t)$ for $x = \text{location of charge } q$.

So at some time $t'$, we want $X = Y(\lambda)$ with $\lambda$ determined by $x^0 = Y(\lambda)$. In terms of $z$, $\lambda$ is $z = 0$. So $x = y = Y(\lambda) - Y(\lambda')$.

Since the divergences are associated with the field at $x = x_{\text{electron}} = Y(\lambda)$, we expand the integrand in powers of $z'$.

Note that

$$\int_{0}^{\infty} dt' f (\lambda t) e^{-\alpha t'} = \frac{1}{\alpha} \int_{0}^{\infty} dt' f (\lambda t) e^{-\alpha t'} = \frac{1}{\alpha} \int_{0}^{\infty} dt' f (\lambda t) e^{-\alpha t'}$$

So only a finite number of terms need be retained: beyond some power the expansion terms vanish as $\lambda \to \infty$. This will leave us with some divergent terms (expected, like self-energy), and some $\lambda$-independent terms, the big pay-off of this long computation.

In fact, since there is a $\lambda^2$ in front we need include only $n=1$ above.
So we have

\[
F_{\mu\nu}(x) = 4\pi g \Lambda^2 \int_0^\infty d\bar{z} f(\bar{z}) \left[ \frac{(y - y') \dddot{y}_\mu - (y - y') \dddot{y}_\nu}{2} \right] - (\text{terms})
\]

Use \( y''(z) = y'(z) + \frac{2}{z} \int_0^z \frac{d^2 y}{dz^2} dz \) and \( \frac{d^2 y}{dz^2} = \frac{d^2 y}{dz^2} \bigg|_0 + \frac{2}{z} \frac{d^2 y}{dz^2} \bigg|_0 \)

and let \( dz \)denote derivatives at current time, i.e., \( \dot{y}^\mu = \frac{d y^\mu}{dz} \), etc.

\[
F_{\mu\nu}(y,z) = -4\pi g \Lambda^2 \int_0^\infty d\bar{z} f'(\bar{z}) \left[ \left( \dot{y}^\mu + \frac{2}{z} \ddot{y}_\mu \right) \left( \dot{y}^\nu + \frac{2}{z} \ddot{y}_\nu \right) - \mu_{\mu\nu} \right] + O\left( \frac{1}{z^3} \right)
\]

Postpone determination of \( \mu_{\mu\nu} \). Instead, we are ready to compute \( F_{\mu\nu}^3 \)

\[
\frac{d}{dz} m_0 \dot{\mu} = \frac{\Lambda}{c^3} F_{\mu\nu} \dot{y}^\nu
\]

\[
m_0 \dddot{y}_\mu = \frac{\Lambda}{c^3} F_{\mu\nu} \ddot{y}^\nu = -\frac{4\pi g \Lambda}{c^3} \left[ c_0 \Lambda \left( \dot{y} \cdot \dot{\dot{y}} - \dot{\dot{y}}^\mu \dot{y}_\mu \right) + c_1 \left( \dddot{y}^\mu \dot{y}_\mu - \dddot{y}^\nu \dot{y}_\nu \right) \right]
\]

Now, as \( z \to 0, \frac{d}{dz} \to 1 \), so we can interpret the derivatives as w.r.t. \( s \) so \( \dot{y}^s = 1 \) and \( \dddot{y} \cdot \dot{y} + \dddot{y} = 0 \) (and \( \dddot{y} = 0 \)). So

\[
\left( m_0 - \frac{4\pi g \Lambda}{c^3} \right) \dddot{y}_\mu = \frac{4\pi g \Lambda}{c^3} c_1 \left( \dddot{y}^\mu + \dddot{y}^\nu \dot{y}_\nu \right)
\]

The divergent self-energy can be combined with a divergent bare mass \( m_0(x) \) to leave a finite mass \( m(x) \), the physical electron mass \( M = m_0 - \frac{4\pi g \Lambda}{c^3} \). (So we don't much care what \( c_0 \) is). So we have

\[
m_0 \dddot{y}^\mu = \frac{4\pi g \Lambda}{c^3} c_1 \left( \dddot{y}^\mu + \dddot{y}^\nu \dot{y}_\nu \right)
\]
In the non-relativistic limit, \( \dot{\chi} \ll 1 \) and we recognize the NR version of \( \mathcal{F}_{RR} \), proportional to \( \dot{\chi} \). Comparing with our simplistic energy conservation-on-average argument we can read off the constant \( c_i \):

\[
\mathcal{F}_{RR} = \frac{\rho}{3} \frac{d^2 \chi}{dt^2} = 4\pi c_i = \frac{2}{3} \left( c_i = \frac{1}{6} \right) . \text{ So finally}
\]

\[
m\ddot{\chi} = \frac{2}{3} \frac{q^2}{\epsilon} \left( \chi'' + \chi' \dot{\chi}' \right)
\]
Electrostatics. Spherical Harmonics. Multiple Expansion

Electrostatics: $\frac{\partial B}{\partial t} = 0$, $\mathbf{B} = 0$, $\mathbf{E} = \mathbf{0}$ in Maxwell's Eqs:

$$\nabla \times \mathbf{E} = \mathbf{0} \quad \nabla \cdot \mathbf{E} = 4\pi \rho$$

$$\mathbf{E} = -\nabla \phi \quad (\phi = \Phi, \mathbf{A} = 0). \quad \Rightarrow \quad \nabla^2 \phi = -4\pi \rho \quad \text{Poisson Equation}$$

We have seen the solution in terms of Green functions:

$$\phi(x) = \phi_{\text{hom}}(x) + \int G(x-x') \rho(x') \, d^3x' \quad \text{where} \quad \nabla^2 G(x) = -4\pi \delta^3(x), \quad \nabla \phi_{\text{hom}} = 0$$

and had determined $G(x)$ by Fourier transform. We can also infer $G$ from our knowledge of Coulomb's law:

$$\phi(x) = \frac{q}{|x|} \quad \text{is for} \quad \rho(x) = q \delta^3(x) \quad \Rightarrow \quad \nabla^2 \phi = -4\pi q \delta^3(x)$$

$$G(x) = \frac{1}{|x|}$$

So

$$\phi(x) = \phi_{\text{hom}}(x) + \int \frac{\rho(x')}{|x-x'|} \, d^3x'$$

Boundary value problems: often concerned with a region of space with boundaries on which we know something about $\mathbf{E}$. Then $\phi_{\text{hom}}$ is chosen to ensure these “boundary conditions” are satisfied.

As in

- Infinite plane
- Infinite cylinder with fixed surface charge
- Conducting boundary: $\phi = \text{constant}$
- Surface charge density: $\Delta \mathbf{E} \cdot \hat{n} = \sigma$ (pill box)

Gauss:

$$\iint \mathbf{E} \cdot d\mathbf{A} = 4\pi \sigma$$

$$\Rightarrow \quad \iint (\mathbf{E} - \mathbf{E}_0) \cdot d\mathbf{A} = 4\pi \sigma$$

$$\Rightarrow \quad \iint (\mathbf{E}_0 - \mathbf{E}_0) \cdot d\mathbf{A} = 4\pi \sigma$$

$$\Rightarrow \quad \iint \mathbf{E}_0 \cdot d\mathbf{A} = 4\pi \sigma$$

$$\Rightarrow \quad \iint \mathbf{E}_0 \cdot d\mathbf{A} = 4\pi \sigma$$
Regardless of the presence of charges, the central problem in electrostatics is then to solve

$$\nabla^2 \phi = 0 \quad \text{(Laplace Equation)}$$

subject to boundary conditions:

1. $\phi$ specified (Dirichlet)
2. $\frac{\partial \phi}{\partial n}$ specified (Neumann)
3. Mixed

Uniqueness of solution of Poisson with boundary: [Garg 16]

If $\phi_1$ and $\phi_2$ are two solutions to $\nabla^2 \phi = 0$ where $\phi_1$ and $\phi_2$ are specified on $\partial V$, then $\psi = \phi_2 - \phi_1$ satisfies $\nabla^2 \psi = 0$ and

$$\psi = 0 \text{ or } \frac{\partial \psi}{\partial n} = 0 \text{ on } \partial V.$$ 

By Gauss's Theorem

$$\int_V \nabla \cdot (\psi \nabla \psi) \, dV = \int_{\partial V} (\psi \frac{\partial \psi}{\partial n}) \, dS = \int_{\partial V} \frac{\partial \psi}{\partial n} \, dS$$

The RHS vanishes by assumption. The LHS

$$\int_V \nabla \cdot (\nabla \psi \nabla \psi + \psi \nabla \psi) \, dV = 0 \text{ by assumption} \Rightarrow \int_V \nabla \psi \, dV = 0 \Rightarrow \nabla \psi = 0$$

$\Rightarrow \psi = \text{Constant} \Rightarrow \phi_2 = \phi_1 + \text{Constant}$

(Note: for 2 functions $\psi_1, \psi_2$

$$\int_V (\psi_1 \nabla^2 \psi_2 + \nabla \psi_1 \cdot \nabla \psi_2) \, dV = \int_{\partial V} \psi_1 \frac{\partial \psi_2}{\partial n} \, dS$$

is "Green's 1st identity"

substituting $\psi_2 = \psi_1$

$$\int_V (\nabla^2 \psi_1 - \nabla \psi_1 \cdot \nabla \psi_1) \, dV = \int_{\partial V} \psi_1 \frac{\partial \psi_1}{\partial n} - \psi_1 \frac{\partial \psi_1}{\partial n} \, dS$$

is "Green's 2nd identity" or "Green's Theorem"
Solving Laplace (PDE) : separation of variables

Cartesian : \( \phi(x,y,z) = X(x) Y(y) Z(z) \)

\[ \frac{1}{\rho} \nabla^2 \phi = \frac{1}{\rho} X''(x) + \frac{1}{\rho} Y''(y) + \frac{1}{\rho} Z''(z) = 0 \]

The three functions of different arguments can add up to zero only if each one is a constant:

\[ \frac{1}{\rho} X''(x) = \alpha^2 \quad \frac{1}{\rho} Y''(y) = \beta^2 \quad \frac{1}{\rho} Z''(z) = \gamma^2 \]

\[ \alpha^2 + \beta^2 + \gamma^2 = 0, \quad X = e^{\pm \alpha x}, \quad Y = e^{\pm \beta y}, \quad Z = e^{\pm \gamma z} \]

The boundary conditions (b.c.'s) limit values of \((\alpha, \beta, \gamma)\). The solution is a linear combination.

Example:

\[ \phi(x,y,z) = \sum_{n,m} \phi_n(x,y,z) \]

\[ \phi_n(x,y,z) = \sin(n \pi x) \sin(m \pi y) \left( a_n \cos(n \pi z) + b_n \sin(n \pi z) \right) \]

b.c. at \( z = 0, \gamma = 0, a_n = \frac{A_n e^{\pm \alpha x} + B_n e^{\pm \beta y}}{a_n} \]

\[ \phi(x,y) = \sum_{n,m} A_{nm} \sin(n \pi x) \sin(m \pi y) \]

\[ A_{nm} = \int_{0}^{L} \int_{0}^{L} dxdy \sin(n \pi x) \sin(m \pi y) \phi(x,y) \]

Finally, as \( \rho \to \infty \) we do not want \( \int_{0}^{\infty} e^{\pm \rho z} \to \infty \) choose \( B_{nm} = -A_{nm} \)

**CLEAR THAT THE SOLUTION IS MOST GENERAL, BUT IMPLEMENTING BOUNDARY CONDITIONS COMPLICATED UNLESS RECTANGULAR SYMMETRY IN PROBLEM -> CONSIDER ALSO CONVEX/CONVEX COORDINATES**
Cylindrical: \( \Phi(r, \phi, z) = R(r) \Theta(\phi) Z(z) \)  
(use \( \phi \) = angle \( \phi \) = potential)

\[
\frac{1}{r} \frac{d}{dr} \left( r \frac{dR}{dr} \right) + \frac{1}{r^2} \Theta'' + \frac{1}{z} Z'' = 0
\]

\( \Rightarrow Z'' = \frac{Z}{z} \),  
\( r \left[ \frac{1}{r} \frac{d}{dr} \left( r \frac{dR}{dr} \right) + \alpha^2 \right] = - \frac{1}{z} \Theta'' = \beta^2 )

\( \Rightarrow \Theta'' - \beta^2 \Theta = 0 \)

\( \Phi \) periodic (except for "conical" configurations)

\( \begin{pmatrix}
\Theta \\
p(\phi, \theta) \\
(\text{not at plane})
\end{pmatrix}
\]

\( \Rightarrow \beta = n, \ m \in Z, \ \Theta = e^{\pm in \phi} \)

\( \phi = e^{ \pm in2} \)

R equation is Bessel's (see 203A, caution on various)

\( \Rightarrow R(r) = J_m(\alpha r) \)  
Won't review here: see last quarter notes.

(Expansion in terms \( J_m(\sin \theta) = 0 \), say \( \int_0^{2\pi} \Phi(\phi = \alpha, \theta, 0) = 0 \)

\( \Phi(r, \phi, z) = \sum \sum c_{nm} J_m(\sin \theta) e^{\pm in \phi} e^{-\frac{\pi z}{a}} \)
Spherical Coordinates $\phi(r, \theta, \phi)$

Appropriate for problems with spherical symmetry: spherical boundaries.

Recall, a scalar has $\phi'(r) = \phi(r)$ with $r' = r$ so $r'^2 = 1$.

With $r = 1 \times e$, an infinitesimal $r'^2 = 1 \implies e' = -e$.

With $\phi'(r) = \phi(r')$, we have $\delta \phi = \phi(r') - \phi(r) = \phi(r') - \phi(r) = \epsilon_{ij} x_j \partial_i \phi$.

The infinitesimal rotation is generated by $-\epsilon_{ij} x_j \partial_i$.

Now an antisymmetric $3 \times 3$ matrix has $\frac{3 \cdot 2}{2} = 3$ independent components, so we can parameterize $\epsilon_{ij}$ as $\epsilon_{ij} = \epsilon^{a} e_{aij} \epsilon^{a}$, $a = 1, 2, 3$ (we parameterize infinitesimally)

(and $e_{aij}$ = completely antisymmetric tensor with $\epsilon_{123} = +1$).

$\delta \phi = \epsilon^{a} e_{aij} x_j \partial_i = \vec{E} \cdot (\vec{x} \times \vec{\phi})$

This should ring a bell! In QM $\hat{\mathbf{L}} = \vec{r} \times \vec{p} = -i\hbar \mathbf{\hat{L}} \times \vec{\phi}$

Setting $\hbar = 1$ (because we are not doing QM) i.e., $\hat{\mathbf{L}} = \vec{r} \times \vec{p}$, we have $\delta \phi = i \vec{E} \cdot \vec{\mathbf{L}} \phi$

We can use our knowledge from QM here:

$[\hat{\mathbf{L}}_i, \hat{\mathbf{L}}_j] = i \epsilon_{ijk} \hat{\mathbf{L}}_k$

(Property: $[\hat{\mathbf{L}}_i, \hat{\mathbf{L}}_j] = (-i)^2 \epsilon_{iim} \epsilon_{jnp} [x_m \partial_n, x_p \partial_i] = (-i)^2 \epsilon_{iim} \epsilon_{jnp} (\delta_{nm} x_p \partial_i - \delta_{np} x_m \partial_i)$

$= (-i)^2 \left( \epsilon_{iim} \delta_{np} x_p \partial_i - \epsilon_{jim} \delta_{np} x_m \partial_i \right) x_p \partial_n$

$= (-i)^2 \epsilon_{ijk} \epsilon_{kmn} x_m \partial_n$

$= -i \epsilon_{ijk} \hat{\mathbf{L}}_k = i \epsilon_{ijk} \hat{\mathbf{L}}_k$)

$\mathbf{L}^2 = \mathbf{L} \cdot \mathbf{L}$, $[\mathbf{L}, \mathbf{L}^2] = 0$ (Property: $\mathbf{L}_j [\mathbf{L}_i, \mathbf{L}_j] + [\mathbf{L}_j, \mathbf{L}_i] \mathbf{L}_j = \epsilon_{ijk} (\mathbf{L}_k \mathbf{L}_x + \mathbf{L}_x \mathbf{L}_k) = 0$).

$\mathbf{L}^2 = \frac{\hbar^2}{j^2} (\mathbf{L}_1^2 + \mathbf{L}_2^2) \Rightarrow [\mathbf{L}^+, \mathbf{L}^-] = \frac{\hbar}{2} (\mathbf{L}_1^+ \mathbf{L}^- + 2 \mathbf{L}_1 \mathbf{L}_2) = \mathbf{L}_3$

$[\mathbf{L}^+, \mathbf{L}_3] = \frac{\hbar}{2} [\mathbf{L}^+ \mathbf{L}_3 + \mathbf{L}_3 \mathbf{L}^+] = \frac{\hbar}{2} (\mathbf{L}_2^+ + \mathbf{L}_1^+) = \mathbf{L}^+$
We can simultaneously diagonalize $L^2$ and one of $L_1, L_2, L_3$, say $L_3$:

The eigenvectors are $Y_{\ell m}$, "spherical harmonics."

Before we find them, let's connect this to $\nabla^2 \phi = 0$.

Note that $L^2 = (\mathbf{r} \cdot \nabla)(\mathbf{r} \cdot \nabla) - \mathbf{r} \cdot \mathbf{r}$ = $- \nabla \cdot (\mathbf{r} \nabla)$ - $\nabla \cdot (\mathbf{r} \nabla)$

$= - \nabla \cdot (\mathbf{r} \nabla) + \mathbf{r} \cdot \mathbf{r} - \mathbf{r} \cdot \mathbf{r}$

$= - \nabla \cdot (\mathbf{r} \nabla) + \mathbf{r} \cdot \mathbf{r}$

$= - \nabla^2 + \mathbf{r} \cdot \mathbf{r}$

Now $\mathbf{r} \cdot \mathbf{r} = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \mathbf{r}^2$

$\Rightarrow \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \mathbf{r}^2$

As we will see, $L^2 \ Y_{\ell m} = \ell (\ell + 1) Y_{\ell m}$, $\ell = 0, 1, 2, ...$

So $\phi = \frac{1}{r} R(r) Y_{\ell m}$ $\Rightarrow \nabla^2 \phi = 0$ $\Rightarrow \frac{1}{r} R'' + \frac{2}{r} R' - \ell (\ell + 1) R = 0$ $\Rightarrow R'' - \ell (\ell + 1) R = 0$

This is homogeneous in $R = r^\ell$ gives $a (\ell - 1) = \ell (\ell + 1)$ $\Rightarrow a = \ell + 1$, $a = -\ell$

By the standard argument, the $Y_{\ell m}$'s form an orthonormal set. They are normalized:

$$\int d\Omega \ Y_{\ell m}^* (\theta, \phi) Y_{\ell m} (\theta, \phi) = \delta_{\ell \ell'} \delta_{m m'}$$

and form a complete basis (in the space of normalizable functions on the unit sphere), so

$$C_{\ell m} r^\ell + d_{\ell m} r^{-\ell-1} = \int d\Omega \ Y_{\ell m}^* (\theta, \phi) \phi (r, \theta, \phi)$$

Either determine $\phi$ at two radii to solve for both $C_{\ell m}$ and $d_{\ell m}$, or, very commonly, use some additional condition, e.g., regularity at origin ($r=0 \Rightarrow d_{\ell m} = 0$) or at $r = \infty$ ($\Rightarrow a = 0$, $b > 0$).
Finding eigenvalues: \(L^2 \psi = \lambda \psi \), \(L_3 \psi = \lambda' \psi\).

To streamline notation, use an notation \(L_\lambda |\lambda\rangle = \lambda |\lambda\rangle\). We really should write \(|\lambda, \lambda'\rangle\) for \(L^2 |\lambda, \lambda\rangle = \lambda |\lambda, \lambda\rangle\) and \(L_3 |\lambda, \lambda\rangle = \lambda' |\lambda, \lambda\rangle\), but for now we concentrate on \(\lambda\) for fixed \(\lambda\), so omit \(\lambda\) in \(|\lambda, \lambda\rangle\).

We also need an inner product \(\langle \psi | \chi \rangle = \int d^2 \mathbf{r} \psi^* \chi\). Note that \(\bar{\mathcal{L}}\) is hermitian w.r.t. this inner product: the hermitian conjugate

\[\langle \chi | \bar{\mathcal{L}}^+ | \psi \rangle = \langle \psi | \mathcal{L} | \chi \rangle^*\]

hence

\[\langle \chi | \bar{\mathcal{L}}^+ | \psi \rangle = \langle \chi | \mathcal{L} | \psi \rangle\]

Note also \(\mathcal{L}_+ = -\mathcal{L}_-\).

We will show that \(|\lambda\rangle\)'s come in discrete set with \(\lambda' = -l, -l+1, \ldots, l-1, l\) for some integer \(l\).

First note: \(L_3 L_\pm |\lambda\rangle = (L_3 L_\pm + L_\pm L_3) |\lambda\rangle = (\lambda' \pm l) L_\pm |\lambda\rangle\)

\(\Rightarrow L_\pm |\lambda\rangle\) is an eigenvector of \(L_3\) with eigenvalue \((\lambda' \pm l)\)

So we have a chain \(\cdots, L^2 |\lambda\rangle, L_+ |\lambda\rangle, L_0 |\lambda\rangle, L_- |\lambda\rangle, L^2 |\lambda\rangle\, \ldots\)

This will terminate if \(L_+ |\lambda\rangle = 0\) for some \(\lambda' = \bar{l}\). Assume this.

Introduce proportionality constants into \(L_+ |\lambda\rangle = a |\lambda\rangle\) (use \(\gamma\) because the prime in \(\lambda\) is tiring). Assume \(\langle \gamma | \gamma \rangle\) is normalized: \(\langle \gamma | \gamma \rangle = 1\) (any \(\gamma\))

\[L_+ |\gamma\rangle = c_\gamma |\gamma + 1\rangle\] and \(L_- |\gamma\rangle = d_\gamma |\gamma - 1\rangle\)

These are not independent: \(\|L_+ |\gamma\rangle\|^2 = c_\gamma^2 \langle \gamma + 1 | \gamma + 1 \rangle = \langle \gamma | \mathcal{L}_{+} | \gamma \rangle = c_\gamma d_{\gamma + 1} \langle \gamma | \gamma \rangle\)

\[\Rightarrow c_\gamma = d_{\gamma + 1}\]
Now we use \( [L_+, L_-] |l\rangle = L_3 |l\rangle = -2l |l\rangle \)

so \( C_{l+1} D_{k} - C_{l} D_{k+1} = 0 \)

\( \Rightarrow C_{l+1} = C_{l} \)

and \( L_+ |l\rangle = 0 \) is \( C_{l} = 0 \). So we have

\[
\begin{align*}
C_{0}^2 &= 0 = l \\
C_{1}^2 - C_{0}^2 &= l - 1 \\
& \vdots \\
C_{k}^2 - C_{k-1}^2 &= l - k \\
C_{k}^2 &= (k!) \frac{(-1)^k}{k!} (2l-k)
\end{align*}
\]

This should not be negative: if we take \( 2l \) = integer, then for \( k = 2l \)

\( C_{2l} = 0 \Rightarrow L_+ |l\rangle = 0 \)

\( \Rightarrow \) The set of functions is \( |l\rangle , |l+1\rangle , \ldots , |l-1\rangle , |l\rangle \) (call them \( |m\rangle \)

for \( l \) = integer or half-integer.

\[
L^2 = L_+ L_- + L_3
\]

Note \( L_+ L_- + L_3 = \frac{1}{2} \left( (L_1 + L_2) (L_1 - L_2) + (L_1 - L_2) (L_1 + L_2) \right) = L_1^2 + L_2^2
\]

so \( L^2 = L_+ L_- + L_3 + L_3 \). \( B_0 \neq 0 \)

\[
L_+ L_- |m\rangle = C_m D_m |m\rangle = c_m |m\rangle \quad L_3 |m\rangle = c_m |m\rangle
\]

\( \Rightarrow L^2 |m\rangle = (c_{m+1} + c_{m-1}) |m\rangle \)

\( B_0 \neq 0 \)

\( c_m = \frac{1}{(m!)} (2m) \) \( k(k+1) \) \( \Rightarrow \) \( c_m = \frac{1}{(m!)} (2m)! \)

\( c_m + c_{m} = \frac{1}{(m!)} (2m)! + \frac{1}{(m+1)!} (m+1)! = \frac{1}{(m+1)!} m! \Rightarrow L^2 |m\rangle = \frac{1}{(m+1)!} m! |m\rangle
\]

So all our functions have the same \( L^2 \) eigenvalue, and are fully (properly) labeled \( |l, m\rangle \), with \( L^2 |l, m\rangle = \ell (\ell + 1) |l, m\rangle \), \( L_3 |l, m\rangle = m |l, m\rangle \)

\( \ell = \frac{1}{2} \mathbb{Z} \), \( m = -\ell , -\ell + 1 , \ldots , \ell - 1 , \ell \)
Find eigenfunctions.

\[ L_+ \psi = \frac{1}{\mu} \psi \left[ \left( x_+ x_+ - 2 \partial_+ \right) + i \left( x_+ \partial_+ - x_+ \partial_+ \right) \right] \]

\[ = \frac{1}{\mu} \left[ (x_+ x_+) \partial_+ + 2 (x_+ \partial_+) \right] \]

\[ \text{Note: } (\partial_+ + i \partial_+) (x_+ y) = 0 \quad \text{and} \quad (\partial_+ - i \partial_+) (x_+ y) = 0 \]

\[ \frac{1}{\mu} (x_+ \partial_+) (x_+ y) = 1 \quad \frac{1}{\mu} (x_+ - i x_+) (x_+ y) = 1 \]

\[ S_0 \mu \left[ (x_+ x_+)^{N_k} \left( x_+ - i y \right)^{N_k} \right] z^{N_k} = \text{replace } z^N \rightarrow \psi \left( \psi \left( x_+ x_+ \right)^{N_k} \right) \]

\[ = \psi \left( \frac{x_+ x_+}{\mu} \right)^{N_k} \left( x_+ - i y \right)^{N_k} z^{N_k} \]

\[ A_{130} \quad L_3 = \tilde{u} (x_+ \partial_+ - y \partial_+) \]

To streamline notation, let \( x_2 = \frac{x_+ x_+}{\mu} \quad \partial_2 = \frac{x_+ \partial_+ - y \partial_+}{\mu} \)

\[ \{ 0, \partial x_2 = i, \partial_2 x_2 = 0 \} \]

Then \( x = \frac{1}{2} (x_+ + x_2) \quad y = \frac{1}{2} (x_+ - x_2) \quad \partial x = \frac{1}{\mu} (x_+ + x_2) \quad \partial y = \frac{1}{\mu} (x_+ - x_2) \)

\[ L_3 = \frac{1}{\mu} \left[ \psi \left( x_+ x_+ \right)^{N_k} \right] \left( x_+ - i y \right)^{N_k} \]

\[ = \psi \left( \frac{x_+ x_+}{\mu} \right)^{N_k} \left( x_+ - i y \right)^{N_k} \]

\[ \text{and in this notation } \quad L_3 = \frac{1}{\mu} \left[ \psi \left( x_+ x_+ \right)^{N_k} \right] \left( x_+ - i y \right)^{N_k} \]

\[ \text{This suggests } \quad |l, k \rangle = N_k X_+^l \quad \text{with } N_k \text{ a normalization constant} \]

\[ \text{Clearly } \quad L_+ |l, k \rangle = 0 \quad L_3 |l, k \rangle = l |l, k \rangle \]

\[ L_2 |l, k \rangle = \left( L_+ + L_+ + L_3 + L_3 \right) N_k X_+^l = N_k \left( L_+ (l + 2 \partial_2 \partial_2 + \partial_2) + \partial_2 \partial_2 \right) \]

\[ = N_k \left[ -l (x_+ x_+) + 0 \right] X_+^l = l \left(l+1 \right) X_+^l \]

\[ \Omega \langle m | \quad \text{We have } \quad |l, l-k \rangle = N_{l-k} X_+^l \]

\[ \text{For example } \quad |l, l-k \rangle = N_{l-k} \left( l - x_+ x_+ \right) \}

\[ \text{and } \quad |l, l-k \rangle = N_{l-k} \left( -l - x_+ x_+ \right) \}

\[ = N_{l-k} \left( l - x_+ x_+ \right) \left( -l - x_+ x_+ \right) \]
In terms of \( \theta, \phi \),
\[
    x_z = \frac{1}{r} \sin \theta (\cos \phi z \sin \theta) = \frac{1}{r} \sin \theta e^{i \phi}
\]
\[
    z = r \cos \theta
\]

Since \( |\ell, m, \phi\rangle \) defined above is proportional to \( r^\ell \) for all functions, we can take it out and replace \( |\ell, m, \phi\rangle \rightarrow \frac{1}{r^\ell} |\ell, m, \phi\rangle \), a function of \( \theta \) and \( \phi \) only.

This gives \( Y_{\ell m} \), up to normalization,
\[
    Y_{\ell m}(\theta, \phi) = \frac{1}{r^\ell} N_{\ell m} L^{\ell m} x_z
\]

Normalization:
\[
    \int d\Omega \ Y_{\ell m}^* Y_{\ell m} = \delta_{\ell 1} \delta_{m 0}
\]

Example: \( \ell=1 \). From the above \( x_z, x_\theta, x_\phi \):
\[
    S_{10} \quad Y_{10} = N_{10} \sin \theta \ e^{i \phi} \quad Y_{10} = N_{10} \cos \theta
\]
\[
    \int d\Omega \ |Y_{10}|^2 = \frac{2}{\pi} N_{10}^2 \int_0^\pi \sin^2 \theta \ d\theta = \frac{3}{2} (N_{10})^2 \quad |N_{10}| = \sqrt{\frac{2}{3\pi}}
\]
\[
    \int d\Omega \ |Y_{10}|^2 = 2\pi |N_{01}|^2 \int_0^\pi \sin \theta \cos \theta \ d\theta = \frac{4\pi}{3} |N_{01}|^2 \quad |N_{01}| = \sqrt{\frac{3}{4\pi}}
\]

The phase is by convention: \( Y_{10} = \frac{2}{3\pi} \cos \theta \) and
\[
    \text{get} \quad Y_{1+1} = \frac{1}{4\pi} L^+ \ Y_{10} \quad Y_{1-1} = \frac{1}{4\pi} L^- \ Y_{10}
\]
\[
    = \quad Y_{10} + \sqrt{\frac{2}{3\pi}} \sin \theta \ e^{i \phi}
\]

* Note that the eigenfunctions with \( \ell = \text{odd integer} \), \( \ell = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \)

are periodic in \( \phi \) with period \( 4\pi \), not \( 2\pi \) : \( e^{i m \phi(q+2\pi)} \rightarrow e^{i m \phi} \) Here they do not play a role in solving Laplace’s equation, but they do play a role in other physics (spinors).

* Note: derivation of eigenvalues only depends on commutation relations and normalizable vectors. Applies equally to \( \Sigma \) matrices. Then \( \Sigma \) are \( 2\pi \) on space of \( (l, m) \) vectors.
This presentation emphasizes the connection to the rotation group and angular momentum. There are many other ways to introduce $Y_{lm}$ and many additional developments. Here we list some facts:

- $Y_{lm}(-\hat{r}) = (-1)^l \ Y_{lm}(\hat{r})$
- $Y_{lm}(\hat{e}) = (-1)^m \ Y_{lm}(\hat{e})$

- Completeness:
  $$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}^*(\theta, \phi) Y_{lm}(\theta', \phi') = \delta(\cos \theta - \cos \theta') \delta(\phi - \phi')$$

- Addition Theorem: Let $\cos \theta_{lc} = \hat{r}_l \cdot \hat{r}_c$ (and recall $Y_{l0}$ is $\phi$ independent).
  $$Y_{l0}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} Y_{lm}(\hat{r}_l \cdot \hat{r}_c)$$

- Relation to Legendre polynomials $Y_{l0}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos \theta)$

  (and, more generally, to associated Legendre functions)

- Generating function for $P_l(x)$:
  $$\frac{1}{\sqrt{1 + \ell^2 - 2\ell x}} = \sum_{l=0}^{\infty} \ell^2 \ P_{l}(x) \quad x \in [-1, 1]$$

- Generating function for $P_{lc}$ + addition theorem:
  $$\frac{1}{|\hat{r}_l - \hat{r}_c|} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \ \frac{\ell^2}{(2\ell+1)} \ Y_{lm}^*(\hat{r}_l) \ Y_{lm}(\hat{r}_c)$$

  where $\ell_1 = \min(|\ell_1|, |\ell_2|)$ and $\ell_2 = \max(|\ell_1|, |\ell_2|)$. I have switched notation from $\mathbf{x}$ to $\hat{r}$ to stay closer to textbook (Garg).
Example:

Conducting Sphere of radius $a$ with upper/lower hemispheres at potential $+V/-V$.

He has axial symmetry, so only $m=0$ contributes to expansion of $\phi$:

$$\phi(r, \theta) = \sum_{l=0}^{\infty} (\cos l \theta + \sin l \theta) Y_{l0}$$

For $\phi$ inside sphere

$$d_{20} = 0 \quad \text{(regularity at origin)}.$$

Inverting

$$c_{20} \hat{r}^2 = \int 4\pi Y_{20}^* \phi(r, \theta) \, dV$$

or, evaluating at $r=a$

$$c_{20} = \frac{1}{a^2} 2\pi \int_0^{2\pi} \sin^2 \theta \, d\theta \int_0^a \rho(r, \theta) \sqrt{r^2 + \rho^2(r, \theta)} \, r \, dr \, d\theta$$

$$= \frac{V}{a^2} \frac{2\pi}{4\pi} \int_0^a \rho \sqrt{r^2 + \rho^2} \, r \, dr \int_0^{2\pi} \left[ \rho^2(x) - \rho^2(-x) \right] \, d\theta$$

This vanishes for $l$ even, and we need to do the integral

$$\int_0^a dx \, \rho(x) \quad \text{for odd } l.$$
From the generating function:
\[
\int_0^\infty \sum t^k p_k(x) \, dt = \sum_{k=0}^\infty t^k \int_0^1 p_k(x) \, dx
\]
which equals
\[
\frac{1}{\sqrt{1+2tx+tx^2}} = \sum_{k=0}^\infty \frac{(-1)^k (2k)!}{k!} \frac{t^k}{(1+t)^{k+1}}
\]
Use the Taylor expansion
\[
(1+x)^S = \sum_{k=0}^\infty \frac{S(S-1)\cdots(S-k+1)x^k}{k!} \quad \text{with} \quad S = \frac{1}{2}, \quad x = t^2
\]
So
\[
\frac{1}{\sqrt{1+2tx+tx^2}} = \sum_{k=1}^\infty \frac{(-1)^k (2k-1)(2k-2)\cdots(2k-2k+1)}{k!} t^{k-1}
\]
and
\[
\rho_{2l-1}(x) = \int_0^1 p_{2l-1}(x) \, dx = \frac{(-1)^{k+1}}{2^kk!} \left( \frac{\pi}{2} \right)^{l-\frac{3}{2}} \Gamma(2l-3) \Gamma(l)
\]
So we have
\[
\phi(r, \theta) = \sum_{k=1}^\infty \sqrt{\frac{4\pi}{2k+1}} \frac{(-1)^{k+1}}{2^kk!} \left( \frac{\pi}{2} \right)^{l-\frac{3}{2}} \Gamma(2l-3) \Gamma(l) Y_{l-1,0}(\theta, \phi)
\]
\[
= \sum_{k=1}^\infty (2k-1) \frac{(-1)^{k+1}}{2^kk!} \left( \frac{\pi}{2} \right)^{l-\frac{3}{2}} \Gamma(2l-3) \Gamma(l) Y_{2k-1}(\cos \theta)
\]
\[
= \sqrt{\frac{2}{\pi}} \frac{\Gamma}{a} \rho_1(\cos \theta) - \frac{3}{8} \frac{\Gamma^3}{a^3} \rho_3(\cos \theta) + \frac{11}{16} \frac{\Gamma^5}{a^5} \rho_5(\cos \theta) - \cdots
\]
Multipole expansion

Localized charge distribution $\rho$:

$$\phi (\vec{r}) = \sum_{\alpha} \frac{q_\alpha}{|\vec{r} - \vec{r}_\alpha|} = \int d\vec{r'} \frac{\rho(\vec{r'})}{|\vec{r} - \vec{r'}|}$$

Physical idea of multipole expansion: at $r \gg R$ we should have $\phi \approx \frac{\rho}{r}$, where $\rho = \oint d\vec{l'} \phi(\vec{l'})$. Coefficient are by an expansion in

$$D(R) = \rho \approx \rho \left(1 + \frac{\rho R^2}{r} + \frac{\rho R^4}{r^2} + \cdots \right)$$

Expand in powers of $|\vec{r} - \vec{r'}| < 1$

We can do this in one swoop by using $Y_{\ell m}$ expansion of $1/|\vec{r} - \vec{r'}|$. But let's get some intuition of what this is by expanding by hand first. We'll use a Taylor series for $x$ about 0:

$$f(x) = f(0) + x \frac{df}{dx}(0) + \frac{x^2}{2} \frac{d^2f}{dx^2}(0) + \cdots$$

$$= \left. \frac{1}{|\vec{r} - \vec{r'}|} \right|_{\vec{r'} = 0} = \left. \frac{1}{|\vec{r'} - \vec{r}|} \right|_{\vec{r'} = 0} + \frac{1}{2!} \left. \frac{d^2}{d\vec{r'}^2} \right|_{\vec{r'} = 0} = \frac{1}{|\vec{r} - \vec{r'}|} + \cdots$$

Stick this into $\int \frac{\rho(\vec{r'})}{|\vec{r} - \vec{r'}|} \, d\vec{r'}$.

Charge

Moment

"monopole"

"dipole"

"quadrupole"
More precisely:

Define \( q = \int \rho(r') d^3r' \) 

\( \mathbf{J} = \int \rho(r') \mathbf{r}' d^3r' \)  

“dipole moment”

\( D_{ij} = \int \rho(r') \left( 3x_i x_j - r^2 \delta_{ij} \right) \)  

“quadrupole moment”

The extra term \( \delta_{ij} r^2 \) in the definition of \( D_{ij} \) is included so that \( \delta_{ij} D_{ij} = 0 \) (i.e., it is traceless). This can be done freely because the coefficient \( f_{2ij}(r) \) is traceless as well, as we will see. The “3” is just an arbitrary normalization in the definition of \( D_{ij} \).

Note that under rotations:

\( q \rightarrow \text{scalar \ (l=0)} \)  

\( \mathbf{J} \rightarrow \text{vector \ (l=1)} \)  

\( D_{ij} \rightarrow l=2 \)

I.e., 2-index symmetric tensors transform in 1-to-1 correspondence with \( l, m \geq 0, l \geq 2 \). Note both have 5 (independent) components.

This is why we subtract the trace in the definition of \( D_{ij} \): a \( 3 \times 3 \) matrix \( M_{ij} \) in general has 9 components:

- Trace: \( M_{ii} \rightarrow l=0 \) 

- Anti-symmetric: \( M_{ij} - M_{ji} \rightarrow l=1 \) 

- Symmetric-tensors: \( M_{ij} + M_{ji} - \frac{2}{3} \delta_{ij} M_{kk} \rightarrow l=2 \) 

\[ \sum_{l=0}^{\infty} \sum_{m=-l}^{l} c_l^m \psi_l^m(\mathbf{r}) = - \frac{q}{4\pi \varepsilon_0} \delta^3(\mathbf{r}) - \frac{\mathbf{J}}{4\pi \varepsilon_0} \cdot \mathbf{r} \]

\[ f_{2ij} \]

and \( \partial_i \mathbf{J} / r \mathbf{J} = \partial_i \mathbf{J} / r^2 \mathbf{J} = - \frac{2}{r^3} \frac{x_i x_j}{x^2} \) and \( \partial_i \mathbf{J} / r \mathbf{J} = - \frac{\delta_{ij} \mathbf{J} / r^2}{r^2} + \frac{3x_i x_j}{r^5} \)

\[ \int_{\text{S}} = \frac{x_i}{r^2} \quad f_{2ij} = \frac{3x_i x_j - \delta_{ij} r^2}{r^5} \] 

Note \( \delta^{3} f_{2ij} = 0 \) as advertised
So we have

\[ \Phi (\vec{r}) = \frac{q}{r} + \frac{\hat{r} \cdot \vec{E}}{r^2} + \frac{1}{r^3} \sum_{j} x_j y_j - \frac{1}{2} \delta_{j,1} \frac{1}{r^5} D_{ij} + \ldots \]

or

\[ \Phi (\vec{r}) = \frac{q}{r} + \frac{\hat{r} \cdot \vec{E}}{r^2} + \frac{n \cdot \vec{r}}{r^3} D_{ij} + \ldots \]

where \( \vec{n} = \frac{\vec{r}}{r} \) has \( n^2 = 1 \).

Clearly \( |\vec{d}| \leq q \hat{r} \), \( |D_{ij}| \leq q \hat{r}^3 \) as expected.

Systematize:

\[ \Phi (\vec{r}) = \int d^3 p(\vec{r}') \frac{1}{|\vec{r} - \vec{r}'|} = \int d^3 p(\vec{r}') \sum_{l,m=0}^{\infty} \frac{2l+1}{2l+1} \frac{r^l}{l+1} Y^*_{lm}(\vec{r}') Y_{lm}(\vec{r}) \]

\[ = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{2l+1}{2l+1} \frac{q_{lm}}{r^{l+1}} Y^*_{lm}(\vec{r}) \]

where \( q_{lm} = \sqrt{\frac{2l+1}{2l+1}} \int d^3 p(\vec{r}') r^l Y^*_{lm}(\vec{r}) Y_{lm}(\vec{r}) \) is the \( l \)-pole moment.

Note that \( \Phi (\vec{r})^* = \Phi (\vec{r}) \Rightarrow \sum_{l,m=0}^{\infty} q_{lm} Y_{lm}(\vec{r}) = \sum_{m=-l}^{l} q^*_{lm} Y^*_{lm}(\vec{r}) = \sum_{m=-l}^{l} (-)^m q^*_{lm} Y^*_{lm}(\vec{r}) = \sum_{m=-l}^{l} q_{lm} Y_{lm}(\vec{r}) \)

\( q_{lm} = (-)^m q^*_{lm} \) which is verified by its definition in terms of the integral over the real charge density \( p(\vec{r}') \).

Relating to \( \hat{d} \):

\[ Y^*_{l1}(\vec{r}) = \frac{3}{8\pi} \left(x y - iy x \right) \Rightarrow \tilde{q}_{l1} = \frac{3}{8\pi} \int d^3 r p(\vec{r})(x y - iy x) = \frac{3}{8\pi} (x^2 + y^2) \]

Similarly \( q_{10} = d_a \)
Exercise: show 19.25 in Garg

\[ q_{20} = D_{x^2} \quad q_{21} = \pm \frac{1}{\sqrt{2}} \left( D_{x^2} \mp i D_{y^2} \right) \]

\[ q_{22} = \frac{1}{\sqrt{6}} \left( D_{x^2} - D_{y^2} \mp 2i D_{xy} \right) \]

Assignment: Read about Earnshaw's theorem in Garg.

In a charge-free region \( \phi \) over a sphere equals \( \phi_{\text{center}} \).
Charge distributions in external fields

A charge distribution characterized by \( q, \vec{d}, \vec{N}, \ldots \) that give its field away, in an external \( \vec{E} \) field experiences forces and torques. What are they?

Take now \( \phi(\vec{r}) = \) potential due to external sources

\[ \vec{E} = -\nabla \phi \]

The energy in this configuration is (with \( q_a = d^3 r(\vec{r}^a) \))

\[ U = \sum_a q_a \phi(\vec{r}_a) = \int d^3 r \rho(\vec{r}) \phi(\vec{r}) \]

Choose coordinate system with origin within \( \rho \) (hect, choose the same one as was used to define multipoles):

Expanding \( \phi(\vec{r}) \)

\[ \phi(\vec{r}) = \phi(0) + x_i \partial_i \phi |_0 + \frac{1}{2} x_i x_j \partial_i \partial_j \phi |_0 + \cdots \]

So

\[ U = \int d^3 r \rho(\vec{r}) [\phi(0) + \cdots] = q_0 \phi(0) + d_1 \partial_i \phi |_0 + \frac{1}{2} D_{ij} \partial_i \partial_j \phi |_0 + \cdots \]

(using \( \nabla^2 \phi |_0 = 0 \) since the external field is due to remote charges).
Examine result: contributions to $U$:

(i) Lowest: $q \phi(0)$

If we were to move the charge to a new location, $\vec{r}$, we would have instead $q \phi(\vec{r})$. The force on this is

$$\vec{F} = -\nabla U |_{\vec{r}} = q (-\nabla \phi) = q \vec{E}(\vec{r})$$

No surprise!

(ii) 1st correction: $d_j \phi, \vec{E}_j = -\vec{d} \cdot \vec{E}$

As above, a rigid translation $\Rightarrow -\vec{d} \cdot \vec{E}(\vec{r})$

Force $\vec{F} = -\nabla \times (d \vec{E}) = d_j \nabla E_j$

or $F_j = d_j \Theta_j E_j = d_j (\partial_j E_i - \partial_i E_j) + (\vec{d} \cdot \vec{E}) E_j$

But in static situation $\nabla \times \vec{E} = 0 \Rightarrow \vec{F} = (\vec{d} \cdot \vec{E}) \vec{E}$

Even if $\partial_i E_j = 0$ ($\vec{E} =$ uniform) there is a torque

$$\vec{N} = \int \vec{r} \times (d \vec{r} \rho(r) \vec{E}(\vec{r})) = \int d^3 r \rho(r) \vec{r} \times [\vec{E}(\vec{r}) + \vec{E}(0) + \ldots]$$

lowest term

$$\vec{N} = -\vec{d} \times \vec{E}(0)$$

(iii) 2nd correct: $\begin{pmatrix} \frac{1}{\epsilon} D \end{pmatrix}_{ij} \phi_j \phi_i |_{\vec{r}} = -\frac{1}{\epsilon} D_{ij} \phi_j \phi_i (0)$

$$\vec{F}_k = \frac{1}{\epsilon} \sum_{ij} \partial_{ij} \vec{E}$$

and

$$N_i = -\epsilon \delta_{ijk} \left( \int d^3 r \rho(r) x_j' x_m' \right) \partial_k E_k = \frac{1}{3} \epsilon \delta_{ijk} D_{jm} \partial_m E_k$$

where we used $\nabla \times \vec{E} = 0$ to include the $\delta_{ijm}$ term at no price.
Charge on charge:

\[ f_1 \rightarrow R \rightarrow f_2 \]

Energy of configuration: use \( f_1 \) as source of external field and \( f_2 \) in presence of this

\[
\phi = \frac{q_1}{r} + \frac{d_1 \hat{r}}{r^2} + \frac{1}{2} D_{ij}^0 \nabla_i \phi \nabla_j \phi + \ldots \tag{1}
\]

and

\[
U = \frac{q_2 \phi(0)}{r} - d_2 \cdot \vec{E}(0) - \frac{1}{6} D_{ij}^0 \nabla_i \phi \nabla_j \phi + \ldots \tag{2}
\]

Here "0" is in the \( f_2 \) distribution, so take that as \( R \) from "center" of \( f_1 \); set \( \mathbf{r} = \mathbf{R} \) in (1). Then stick (1) into (2). Expanding in \( 1/R \), we have

\[
U = U^{(0)} + U^{(1)} + U^{(2)} + \ldots \quad \text{with} \quad U^{(0)} = \frac{q_2 \phi(0)}{R} = \frac{q_1 q_2}{R}
\]

the potential energy of two point charges.
Next

$$U^{(0)} = q_2 \left[ \frac{\vec{d}_2 \cdot \vec{R}}{R^3} \right] + \left[ -\vec{d}_2 \cdot \vec{E} \right]$$

where

$$\vec{E} = -\nabla \left( \frac{q_1}{r} \right) \bigg|_{r=R}$$

$$= \frac{q_1 \vec{R}}{R^3}$$

$$\Rightarrow U^{(1)} = q_2 \frac{\vec{d}_2 \cdot \vec{R}}{R^3} - \frac{q_1 \vec{d}_2 \cdot \vec{R}}{R^5}$$

(Looks asymmetric, but isn’t: exchange $q_1 \leftrightarrow q_2$
also exchanges $\vec{R} \leftrightarrow -\vec{R}$).

At order $1/R^3$ there are $4 D_{ij}$ terms and $d_1 d_2$ terms.

The latter are

$$U^{(2)} = -\vec{d}_2 \cdot \vec{E}$$

with

$$\frac{\vec{E}}{r^3} = \frac{-3 \frac{\vec{d}_1 \cdot \vec{R}}{R^3}}{r^3} = \frac{-3 R_i R_j - \delta_{ij} R^2}{R^5} \vec{d}_{ij}$$

$$\Rightarrow U^{(2)} = \frac{\vec{d}_1 \cdot \vec{d}_2 R^2 - 3 (\vec{d}_1 \cdot \vec{d}_2) (\vec{d}_1 \cdot \vec{d}_2)}{R^5} \vec{d}_{ij}$$

Min for $\vec{d}_1 \parallel \vec{d}_2 R^2$, max for $\vec{d}_1 \perp \vec{R}$, $\vec{d}_2 \perp (\vec{R})$ or vice versa.
Garg Ch. 10: Radiation From Localized Sources.

Private Notes

\[ \theta \cdot A = 0 \rightarrow \Delta A_{\omega} = \frac{\omega^2}{c^2} A - A = \frac{1}{c} \int \frac{d\mathbf{r}'}{|\mathbf{r}' - \mathbf{r}|} \frac{\Delta A_{\omega}(\mathbf{r}', \mathbf{r})}{|\mathbf{r}' - \mathbf{r}|} \] \[ k = \frac{\omega}{c} \]

\[ A_0 = \phi = \int \frac{d\mathbf{r}}{4\pi} \frac{\rho_i(\mathbf{r}, \omega)}{|\mathbf{r}|} \]

\[ E(r, \omega) = \int \frac{d\mathbf{r}}{4\pi} e^{-i\omega t - i\mathbf{k} \cdot \mathbf{r}} \tilde{E}(\mathbf{r}, \omega) \quad \text{ where } (\mathbf{r}, \omega) \in \text{Garg}. \]

\[ E = -\frac{\partial \Phi}{\partial \mathbf{r}} \quad \tilde{E} = -\frac{\partial \Phi}{\partial \mathbf{r}} + i\mathbf{k} \tilde{A} = -\frac{\partial \Phi}{\partial \mathbf{r}} + i\mathbf{k} \tilde{A} \quad k = \frac{\omega}{c} \]

For large \( \lambda \ll R \), \( \lambda \ll r \)

\[ \lambda = \frac{\omega}{c} \quad \text{and consequently } \lambda \ll r \quad \text{so NR approx is } \frac{\omega}{c} \ll \omega < \omega, c. \]

\[ \tilde{A}(\mathbf{r}, \omega) = \int dt e^{i\omega t} A_0(\mathbf{r}, t) = \frac{1}{c} \int \frac{d\mathbf{r}'}{4\pi} \frac{e^{-i\omega t - i\mathbf{k} \cdot \mathbf{r}'}}{1 - \frac{c^2}{1 - \frac{\mathbf{k}^2}{c^2}} |\mathbf{r}' - \mathbf{r}|} \tilde{E}(\mathbf{r}', \omega) \]

\[ = \frac{1}{c} \int d\mathbf{r}' \frac{e^{ik|\mathbf{r} - \mathbf{r}'|}}{R} \tilde{E}(\mathbf{r}', \omega) \]

(Note: LHS is \( A_0(\mathbf{r}, \omega) \) \( \mathbf{r} \neq \mathbf{r}' \) in this case)

\[ \text{while RHS is } \tilde{A}(\mathbf{r}, \omega) \text{ \( \mathbf{r} \neq \mathbf{r}' \text{ in this case} \) \]
\[ \mathcal{F}(\mathbf{k}, \omega) : \mathbf{E}(\mathbf{r}, \omega) = -\frac{\omega}{c} \mathbf{A}(\omega) + i k \mathbf{A}(\omega) \]

\[ = -\frac{1}{c} \nabla \left( \frac{e^{i k \mathbf{r} \cdot \mathbf{r}}}{\mathbf{r}} \right) + \frac{e^{i k \mathbf{r} \cdot \mathbf{r}}}{\mathbf{r}} \frac{1}{c} \mathbf{E}(\mathbf{k}, \omega) \]

\[ \mathbf{S}(\mathbf{k}) \cdot \frac{e^{i k \mathbf{r} \cdot \mathbf{r}}}{\mathbf{r}} \frac{1}{c} \overrightarrow{\mathbf{p}}(\mathbf{k}, \omega) = \frac{e^{i k \mathbf{r} \cdot \mathbf{r}}}{\mathbf{r}} \frac{1}{c} \mathbf{E}(\mathbf{k}, \omega) \]

\[ = -i k \frac{e^{i k \mathbf{r} \cdot \mathbf{r}}}{c} \left[ -\mathbf{p}(\mathbf{k}, \omega) + \frac{\omega}{c} \mathbf{E}(\mathbf{k}, \omega) \right] \]

\[ \mathbf{A}(\mathbf{r}, \omega) = -\frac{1}{c} \mathbf{E}(\mathbf{k}, \omega) - \frac{e^{i k \mathbf{r} \cdot \mathbf{r}}}{c} \mathbf{E}(\mathbf{k}, \omega) \]

\[ \frac{\partial \mathbf{A}}{\partial t} + \nabla \mathbf{A} = 0 \Rightarrow -i \omega \mathbf{A}(\mathbf{k}, \omega) + i \frac{\omega}{c} \mathbf{E}(\mathbf{k}, \omega) = 0 \]

\[ \frac{\partial \mathbf{E}}{\partial t} + \nabla \mathbf{E} = \omega \frac{\mathbf{A}(\mathbf{k}, \omega)}{c} = -i k \frac{e^{i k \mathbf{r} \cdot \mathbf{r}}}{c} \frac{1}{c} \mathbf{E}(\mathbf{k}, \omega) \]

\[ \mathbf{E}(\mathbf{r}, \omega) = -i k \frac{e^{i k \mathbf{r} \cdot \mathbf{r}}}{c} \frac{1}{c} \mathbf{E}(\mathbf{k}, \omega) \]

\[ \mathbf{A}(\mathbf{r}, \omega) = \frac{e^{i k \mathbf{r} \cdot \mathbf{r}}}{c} \mathbf{E}(\mathbf{k}, \omega) \]

\[ \Rightarrow \quad \mathbf{E}(\mathbf{r}, \omega) = \frac{e^{i k \mathbf{r} \cdot \mathbf{r}}}{c} \frac{1}{c} \mathbf{E}(\mathbf{k}, \omega) \]

\[ \mathbf{A}(\mathbf{r}, \omega) = \frac{e^{i k \mathbf{r} \cdot \mathbf{r}}}{c} \mathbf{E}(\mathbf{k}, \omega) \]

\[ \text{(only used for } \mathbf{r} \approx \mathbf{k} \mathrm{so} \frac{1}{c} \text{ no problem on } \omega \text{ or } \mathbf{k} \text{ but for } \mathbf{r} \text{ far from } \mathbf{k} \text{ apply to point particle recover old math).} \]

The domain: \[ \mathbf{E}(\mathbf{r}, t) = \int \frac{d^3 \mathbf{k}}{2\pi^3} e^{i \mathbf{k} \cdot \mathbf{r}} \mathbf{E}(\mathbf{k}, \omega) \]

\[ = \int \frac{d^3 \mathbf{k}}{2\pi^3} e^{i \mathbf{k} \cdot \mathbf{r}} \frac{e^{i k \mathbf{r} \cdot \mathbf{r}}}{c} \frac{1}{c} \mathbf{E}(\mathbf{k}, \omega) \]

\[ = \int \frac{d^3 \mathbf{r}}{2\pi^3} \int \frac{d^3 \mathbf{r}}{2\pi^3} e^{i \mathbf{k} \cdot \mathbf{r}} e^{i (t - \mathbf{r} + \mathbf{k} \cdot \mathbf{r})} \frac{1}{c} \mathbf{E}(\mathbf{k}, \omega) \]

\[ \text{The factor of } i k = \frac{e^{i k \mathbf{r} \cdot \mathbf{r}}}{c} \Rightarrow -\frac{1}{c^2} \mathbf{E}(\mathbf{k}, \omega) \frac{d^2}{dt^2} \left( \mathbf{E}(\mathbf{k}, \omega) \right) \]

\[ = \mathbf{E}(\mathbf{r}, t) \text{ along } \mathbf{E}, \text{ so } \mathbf{E} \text{ if } \mathbf{p} \text{ points along line of sight } \frac{\mathbf{p}}{c} \mathbf{E} \]

\[ \text{steady current do not radiate } (\mathbf{J} = \mathbf{q} \times \mathbf{E}) \Rightarrow \mathbf{j} = \mathbf{q} \times \mathbf{E} \text{ at observation) } \]

2020-06-04 11:09:33
$$\text{Power} \quad S = \frac{E}{\nu} \quad \vec{E} \cdot \vec{B} = \frac{1}{\nu} \hat{n} \cdot \vec{E}^2$$

$$\frac{dP}{dn} = R^2 \hat{n} \cdot \vec{S} = \frac{1}{2} \gamma^2 \left( \frac{\partial}{\partial E} \right) \left( \frac{\partial}{\partial \nu} \right) \left[ \frac{E}{\nu} \right]$$

\text{Spectrum: We already have} \quad \vec{E}(k, \omega) = \frac{2 \pi \nu}{c k} \hat{n} \cdot \vec{E}(k, \omega)$$

$$\text{Now:} \quad \frac{dI}{dn} = \int_0^1 \frac{dE}{E} \left[ \frac{E}{\nu} \right] = \frac{2 \pi \nu}{c k} \left( \frac{E}{\nu} \right) \int_0^1 \frac{d\omega}{\nu} \left( \frac{E}{\nu} \right)^2 \frac{d\omega}{\nu}$$

$$= \frac{2 \pi \nu}{c k} \left( \frac{E}{\nu} \right) \int_0^1 \frac{d\omega}{\nu} \left( \frac{E}{\nu} \right)^2$$

\text{(This is the case "amplitude" in Garg, but must be "localized in time".)}

\text{Periodic: Fourier} \quad \omega \rightarrow \frac{dP}{dn} = \sum_n c_n e^{i\omega t} \quad \text{(Note: different treatment than in Garg)}$$

$$\frac{1}{T} \int_0^T e^{i\omega t} \frac{dP}{dn} = c_0 \left( \frac{T}{2\pi} \right) \frac{dE}{\nu}$$

\text{Now: instead of} \quad \psi(t) = \int \frac{d\omega}{2\pi} e^{i\omega t} \psi(\omega) \Rightarrow e^{i\omega t} \rightarrow e^{i\omega t} \psi(\omega)$$

$$\text{So:} \quad \langle \vec{E}^2 \rangle = \frac{1}{T} \int_0^T \vec{E} \cdot \vec{E} e^{i\omega t} \mathrm{d}t = \sum_n \frac{E_n}{\nu} E_n = \sum_n \frac{E_n^2}{\nu}$$

$$\Rightarrow \langle \frac{dP}{dn} \rangle = \frac{2 \pi \nu}{c k} \left( \frac{E}{\nu} \right)^2 \sum_n \frac{E_n^2}{\nu} = \frac{2 \pi \nu}{c k} \left( \frac{E}{\nu} \right)^2 \sum_n \frac{E_n^2}{\nu} \delta(\omega - \nu)$$

\text{Connect with Garg:}$$
\text{A Fourier exp can be written as a F-T:}$$
\text{i.e.,} \quad \delta(\omega - \nu) \Rightarrow \frac{1}{2\pi} \int_0^1 \frac{d\omega}{\nu} \left( \frac{E}{\nu} \right)^2 \delta(\omega - \nu)$$

$$= \frac{dP}{dn} \rightarrow \int \frac{d\omega}{\nu} \left( \frac{E}{\nu} \right)^2 \delta(\omega - \nu)$$

\text{Garg has a constant} \quad \frac{1}{2} e^{i\omega t} + \text{c.c.} \quad \Rightarrow \text{an extremum} \frac{\partial}{\partial \nu} = \frac{dP}{dn}$$
Stochastic

Underlying process (unspecifed) gives rise to random movement of charges in the confined region. \( \xi \) becomes a random variable.

If \( f(t) \) is random (assume \( \langle f(t) \rangle = 0 \)) we can take one instance of the function \( f(t) \) and look at widely separated intervals \((t_k, t_{k+1})\), \( (t_{k+1}, t_{k+2}) \), ... where \( t_k < t_{k+1} \).

With \( T \) large, but \( T \ll t_{mn} - t_n \), each segment will be a sample of \( f(t) \) over an interval \((0, T)\) taken from a random distribution.

(This plot is some \( f \neq 0 \), but if \( T \) gets larger than eventually \( \langle f \rangle = 0 \).

Then we can use a single function to compute correlations:

\[
\langle f(t) f(t + \tau) \rangle = \frac{1}{T} \int_{0}^{T} dt \langle f(t) f(t + \tau) \rangle
\]

Note: I am not sure how to prove this. I'd like

\[
\langle f(t) f(t + \tau) \rangle = m(t) \langle f(t) f(t + \tau) \rangle \text{ for some measure } m(t), \text{ say, } m(t) = e^{-\int \tau \omega(t) dt}. \text{ End note.}
\]

We assume \( \xi^t \) is stochastic. To obtain the spectrum, start from the above formula

\[
\frac{dI}{d\omega} = \int_{-\infty}^{\infty} \left| \int_{0}^{T} \langle \xi^t \xi^{t+\tau} \rangle dt \right|^2 d\omega
\]

where we have taken the expectation value of the stochastic variable.
Now we undo the time shift:

\[ \langle \hat{\mathcal{E}}^2(t,\omega) \rangle = \int_0^\infty dt_1 e^{i\omega t_1} \hat{\mathcal{E}}(t_1,\omega) \left( \int_0^\infty dt_2 e^{i\omega t_2} \hat{\mathcal{E}}^*(t_2,\omega) \right)^* \]

Change variables \( t_1 = \frac{t + \xi}{2} \quad t_2 = \frac{t - \xi}{2} \)

\[ dt_1 dt_2 = \left( \frac{\partial (t,\xi)}{\partial (t,\xi)} \right) dt_1 dt_2 = \left| \frac{1}{2} \right| dt_1 dt_2 = dt_1 dt_2 \]

\[ \int_0^\infty dt_1 e^{i\omega t_1} \left( \int_0^\infty dt_2 e^{i\omega t_2} \hat{\mathcal{E}}^*(t_2,\omega) \right) \]

So,

\[ \int_0^\infty dt \frac{d\mathcal{P}}{d\omega} = \int_0^\infty dt \left[ \frac{\hat{\mathcal{E}}^2}{\hbar^2 c^2} \int_{-\infty}^\infty d\xi e^{i\omega \xi} G_j^L(t) \right] \]

and we take a leap of faith regarding the integrands and interpret this as LHS

as an instantaneous \( \frac{d\mathcal{P}}{d\omega}(t) \):

\[ \frac{d\mathcal{P}}{d\omega}(t) = \frac{\hbar^2 c^2}{\omega^2} \int_{-\infty}^\infty dt e^{i\omega t} G_j^L(t) \]
The long wavelength, non-relativistic, electric dipole approximation

\[ \text{if } \beta \ll 1 \text{, then the frequency } \omega = \gamma \omega_0 \text{ and we will neglect } \frac{\omega}{c} \text{ terms.} \]

\[ \lambda = \frac{2\pi e}{\omega} \approx \frac{\omega_0}{\gamma \omega_0} = \frac{\omega_0}{\sqrt{1 - \beta^2}}. \]

Now \( \vec{J}(r, \omega) = \int d^3r \ e^{i\vec{k} \cdot \vec{r}} \hat{\vec{J}}(r, \omega) \)

The multipole expansion is \( \sum Q_n \vec{e}_n (\vec{r}) \), i.e., small \( |\vec{r}| \ll \lambda \)

Lowest order: \( \vec{E}(\vec{r}, \omega) = \frac{i k e^{ikr}}{cR} \vec{J}(\vec{r}, \omega) = \vec{B} \times \vec{H} \times \vec{r}, \quad \text{for } \left| \frac{k}{c} \right| \ll 1 \)

Interpretation: \( \vec{E}(\vec{r}, \omega) = \sum Q_n \vec{e}_n (\vec{r}) \), \( \vec{e}_n = \frac{\partial \vec{e}_m}{\partial \vec{r}} \).

Then \( \vec{E}(\vec{r}, \omega) = \frac{i k e^{ikr}}{cR} \vec{J}(\vec{r}, \omega) = \frac{k}{c} \vec{E} \times \vec{H} \times \vec{r}, \quad \text{for } \left| \frac{k}{c} \right| \ll 1 \)

For discrete particles, \( \gamma \ll 1 \), from Lorentz invariant, we get:

\[ \vec{E}(\vec{r}, \omega) = \int d\omega d^3k \left[ \frac{k^2}{c^2} \epsilon_{\omega k} \vec{e}_{\omega k} \right] = \frac{-1}{c^2} \delta(\omega - \omega_0) \int d^3k e^{-i\omega k} \vec{J}(\vec{r}, \omega) = \frac{1}{4\pi} \frac{k^2}{c^2} \int d^3k \delta(\omega - \omega_0) \]

Now \( \vec{d} = \vec{x} \times \vec{\delta} \) and the correspondence follows.

Dipole spectrum: \( \frac{d^4\vec{J}}{d\omega d^3k} = \frac{\omega^2}{4\pi c^2} \left( \frac{\vec{d}}{\omega} \right)^2 = \frac{\omega^2}{4\pi c^2} \left[ \vec{d} \times \left( \vec{d} \times \vec{\delta} \right) \right] \)

\[ x, \ \frac{d^2\vec{J}}{d\omega^2} = \frac{\omega^2}{4\pi c^2} \int d^3k \left( \vec{d} \cdot \vec{\delta} \right) \frac{d^3k}{k^2} \left( \vec{d} \times \vec{\delta} \right) \times \left( \vec{d} \times \vec{\delta} \right) = \frac{\omega^2}{4\pi c^2} \vec{d} \times \left( \vec{d} \times \vec{\delta} \right) \]

\[ \frac{d^3\vec{J}}{d\omega} = \frac{\omega^2}{4\pi c^2} \vec{d} \times \vec{\delta} \]

What kind of pattern? It depends on \( \vec{d} \). Example: \( \vec{d} = \vec{e}_3 \), take \( \vec{d} = \vec{z} \).

\[ d^2 - (\vec{d} \times \vec{\delta})^2 = d^2 - (\vec{d} \times \vec{\delta})^2 = d^2 (\cos^2 \theta - \sin^2 \theta) = \sin^2 \theta \]
\[ S_0 \frac{d^3 \hat{S}}{d\Omega \, d\omega} = \frac{e^2}{4\pi} \omega^2 \sin \theta \]

The radiation pattern (whereby \( \frac{d^3 \hat{S}}{d\Omega \, d\omega} \) is represented as distance from origin) is

\[ \vec{d} = \frac{\hat{z}}{\lambda} \]
\[ l = 1 \]
\[ m = 0 \]

For \( l = 1, m = 1 \)

\[ \vec{d} = \vec{d}_e \left( 1 + \sin \theta \right) \]
\[ \vec{d} \cdot \vec{e} = \frac{d_0}{r_0} \sin \theta \cos \phi \]
\[ \left| \vec{d} \right|^2 = \left| \vec{d}_e \right|^2 \left| 1 + \sin \theta \cos \phi \right|^2 = \frac{d_0^2}{r_0^2} \left( 1 + \sin^2 \theta \right) \left( 1 + \cos^2 \theta \right) \]

\[ \vec{d} = \frac{\hat{z}}{\lambda} \cos \phi \]

\( l = 1, m = 1 \)

\[ \text{Make \textit{black}: } q(c \cos \phi, -c \sin \phi, c) \Rightarrow \text{circular motion in } xy \text{ plane.} \]

Dipole:
\[ N \text{or dec. } \quad \overrightarrow{\mathbf{E}}^{(1)}(\mathbf{r}, \omega) = \int \left( -\hat{\mathbf{r}} \cdot \hat{\mathbf{F}} \right) \overrightarrow{\mathbf{F}}(\mathbf{r}', \omega) \, d^3r' \]

\[ \overrightarrow{\mathbf{F}}(\mathbf{r}', \omega) = -i \omega \epsilon \frac{\partial}{\partial t} \int \frac{\overrightarrow{\mathbf{E}}(\mathbf{r}', \omega)}{\epsilon} \, d^3r' \]

which we will use \( \overrightarrow{\mathbf{F}}(\mathbf{r}, \omega) = \frac{1}{4\pi} \int \cdots \delta^3(\mathbf{r} - \mathbf{r}'(\omega)) \) again (put our FT in \( \omega \)):

\[ \int \frac{\overrightarrow{\mathbf{E}}_a \overrightarrow{\mathbf{E}}_b \delta^3(\mathbf{r} - \mathbf{r}_a) \delta^3(\mathbf{r} - \mathbf{r}_b)}{\epsilon} = \frac{Z}{a_c} \epsilon_a \epsilon_b \overrightarrow{\mathbf{E}}_a \overrightarrow{\mathbf{E}}_b \]

\[ \mathbf{V}_a \cdot \frac{\partial}{\partial t} \overrightarrow{\mathbf{r}}_a = \mathbf{v}_a \overrightarrow{\mathbf{r}}_a + \mathbf{r}_a \mathbf{v}_a \]

\[ \Rightarrow \quad \mathbf{v}_a \mathbf{r}_a = \frac{1}{2} \left( \mathbf{v}_a \mathbf{r}_a + \mathbf{r}_a \mathbf{v}_a \right) + \frac{1}{2} \mathbf{r}_a \mathbf{v}_a \mathbf{r}_a \]

\[ = \frac{1}{2} \frac{d}{dt} \left( \mathbf{r}_a \mathbf{v}_a \right) + \frac{1}{2} \epsilon_{ijk} \epsilon_{kmn} \mathbf{r}_a \mathbf{v}_i \mathbf{v}_j \]

The 1st lemma gives

\[ \frac{d}{dt} \overrightarrow{\mathbf{E}}_a \mathbf{r}_a \overrightarrow{\mathbf{E}}_b \mathbf{r}_b = \frac{d}{dt} \frac{1}{2} \sum_a Z_a \epsilon_a \left( 3 \epsilon_a - \epsilon_i \epsilon_j \right) \mathbf{r}_a \mathbf{v}_a \mathbf{r}_b \mathbf{v}_b \]

\[ + \frac{1}{2} \delta_{ij} \frac{d}{dt} \sum_a Z_a \epsilon_a \mathbf{r}_a \]

\[ D_{ij} \]

The 2nd gives

\[ \sum_a Z_a \epsilon_a \mathbf{r}_a \mathbf{v}_i \mathbf{v}_j = \sum_a Z_a \epsilon_a \mathbf{r}_a \mathbf{v}_a = -2 \overrightarrow{\mathbf{m}} \quad \text{(magnetic dipole moment).} \]

So, with \( \overrightarrow{\mathbf{D}} = \overrightarrow{\mathbf{D}}_0 + \mathbf{D} \) and taking \( -i \omega \to \frac{d}{dt} \)

\[ \overrightarrow{\mathbf{E}}^{(1)}(\mathbf{r}, \omega) \approx \int dt e^{i \omega t} \left[ \mathbf{v}_x \mathbf{D}_y + \frac{1}{2} \mathbf{v} \times \mathbf{D} + \frac{1}{2} \left( \frac{d}{dt} \sum_a Z_a \epsilon_a \mathbf{r}_a \right) \mathbf{r} \right] \]

When taking \( \int d^3r \), the last term never contributes.\( \overrightarrow{\mathbf{E}}^{(1)} \).

\[ \Rightarrow \quad \overrightarrow{\mathbf{E}}^{(1)}(\mathbf{r}, \omega) \approx \mathbf{E}^{(1)}(\mathbf{r}, \omega) \approx \frac{i}{C} \omega \epsilon_0 \left[ \frac{1}{C} \mathbf{D}(\mathbf{r}) - i \omega \overrightarrow{\mathbf{m}} \right] \]

or in time domain \( \mathbf{D} + \frac{1}{C} \mathbf{E} \) or \( e^{i \omega t} \) is \( t + t_{mix} \). Putting \( (\alpha) \leftrightarrow (\beta) \) together:

\[ \mathbf{E}(\mathbf{r}, t) = \frac{1}{C} \left[ \mathbf{R} \times (\mathbf{F} \times \mathbf{D}) + \mathbf{R} \times \mathbf{F} + \frac{1}{C} \mathbf{R} \times (\mathbf{R} \times \mathbf{D}) \right] \mathbf{D} \]

\[ \overline{\mathbf{D}}(\mathbf{r}, t) = \left[ \frac{1}{C} \mathbf{R} \frac{\partial}{\partial t} \left( \mathbf{R} \times \mathbf{D} + \mathbf{R} \times (\mathbf{F} \times \mathbf{D}) \right) - \frac{1}{C} \mathbf{R} \times \left( \mathbf{R} \times \mathbf{D} + \mathbf{R} \times (\mathbf{F} \times \mathbf{D}) \right) \right] \mathbf{R} \]

Terminology radiation from \( \mathbf{D}, \mathbf{E}, \ldots \) is \( \mathbf{E}, \mathbf{D}, \mathbf{m}, \mathbf{p}, \mathbf{\overline{D}} \)
\[
\frac{dP}{d\Omega} : \text{patterns have interference between these terms}
\]

\[P = \int \frac{d\Omega}{d\Omega} \text{ is sum of individual powers}^3\]

To see this, we need averages of powers of \( \hat{\vec{e}} \) : let \( \langle \cdot \rangle = \frac{1}{4\pi} \int d\Omega \cdot \)

\[\langle i \rangle = 1, \quad \langle \hat{\vec{e}} \rangle = 0 = \langle \hat{\vec{e}}_1 \hat{\vec{e}}_2 \hat{\vec{e}}_3 \hat{\vec{e}}_4 \rangle
\]

\[\langle \hat{\vec{e}}_i \hat{\vec{e}}_j \rangle = \frac{i}{2} \delta_{ij}
\]

\((\text{Abnd)}: \quad \langle \hat{\vec{e}}_i \hat{\vec{e}}_j \rangle = c \delta_{ij} \text{ by orthogonality symmetry} \Rightarrow \langle \hat{\vec{e}} \rangle = 2c \Rightarrow c = \frac{1}{8} \sqrt{i}
\]

\[\langle \hat{\vec{e}}_i \hat{\vec{e}}_j \hat{\vec{e}}_k \hat{\vec{e}}_l \rangle = \frac{1}{i^2} \left( \delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) \quad \left(\langle \hat{\vec{e}} \hat{\vec{e}} \hat{\vec{e}} \rangle = \frac{1}{5} (q^{12})\right)
\]

So for \( \frac{dP}{d\Omega}(\mathbf{r}, \mathbf{p}) \) need \( \langle E^3 \rangle \); keep in mind \( \hat{\vec{e}}_i \hat{\vec{e}}_j \). So the number of \( \hat{\vec{e}}_i \)'s in \( \tilde{\mathbf{m}} \) has to be \# cross terms is odd. Moreover, the \( \hat{\vec{e}}_i \)s in \( \tilde{\mathbf{m}} \) comes from four \( \hat{\vec{e}}_i \)'s \( \rightarrow \) product of \( \delta_{ij} \)'s and \( \hat{\vec{e}}_i \)'s. 

Need to contract with \( \mathbf{w}_k \hat{\vec{e}}_i \) : only possibility is \( \epsilon^{ijk} \mathbf{w}_k \hat{\vec{e}}_j = 0 \) \((\mathbf{p}_l = 0_{l, k})
\]

\[\frac{dP}{d\Omega} \sim d^4, \quad \mathbf{m}^2 \text{ and } \hat{\vec{e}}_i \text{ terms}
\]

In fact: 

\[
P = \frac{2}{3c^2} \tilde{\mathbf{m}}^2 + \frac{2}{3c^2} \tilde{\mathbf{m}}^2 + \frac{1}{180c^5} \epsilon^{i j k} \mathbf{w}_i \hat{\vec{e}}_j \hat{\vec{e}}_k
\]

Do \( \tilde{\mathbf{m}} \) first: 

\[
\frac{dP}{d\Omega} = \frac{\tilde{\mathbf{m}}^2}{4c^3} E^2 = \frac{1}{4c} \left( \mathbf{R} \times \hat{\vec{e}} \right)^2 = \frac{1}{4c} \left( \mathbf{R} \times \hat{\vec{e}} \right)^2
\]

\[
\Rightarrow \quad P = \frac{1}{c} \left( \left| \hat{\vec{e}} \right|^2 - \langle \hat{\vec{e}}_i \hat{\vec{e}}_j \rangle \hat{\vec{e}}_i \hat{\vec{e}}_j \right) = \frac{1}{c} \left( \left| \hat{\vec{e}} \right|^2 - \frac{1}{3} \delta_{ij} \hat{\vec{e}}_i \hat{\vec{e}}_j \right) \hat{\vec{e}}_i \hat{\vec{e}}_j = \frac{2}{3c} \tilde{\mathbf{m}}^2
\]

Clearly \( \tilde{\mathbf{m}} \) is the same.

Exercise: Do \( \tilde{\mathbf{m}}^2 \) term.
See text for pure quadrupole patterns.

\[
\frac{dL}{d\Omega} = \frac{e}{\mu_0 m c^2} \left( \mathbf{R}(x, y, z) \right) \hat{R} \left( \mathbf{R}(x, y, z) \right) \mathbf{P} = \frac{1}{\mu_0 ec} \left( \mathbf{l} \mathbf{d} - \mathbf{d} \mathbf{l} \right)
\]

\[
= \frac{1}{\mu_0 ec} \left( \mathbf{\hat{l}} \cdot \mathbf{\hat{d}} \cdot \delta_{ij} \mathbf{P}_i \mathbf{P}_j - \mathbf{\hat{R}} \mathbf{\hat{R}} \mathbf{\hat{R}} \mathbf{\hat{R}} \mathbf{\hat{P}}_i \mathbf{\hat{P}}_j \mathbf{\hat{P}}_i \mathbf{\hat{P}}_j \right)
\]

Then pick particular \( \mathbf{P}_i \) (unions, symmetries). Tentatively used to be \( q_{2m} \). For \( l \leq 2 \):

\[
\begin{align*}
\ell = 2, \quad m = 2 & \quad \text{(bent)} \\
\ell = 2, \quad m = 1 & \quad \text{(bent)} \\
\ell = 2, \quad m = 0 & \quad \text{(unions)}
\end{align*}
\]
Antenna

Centered linear antenna: \[ \begin{array}{c}
\text{antenna on } \xi \text{-axis, origin at middle point.}
\end{array} \]

Crude model: \[ I(t) = I_0 \cos \omega t \] (independent of \( z \)) \text{(unrealistic)}.

\[ R_{\text{echo}}(\omega) = \frac{2}{
\int \nabla \cdot \mathbf{E}(\mathbf{r}, \omega)
\] and \[ \mathbf{E}(\mathbf{r}, \omega) = \int d\mathbf{r}' \mathbf{E}(\mathbf{r}', \omega) \]

So \[ E^{(\omega)} = \int d\omega \cdot \mathbf{E} \cdot I(\omega) = \frac{1}{2} \text{A} \int dt \cdot \mathbf{E} \cdot \mathbf{E} = \frac{1}{2} \text{A} \int dt (\delta(\omega - \omega) + \delta(\omega + \omega)) \]

\[ \Rightarrow \quad \frac{1}{2} \left| E^{(\omega)} \right|^2 = \frac{1}{8} \text{A}^2 \sin(\omega t) \sin \theta \quad (\sin \theta = \sqrt{1 - (\delta k)^2} = \sqrt{1 - (\omega \tau)^2}) \]

So the power follows:

\[ \frac{dP}{d\omega} = \frac{1}{8} \text{A}^2 \sin(\omega t) \sin \theta \]

or in terms of wavelength \( \lambda = \frac{2\pi}{\omega} \),

\[ \frac{dP}{d\omega} = \frac{\pi}{2} \left( \frac{\text{A}}{\lambda} \right)^2 \int_0^\lambda \sin(\omega t) \sin \theta \]

and averaging over one cycle, \( \sin(\omega t) \to \frac{1}{2} \quad \frac{dP}{d\omega} = \frac{\pi}{2} \left( \frac{\text{A}}{\lambda} \right)^2 \int_0^\lambda \sin \theta \]

We could have guessed \( I_0 \left( \frac{q}{\lambda} \right)^2 \) \text{ (dipole multipoles, } d = \frac{q}{\lambda} \)

the \( \sin \theta \) \( \mathbf{d} \) along \( \mathbf{e} \) and \( I_0 \left( \frac{d}{\lambda} - 1 \right) \). The \( \frac{1}{2} \) is from dimensional analysis. Let \( \mathbf{d} \) with \( \mathbf{e} \), for which we needed a calculation.
A more realistic model needs $I(x,y) = 0$. For a very thin antenna we take $\mathbf{J} = \delta(x) \delta(y)$. So we propose

$$
\mathbf{J} = J_0 \sin(\frac{k}{2} k_a \cdot k_{el}) \cos(w t) \delta(x) \delta(y) \mathbf{h}
$$

$I_0$ is the max current at center, $w_0$ fed current is $I_0 = I_0 \sin(\frac{k}{2} k_a)$

Calculate:

$$
\mathbf{J}(x,y,z) = \int dt e^{i\omega t} \int d\mathbf{r} \mathbf{J}(r,t)
$$

$$
= 2 \int d\mathbf{r} e^{i\omega t} \mathbf{J}(r,t) = \frac{2}{k_0} \int dt e^{i\omega t} \mathbf{J}(r,t) = \frac{2}{k_0} \int dt e^{i\omega t} \mathbf{J}(r,t) = \frac{2}{k_0} \int dt e^{i\omega t} \mathbf{J}(r,t) = \frac{2}{k_0} \int dt e^{i\omega t} \mathbf{J}(r,t)
$$

$$
\mathbf{d} = \int d\omega \mathbf{e}^{i\omega t} \mathbf{J}(\omega) = \frac{2}{k_0} \int d\omega \mathbf{e}^{i\omega t} \mathbf{J}(\omega)
$$

$$
\frac{\mathbf{d}}{dt} = \frac{2}{k_0} \int d\omega \mathbf{e}^{i\omega t} \mathbf{J}(\omega)
$$

$$
\mathbf{E} = \frac{2}{k_0} \int d\omega \mathbf{e}^{i\omega t} \mathbf{J}(\omega) = \mathbf{E} = \frac{2}{k_0} \int d\omega \mathbf{e}^{i\omega t} \mathbf{J}(\omega) = \mathbf{E} = \frac{2}{k_0} \int d\omega \mathbf{e}^{i\omega t} \mathbf{J}(\omega)
$$

$$
\mathbf{E} = \frac{2}{k_0} \int d\omega \mathbf{e}^{i\omega t} \mathbf{J}(\omega) = \mathbf{E} = \frac{2}{k_0} \int d\omega \mathbf{e}^{i\omega t} \mathbf{J}(\omega)
$$

For small $k_a$ (dipole approximation, which we are using) $1 - \cos(\frac{1}{2} k_a) = \frac{1}{2} (\frac{1}{2} k_a)^2$

$$
\frac{\mathbf{d}}{dt} = \frac{2}{k_0} \int d\omega \mathbf{e}^{i\omega t} \mathbf{J}(\omega)
$$

This problem can be solved without use of multipole expansion: recall

Recall for monochromatic source

$$
\frac{\partial P_n}{\partial t} = \frac{n^2 \omega^2}{20 c^2} \left[ \mathbf{E}_n \cdot \mathbf{E}_n \right]
$$

Where we only used $n = 1$, and $\mathbf{E}_n = \frac{2}{k_0} \int d\omega \mathbf{e}^{i\omega t} \mathbf{J}(\omega)$

$$
S_\omega \mathbf{E}_n(\mathbf{r}) = \frac{1}{t_t} \int dt e^{i\omega t} \mathbf{e}^{i\omega t} \mathbf{J}(\mathbf{r},t) = \frac{1}{t_t} \int dt e^{i\omega t} \mathbf{e}^{i\omega t} \mathbf{J}(\mathbf{r},t) = \frac{1}{t_t} \int dt e^{i\omega t} \mathbf{e}^{i\omega t} \mathbf{J}(\mathbf{r},t) = \frac{1}{t_t} \int dt e^{i\omega t} \mathbf{e}^{i\omega t} \mathbf{J}(\mathbf{r},t)
$$

The time (in) gives $\frac{1}{t_t}$ (and here it is a $\mathbf{J}_n$ which also has $\frac{1}{t_t}$ but we have accounted for it, recall $\frac{2}{k_0} \mathbf{e}^{i\omega t} \mathbf{J}_n$...).
\[
N_{\kappa=\ell} = \int_{-\ell}^{\ell} e^{-ikz} s_{1\kappa}(\frac{1}{2}k_\alpha - k_\beta) dz
\]

\[
= \int_{-\ell}^{\ell} e^{-ik_z} s_{1\kappa}(\frac{1}{2}k_\alpha - k_\beta) + \int_{-\ell}^{\ell} e^{-ik_z} s_{1\kappa}(\frac{1}{2}k_\alpha + k_\beta)
\]

\[
= \text{Im} \left[ \int_{-\ell}^{\ell} e^{-ik_z} e^{i\left(\frac{1}{2}k_\alpha - k_\beta\right) z} + \int_{-\ell}^{\ell} e^{-ik_z} e^{i\left(\frac{1}{2}k_\alpha + k_\beta\right) z} \right]
\]

\[
= \text{Im} \left[ e^{-i\frac{1}{2}k_\alpha z} e^{i\frac{1}{2}k_\beta z} \left( e^{-i\left(k_\alpha + k_\beta\right) z} - 1 \right) + e^{i\frac{1}{2}k_\alpha z} e^{-i\frac{1}{2}k_\beta z} \left( 1 - e^{-i\left(k_\alpha - k_\beta\right) z} \right) \right]
\]

\[
= -\text{Re} \left[ \frac{1}{k_\alpha - k_\beta} \left( e^{-i\frac{1}{2}k_\alpha z} - e^{-i\frac{1}{2}k_\beta z} \right) + \frac{1}{k_\alpha + k_\beta} \left( e^{i\frac{1}{2}k_\alpha z} - e^{i\frac{1}{2}k_\beta z} \right) \right]
\]

\[
= -\frac{1}{k_\alpha + k_\beta} \left( \cos\left(k_\alpha z\right) - \cos\left(k_\beta z\right) \right) - \frac{1}{k_\alpha - k_\beta} \left( \cos\left(k_\alpha z\right) - \cos\left(k_\beta z\right) \right)
\]

This is for arbitrary \( k \). But we need only \( k \approx k_\beta \to \kappa = \alpha \sigma k_\beta \)

\[
= \frac{\cos(k_\alpha z) - \cos(k_\alpha z)}{k_\alpha - k_\beta} - \frac{2k_\alpha}{k_\alpha sin^2 \theta} \left[ \cos\left(k_\alpha z\right) - \cos\left(k_\alpha z\right) \right]
\]

\[
\text{For } \left[ \frac{1}{2} \right]^2 \text{ and } \left[ x - \hat{\mathbf{P}} \cdot \hat{\mathbf{r}} \right]^2 = 1 - \cos^2 \theta = \sin^2 \theta
\]

\[
\text{It follows } \int \cos^2 \theta = \frac{1}{2} \left( \cos^2 \theta + \sin^2 \theta \right)
\]

\[
\frac{d\Omega}{d\Omega} = \frac{\frac{1}{2} \cos^2 \theta}{27 \pi c} \left( \cos\left(k_\alpha z\right) - \cos\left(k_\alpha z\right) \right)^2
\]

\[
\text{For } \ell < \frac{1}{2}, \quad \cos\left(k_\alpha \cos \theta\right) - \cos\left(k_\alpha \cos \theta\right) = -\frac{1}{2} \left( k_\alpha \cos \theta \right)^2 + \frac{1}{2} \left( k_\alpha \cos \theta \right) \cos \theta = \left( k_\alpha \cos \theta \right)^2 \sin \theta
\]

\[
= \frac{d\Omega}{d\Omega} \left( k_\alpha \right)^2 \sin \theta
\]
Radiation pattern: we've seen in dipole approx we have $\frac{d\Phi}{d\Omega} \propto \sin^2 \theta$

But (for this model) in exact case we have

$$\frac{d\Phi}{d\Omega} \propto \frac{(\cos(\frac{kz}{2} \cos \theta) - \cos(\frac{kz}{2}))^2}{\sin^2 \theta}$$

At small $\theta$,

$$\cos(\frac{kz}{2} \cos \theta) \approx \cos(\frac{kz}{2}) \cos(\frac{kz}{2} \theta)$$

$$\approx \cos(\frac{kz}{2}) + \sin(\frac{kz}{2}) \sin(\frac{kz}{2} \theta)$$

So $\frac{d\Phi}{d\Omega} \propto \theta^2$ so $\theta = 0$ (mimic) is a null direction, just as in dipole approximation.

Additional null directions arise if $kz$ is not small,

hence $kz = \frac{1}{2} (\frac{2\pi}{\lambda})^3 \approx \frac{k^2 z^3}{\lambda^4}$.
Define radiation resistance \( R_{rad} \) of an antenna such that the power radiated is the "dissipated" power: \( P = \frac{1}{2} \int_0^L R_{rad} \).

In the dipole approximation, \( I_0 = I_0 \sin \left( \frac{k_0 R}{2} \right) \approx I_0 \frac{k_0 R}{2} \)
and \( P = \int \frac{dP}{d\Omega} \, d\Omega = \int \frac{1}{158 \pi c} \left( \frac{k_0 R}{2} \right)^2 \sin^2 \theta \, d\Omega \)

\[
\left( \frac{k_0 R}{2} \right)^4 \left( \frac{I_0}{158 \pi} \right)^2 \times 2 \pi \int_0^1 (1 - s^2)^{1/2} \, ds = \frac{(k_0 R)^2}{12 \pi} \int_0^1 \frac{I_0^2}{s^2} \, ds \quad R_{rad} = \frac{(k_0 R)^2}{C}
\]

Units? : real \( F = \frac{q^2}{r} \quad |P| = |\mathcal{E}| \cdot |\mathcal{H}| \cdot |\mathcal{E}| \cdot |\mathcal{H}| \cdot |\mathcal{H}| \cdot |\mathcal{E}| \cdot |\mathcal{E}| \\
So what is \( \frac{1}{C} \) in ohms? We can look in tables, or use our translation instructions.

Recall : \( \varphi_0 = \frac{q^2}{4 \pi e_0} \Rightarrow \varphi_0 = \frac{q^2}{4 \pi e_0} \cdot C = \frac{1}{e_0} \)

\[
P = \frac{(k_0 R)^2}{12 \pi} \frac{\varphi_0}{4 \pi e_0} = \frac{(k_0 R)^2}{4 \pi \varphi_0 e_0}
\]

The quantity \( Z_0 = \frac{\varphi_0}{e_0} = 377 \) ohms is known as the "impedance of free space.

and \( R_{rad} = Z_0 \left( \frac{\varphi_0}{4 \pi \varphi_0} \right) = \frac{\varphi_0}{4 \pi} \left( \frac{\varphi_0}{Z_0} \right) = \frac{1}{2} \varphi_0 \left( \frac{4 \pi}{Z_0} \right)^2 = 197 \left( \frac{\varphi_0}{Z_0} \right)^2 \)

Since \( a \) is in this approximation, this is a fairly small number, hence low radiation efficiency. For \( a \approx 0.2 \) \( R_{rad} \approx 8 \) \( \Omega \), while \( a = \frac{1}{2} \) ("half-wave antenna") \( R_{rad} = 49 \) \( \Omega \). Better yet, \( a = \lambda \) ("half-wave antenna") but then we need to integrate \( \int \frac{dP}{d\Omega} \, d\Omega \) with post \( \lambda < a \) approximation.
Near Zone Fields: Very briefly with a mis. The region $R \ll \lambda$ is called "near zone." For NR sources one has in addition acc. $\lambda$.

Then in $A^\mu(\vec{r},t) = \frac{1}{c} \int d^4x' \frac{\mathcal{J}(\vec{x}',t'|t)}{|\vec{x}' - \vec{r}|}$

one has $t_{\mu} = t - \frac{R}{c} \to t$ and $A^\mu(\vec{r},t) = \frac{1}{c} \int d^4x' \frac{\mathcal{J}(\vec{x}',t')}{|\vec{x}' - \vec{r}|}$

The instantaneous field.

One can justify this ($t_{\mu} \to t$) more precisely by going to Fourier space:

Recall $\tilde{A}^\mu(\vec{r},t) = \frac{e^{ik\cdot r}}{cR} \mathcal{F}(\vec{k},\omega)$

Now $e^{ik\cdot r} = 1 + ik\cdot r + \cdots$ with $k^2 = \frac{\omega^2}{c^2} < 1$. Replace $e^{ik\cdot r} \to 1$. Now.
Maxwell equations in media (Gary: Chap 13, Sec 81).

Model media as made up of charges that are fixed (as in molecules, which may not be neutral as they may have deficit or excess of electrons) plus free charges (as in conduction electrons in conductors, or added charges in insulators). Denote by \( \vec{e} \) \( \vec{b} \) the microscopic fields, i.e., the fundamental fields. Then change over atomic distance scales. So, we have

\[
\nabla \cdot \vec{e} = \rho_{\text{free}} \quad \nabla \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0 \quad \nabla \times \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} = \frac{\rho_{\text{free}}}{\epsilon_{\text{free}}} \quad \nabla \cdot \vec{B} = 0
\]

Now, smooth these out over "macroscopic" distances (where "macroscopic" depends on context, but can be as short as \( \approx 10 \) atomic distance, say). To this end, use a smoothing (averaging) function \( \delta(r') \) : we want

\[
\int d^3r' \delta(r') = 1
\]

and

\[
\delta(r') \to \frac{1}{d} \quad \text{with } d \to \text{typical test variation of } \vec{e} \text{ and } \vec{b} \text{ on "atomic" scale}
\]

Then let \( \vec{E}(r,t) = \int d^3r' \delta(r-r') \vec{E}(r',t) \)

\( \text{and } \vec{B}(r,t) = \int d^3r' \delta(r-r') \vec{B}(r',t) \)

Now \( \partial_t \vec{E}_j = \int d^3r' \partial_t \delta(r-r') \vec{E}_j(r',t) = \int d^3r' (\vec{E}_j - \vec{E}_j(r',t)) \),

\[
\text{So all diff ops go } \text{through and:}
\]

\[
\nabla \cdot \vec{E} = \rho_{\text{free}} \quad \nabla \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0 \quad \nabla \times \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} = \frac{\rho_{\text{free}}}{\epsilon_{\text{free}}} \quad \nabla \cdot \vec{B} = 0
\]

where \( \rho_{\text{free}} = \int d^3r' \rho(r') \delta(r-r') \) as above.

Now best is to break \( \langle \rho \rangle \) \( \langle \rho \rangle \) into pieces in a useful/current way.
Text discusses various physics at atomic scales, including QM. This is one of those cases where "it's not relevant except when it is." So ignore for now and deal with QM as needed.

(Sec 82): $\rho$ and $\mathbf{P}$

- Charges bound to atoms/molecules
- Charges not bound to atoms/molecules

Bound charges/molecules are neutral; effect it displaces large compared to size $\rightarrow$

multipole expansion $\rightarrow$ usually dipole suffices. Both for its field and its response to applied field.

Let $\mathbf{P} = \text{dipole moment/volume, a local quantity}$ (i.e. $\mathbf{P} = \mathbf{P}(r)$)

\[
\mathbf{P} \sim \frac{\mathbf{E}}{\varepsilon_0} \text{ in some } \delta V, \quad \mathbf{P} = \frac{\mathbf{E}}{\varepsilon_0} : \delta V \text{ is large enough to smooth out atomic scale fluctuation, yet small enough that multiple expansion makes sense.}
\]

Alternatively, text says $\mathbf{P} = n \mathbf{d}$, $n = \# \text{ density.}$

Now $\mathbf{P} \rightarrow$ charge distribution. Recall $\rho(r) = \frac{\mathbf{P} \cdot \hat{r}}{\rho} \text{ from dipole of origin.}$

\[
\rho \rightarrow \phi(r) = \int_V \frac{\rho(r') \cdot (r-r')}{|r-r'|^3} \, dr'
\]

Use $\frac{1}{|r-r'|} = \frac{r-r'}{|r-r'|^3}$, integrate by parts, keep surface term:

\[
\phi(r) = \int_V \frac{\rho(r') \cdot \hat{r}}{|r-r'|} \, dr' - \int_V \frac{\rho(r') \cdot \hat{r}}{|r-r'|} \, dV
\]

$\mathbf{P} = \int_S \rho_{pol} \, dS$

$\mathbf{P} = -\mathbf{P}_{pol}$

$\mathbf{P}_{pol}$ is cancelled by limit of adjacent volume. Not so for surface of material.
So break \( \mathbf{\varepsilon} \mathbf{\varepsilon}_{\text{free}} \) into \( \mathbf{\varepsilon}_{\text{media}} + \mathbf{\varepsilon}_{\text{free}} \)

\[
\nabla \cdot \mathbf{E} = \frac{\mu_0}{\varepsilon_0} (\mathbf{\varepsilon}_{\text{media}} + \mathbf{\varepsilon}_{\text{free}}) \mathbf{\varepsilon}_{\text{media}} = \frac{\mu_0}{\varepsilon_0} \mathbf{\varepsilon}_{\text{media}} \cdot \left( \nabla \cdot \mathbf{D} \right) = \frac{\mu_0}{\varepsilon_0} \left( \nabla \cdot \mathbf{D} \right)
\]

\[
\nabla \cdot \mathbf{D} = 4\pi \mu_{\text{free}}
\]

With this definition then also

\[
\mathbf{\varepsilon}_{\text{media}} \mathbf{\varepsilon}_{\text{media}} + \mathbf{\varepsilon}_{\text{free}} \mathbf{\varepsilon}_{\text{free}} = \left( \mathbf{\varepsilon}_{\text{media}} - \mathbf{\varepsilon}_{\text{free}} \right) \mathbf{\varepsilon}_{\text{media}} = \frac{\mu_0}{\varepsilon_0} \left( \mathbf{\varepsilon}_{\text{media}} + \mathbf{\varepsilon}_{\text{free}} \right) \mathbf{\varepsilon}_{\text{media}} \cdot \left( \nabla \mathbf{D} \right) = \frac{\mu_0}{\varepsilon_0} \left( \mathbf{\varepsilon}_{\text{media}} + \mathbf{\varepsilon}_{\text{free}} \right) \mathbf{\varepsilon}_{\text{media}} \cdot \left( \nabla \mathbf{D} \right)
\]

\[
\left( \nabla \mathbf{D} \right) \cdot \mathbf{\varepsilon}_{\text{media}} = \frac{\mu_0}{\varepsilon_0} \mathbf{\varepsilon}_{\text{free}}
\]

(We used this last quarter in wave propagation in media).

To complete boundary conditions at interfaces, \( \nabla \times \mathbf{B} = 0 \)

\[
\mathbf{B} \cdot \mathbf{n} = 0 \Rightarrow \mathbf{E} \times \mathbf{B} = 0 \Rightarrow E_t = \mathbf{E} \cdot \mathbf{t} \quad \text{"t" = tangential},
\]

Notes:

1. \( \mathbf{\varepsilon} \mathbf{\varepsilon}_{\text{free}} \) is not sourced by \( \mathbf{\varepsilon}_{\text{media}} \): in addition to \( \nabla \cdot \mathbf{D} = 4\pi \mu_{\text{free}} \) we have

\[
\nabla \times \mathbf{D} = \nabla \times (E + \sigma_{\text{free}} \mathbf{J}) = \mu_0 \nabla \times \mathbf{D}
\]

That is, we have to look at \( \mathbf{D}, \mathbf{E} \) and some way of determining \( \mathbf{J} \) to get full picture.

2. Dimensionless Units: in Gaussian units are simple and make sense \( \mathbf{D}, \mathbf{E} \) and \( \mathbf{B} \) have same units. In SI \( \mathbf{D} = \mathbf{E} + \mathbf{P} \), so \( \mathbf{D} \) has units of \( \mathbf{P} \) (Coulomb/m²), different from \( \mathbf{E} \).

See text for translation (or work it out).
\textbf{Section 83: Macroscopic current density}

\textbf{Break into components:}

\[ \vec{J}_{\text{macro}} = \vec{J}_{\text{free}} + \vec{J}_{\text{pol}} + \vec{J}_{\text{conv}} + \vec{J}_{\text{mag}} \]

\( \vec{J}_{\text{free}} \) is from motion of free charges, so \[ \frac{\partial \vec{J}_{\text{free}}}{\partial t} + \vec{\nabla} \cdot \vec{J}_{\text{free}} = 0 \]

\( \vec{J}_{\text{pol}} \) is from + dependence on \( \vec{P} \): \[ \vec{J}_{\text{pol}} = -\vec{\nabla} \vec{P} \Rightarrow \frac{\partial \vec{J}_{\text{pol}}}{\partial t} = -\vec{\nabla} \frac{\partial \vec{P}}{\partial t} \Rightarrow \vec{F}_{\text{pol}} = \frac{\partial \vec{P}}{\partial t} \]

The convection current \( \vec{J}_{\text{conv}} \) (gases & liquids only), carries charge by overall motion of fluid. Artificial breakdown, but useful. If \( \vec{u} \) is fluid velocity field,

\[ \vec{J}_{\text{conv}} = (\vec{P}_{\text{free}} + \vec{P}_{\text{pol}}) \vec{u} \]

Then one has to make sure of no double counting so this is subtracted from \( \vec{J}_{\text{free}} \) & \( \vec{J}_{\text{pol}} \) which then measure current in a moving fluid element.

Most interesting: \( \vec{J}_{\text{mag}} \) magnetization current:

\[ \vec{M}(\vec{r}) = \text{magnetic dipole moment/volume} \]

Let’s recall field due to magnetic dipole; for this read:

\textbf{Brief review of magnetostatics: recall} \[ \delta^3(\vec{d} \cdot \vec{A} - \vec{d} \cdot \vec{A}) = \frac{4\pi}{c} \vec{J} \; \text{Spacial components} \]

\textbf{steady state:} \[ -\vec{\nabla} \times \vec{A} + \vec{\nabla} (\vec{\nabla} \cdot \vec{A}) = \frac{4\pi}{c} \vec{J} \; \text{choose} \; \vec{\nabla} \cdot \vec{A} = 0 \; \text{gauge}, \; \text{and} \]

\[ \vec{A} = \frac{1}{c} \int \frac{\vec{J}^{(s)}}{4\pi} \, d^3 r' \]

\textbf{Magnetostatics:} \[ \frac{\partial \vec{B}}{\partial t} = 0 \Rightarrow \vec{\nabla} \cdot \vec{B} = 0 \].
**Multiply expansion:**

\[ \vec{A}(\vec{r}) = \frac{1}{c} \int d^3 r' \ f(\vec{r}') \left[ \frac{1}{|r'|} + \frac{\vec{r} \times \vec{r}'}{|r'|^3} + \cdots \right] \]

**Trick:** With \( \vec{f} \) localized, given two functions \( f(\vec{r}), g(\vec{r}) \) we have

\[ \int d^3 r \ \vec{\nabla}(fg) = 0 \Rightarrow \int d^3 r \ \left( f \vec{\nabla}g + g \vec{\nabla}f \right) = 0 \quad (\text{used } \vec{\nabla}^2 g = 0) \]

With \( f = 1, \ g = \gamma_i \) so \( \vec{\nabla}f = 0, \ \nabla g = \delta_{ij} \) we have

\[ \int d^3 r |\vec{r}| = 0 \Rightarrow \text{monopole term vanishes.} \]

Next \( f = x_i, \ g = x_j \Rightarrow \int d^3 r \ (r_i \nabla_j + r_j \nabla_i) = 0 \)

For next (dipole) term need \( o(\text{above}) \)

\[ \int d^3 r' \ \gamma_i(x_i, \vec{r}) r_j' = \int d^3 r' \left[ \frac{1}{2} (r_i r_j' + r_j r'_i) + \frac{1}{2} (\delta_{ij} - r_i r_j') \right] \]

\[ \Rightarrow \ A_i(\vec{r}) = \frac{1}{2c} \int \frac{\gamma_i}{|r'|} d^3 r' \left( \frac{r_i r_j' - r_j r'_i}{|r'|^3} \right) \]

Now \( \vec{r}_i r_j' - r_j r_i' = \epsilon_{ijk} \epsilon_{kln} \ r_i r_n' = \epsilon_{ijk} (\vec{r} \times \vec{r}')_k \)

\[ \vec{A}(\vec{r}) = \frac{\vec{m} \times \vec{r}}{|r|^3} \] with \( \vec{m} = \frac{1}{2c} \int d^3 r' \ \vec{r}' \times \vec{r}'(\vec{r}') \) be monopole dipole \( \vec{A} \) of \( \vec{f} \).

\[ \vec{B} = \vec{\nabla} \times \vec{A} \] was an assignment (earlier in course 205A)

Given \( \vec{B} \) in terms of \( \vec{m} \) just as \( \vec{E} \) in terms of \( \vec{A} \).
Back to # \( \mathbf{F} \): now \( \mathbf{A} = \frac{\mathbf{m} \times \hat{\mathbf{n}}}{r^2} \) we have

\[
\mathbf{A}(\mathbf{r}) = \int \frac{d^3 \mathbf{r'}}{V} \frac{\mathbf{m}(\mathbf{r'}) \times (\mathbf{r} - \mathbf{r'})}{|\mathbf{r} - \mathbf{r'}|^3}
\]

Now repeat steps we did for \( \mathbf{A} : \frac{\mathbf{r} - \mathbf{r'}}{|\mathbf{r} - \mathbf{r'}|^3} = \nabla' \frac{1}{|\mathbf{r} - \mathbf{r'}|} \), unk. unk by prb:

\[
\mathbf{A}(\mathbf{r}) = \int d^3 \mathbf{r'} \frac{\mathbf{m}(\mathbf{r'}) \times \hat{\mathbf{n}}}{|\mathbf{r} - \mathbf{r'}|} + \int d^3 \mathbf{r'} \frac{\nabla' \mathbf{m}(\mathbf{r'})}{|\mathbf{r} - \mathbf{r'}|}
\]

The 2nd term, unk.

\[
\mathbf{A}(\mathbf{r}) = \frac{1}{2} \int d^3 \mathbf{r'} \frac{\mathbf{m}(\mathbf{r'})}{|\mathbf{r} - \mathbf{r'}|} \Rightarrow \mathbf{F}_{\mathbf{m}} = \frac{1}{2} \mathbf{\nabla} \times \mathbf{m}(\mathbf{r})
\]

The 1st term is similar but for a surface current density \( \mathbf{J} : \)

\[
\mathbf{J}_{\mathbf{mag}} = \frac{1}{2} \mathbf{m} \times \hat{\mathbf{n}}
\]

Again, in the interior of the material adjacent volume elements give cancelling contribs:

\[ \hat{\mathbf{N}}_1 - \hat{\mathbf{N}}_2 \Rightarrow \mathbf{J}_{\mathbf{mag}} + \mathbf{J}_{\mathbf{mag}} = 0 \]

But not so for boundary surface.

More generally, \( \mathbf{m} \) should include \( \mathbf{m} \)'s from intrinsic magnetic dipole moments from particle spin.

Now

\[
\nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \frac{\mu_0}{c} (\mathbf{J}_{\text{free}} + \mathbf{J}_{\text{mag}} + \mathbf{J}_{\text{conv}} + \mathbf{J}_{\text{mag}})
\]

\[
\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{c} \nabla \times \mathbf{M}
\]

\[
\Rightarrow \nabla \times (\mathbf{B} - \mu_0 \mathbf{M}) = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{P}
\]

\[
\Rightarrow \nabla \times \mathbf{H} = \mathbf{B} - \mu_0 \mathbf{M} \Rightarrow \nabla \times \mathbf{H} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \frac{\mu_0}{c} (\frac{1}{c} \mathbf{J}_{\text{free}} + \mathbf{J}_{\text{conv}})
\]
For Garg, \( \mathbf{H} \) = "magnetizing field", \( \mathbf{B} \) = "magnetic field"

which I like.

\[
\begin{align*}
\text{In free} & \quad \mathbf{H} \cdot \mathbf{\hat{n}}_2 = \mathbf{H} \cdot \mathbf{\hat{n}}_1 = -\mathbf{\hat{n}}_1 \times \mathbf{B} \quad \text{and} \quad \mathbf{\hat{n}}_2 \times \mathbf{B} = \mathbf{B}_0 \quad (\mu_0 - \frac{1}{c} \mathbf{E} = 0) \\
(\mathbf{H}_2 - \mathbf{H}_1) \times \mathbf{\hat{n}}_2 &= \frac{4\pi}{c} \mathbf{J}_{\text{free}}
\end{align*}
\]

(We have given up convection current here).

\[
\text{For dimensions} \quad (\mathbf{H} \times \mathbf{M} \text{ same as } \mathbf{B} \text{ same as } \mathbf{E})
\]

and units (including translation to SI) see Garg.

**Constitutive Relations** (Garg Sec 8.4).

To solve the macroscopic Maxwell equations we need additional relations (giving \( \mathbf{E} \) in terms of \( \mathbf{B} \) or \( \mathbf{H} \), and \( \mathbf{M} \) in terms of \( \mathbf{B} \) or \( \mathbf{H} \)). We also need to know something about separating current/charge into free part.

Conductors: Ohm’s law \( \mathbf{J} = \sigma \mathbf{E} \)

\( \sigma \) = "conductivity".

Not a "law". Fails in semiconductors, or at large fields in conductors. Often frequency dependent, so in Fourier space \( \mathbf{J}(\omega) = \sigma(\omega) \mathbf{E}(\omega) \).
Dielectrics: For insulators
\[ \mathbf{D} = \varepsilon \mathbf{E} \]

\( \varepsilon = \text{“dielectric constant”} \)

Only at small fields. Sometimes need different \( \varepsilon \) in different directions \( \mathbf{D}_i = \varepsilon_i \mathbf{E}_i \) (non-isotropic materials).

Also frequency dependent
\[ \mathbf{D}(\omega) = \varepsilon(\omega) \mathbf{E}(\omega) \]

Permeability: \( \mathbf{B} = \mu \mathbf{H} \)

Not for ferromagnets, nor superconductors.

For ferromagnets, complicated functional relation.

For superconductors \( \mathbf{B} = 0 \) in bulk ("Meissner" effect).

(hypothetical superconductors, and Meissner phase of type II).
**Energetics (Gary 85)**

**Issues:**

- Is $E^2 + B^2 = e^2 + b^2$?

No! There is a lot of $\mathcal{E}$, $\mathcal{M}$ energy in binding charges to form molecules, in making the structure of a solid, and so on. None of this is captured by $E^2 + B^2$ not even when compared to averages $\langle e^2 + b^2 \rangle$ because these are averages over positive definite quantities.

- So there is some internal energy but is not in $E, B$.

- Dissipation: lose energy, need time averages over small enough times to consider internal energy.

So calculate work on free charges. That on bound charges goes into internal energy or lost to heat.

$$\frac{\text{work on free charges}}{\text{time}} = \frac{\partial}{\partial t} \int \vec{E} \cdot \vec{E} = \frac{\mathcal{E}}{\mathcal{M}} \left( \nabla \times \vec{H} - \frac{1}{c} \frac{\partial \vec{B}}{\partial t} \right) \cdot \vec{E}$$

Now

$$\nabla \cdot (\vec{E} \times \vec{H}) = \varepsilon_{ijk} \frac{\partial}{\partial x_j} (E_i H_k) = \varepsilon_{ijk} \left( \frac{\partial}{\partial x_j} E_i \right) H_k + \varepsilon_{ijk} E_i \frac{\partial}{\partial x_j} H_k$$

$$\nabla \cdot (\vec{H} \times \vec{E}) = -\vec{E} \cdot \nabla \times \vec{H}$$

$$= \vec{H} \cdot \left( -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \right) - \vec{E} \cdot (\nabla \times \vec{H})$$

$$= \left[ \nabla \cdot \left( \frac{\mathcal{E}}{\mathcal{M}} \vec{E} \times \vec{H} \right) \right] = \frac{\mathcal{E}}{\mathcal{M}} \cdot \frac{\partial}{\partial t} \vec{E} + \frac{1}{\mathcal{M}} \left( \vec{E} \cdot \frac{\partial \vec{H}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{E}}{\partial t} \right)$$

$$\vec{S} = \frac{\mathcal{E}}{\mathcal{M}} \vec{E} \times \vec{H} \text{ is the macroscopic version of Poynting vector.}$$
Recall $\frac{\partial \psi}{\partial t} + \nabla \cdot \mathbf{S} = \frac{\text{work}}{\text{time}}$ microscopically, but here we have $\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t}$ instead of $(\text{ii}) \frac{\partial}{\partial t}(\mathbf{E}^2 + \mathbf{B}^2)$.

To get beyond this, we need constitutive relations.

Then the last term is

$$\frac{1}{\varepsilon_0 \mu_0} \frac{\partial}{\partial t} (\mathbf{E}^2 + \mu \mathbf{H}^2)$$

But this has limited use/validity.

- $\frac{1}{\varepsilon_0 \mu_0} \left( \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \right)$ includes both energy increase plus power but goes into heat. Cannot generally disentangle, $\Rightarrow$ not $\frac{\partial}{\partial t}$ of a single quantity.

- For validity, $E \mu$ are time independent, really meaning, frequency independent $E = E(\omega) \equiv \text{constant}$, $\mu = \mu(\omega) \equiv \text{constant}$.

- For validity, must be in linear regime so the simple constitutive relations apply.

Keep this in mind!
Electrostatics with conductors (Gary Chp 14).

We have already covered much of this chapter. These notes focus on new material, and even there are mostly supplemental → read text.

Summary of main ingredients:

1. $E = 0$ $\Rightarrow$ $\phi$ constant in conductors.

2. Charges on conductors live on surface.

\[
E_{\phi} \text{ for } E_{\phi} \text{ normal to surface.}
\]

3. Uniqueness: if $\phi$ or $\phi_{\partial}$ is specified at boundaries, then $\phi_{\partial}$ is constant (and constant = 0 if $\phi$ specified anywhere at a boundary).

**Electrostatic energy:** From $V = \frac{k}{2\epsilon} \left( \vec{E}^2 + \vec{B}^2 \right)$, for $\vec{B} = 0$ and $\vec{E} = -\vec{\nabla} \phi$ one has

\[
E = \int_V k \left( \nabla \phi \right)^2 = \int_V \frac{1}{2\epsilon} \vec{\nabla} \phi \cdot \frac{1}{2\epsilon} \vec{\nabla} \phi - \frac{1}{\epsilon} \int_V \vec{\nabla} \phi \cdot \vec{\nabla} \phi.
\]

Then: if the boundary is $r = \infty$, and the fields vanish there, using $\vec{A} = -\vec{\nabla} \phi$,

\[
E = \frac{1}{2} \int d^3r \phi(r) \phi(r). 
\]

(Which you have seen as $E = \frac{1}{2} \int d^3r \frac{q_1 q_2}{r^2}$, $\frac{1}{1^2} \frac{1}{1^2} = \frac{1}{2} \sum_{i=1}^{N} q_i q_j \delta(i,j)$ where $q_i = \sum_{k=1}^{N} \frac{q_k}{r_{ik}}$)

is the potential due to all charges but $q_0 = 0$?

If $\rho = 0$ in $V$ but there are conductors bounding $V$, then $\phi = \phi_a = a_{b=0}$ in each of the $N$ surfaces bounding $V$, and (i) $\phi$ constant on the surface ($\partial V$), and

(ii) $\frac{\partial \phi}{\partial n} = 0$ on the surface (Wigner change is because $\hat{n}$ points into volume)

\[
\Rightarrow E = \int dS \frac{1}{2\epsilon} \phi \frac{\partial \phi}{\partial n} = \frac{1}{2\epsilon} \sum_{a} \phi_a \int dS_{a} (\hat{n} \cdot \hat{e}_a) = \frac{1}{2\epsilon} \sum_{a} \phi_a Q_a \text{ where } Q_a = \int dS_{a}. 
\]
**Capacitance:**

**Problem:** given $\phi_a$'s what are $Q_a$?

OK, given $Q_a$ what are $\phi_a$ (up to an additive constant; assume $\phi(\infty) = 0$).

The basic result is: $\phi_a$ is a linear relation

$$Q_a = \frac{1}{d_{ab}} \cdot C_{ab} \cdot \phi_b$$

$C_{ab} =$ capacitance

or $C_{aa} =$ "capacity" or "capacitance"

$C_{ab}$, $b \neq a =$ coefficient of electrostatic induction.

AND: $C_{ab}$ depend only on geometry ($r$, not on $\phi_a$ nor $Q_a$).

This is proved in a wishy-washy manner in textbook (and not at all in Jackson). Here is my argument: consider the Green's function for the Poisson eq.

$$\nabla^2 G = -4\pi \delta^2 (\vec{r} - \vec{r}')$$

with appropriate boundary conditions (we need Dirichlet, but keep it general for now).

[Note $G = \frac{1}{r} + F(\vec{r}, \vec{r}')$ where $\nabla^2 F = 0$ is chosen to fix boundary conditions.]

Then from Green's 2nd identity:

$$\int_V \left( \psi_x \frac{\partial \phi}{\partial x} - \psi_y \frac{\partial \phi}{\partial y} \right) = \int_{\partial V} \left( \psi_x \frac{\partial \phi}{\partial x} - \psi_y \frac{\partial \phi}{\partial y} \right)$$

with $\psi_x = \phi$ and $\psi_y = G$ we have

$$-4\pi \phi(r) - \int_{\partial V} G(r, r') \nabla \phi = \int_{\partial V} \left( \psi_x \frac{\partial G}{\partial x} - \psi_y \frac{\partial G}{\partial y} \right)$$

In the case of interest $\nabla \phi = 0$ (no charge in V), $G|_{\partial V} = 0$ (Dirichlet), so

$$\phi(r) = -\frac{1}{4\pi} \int_{\partial V} \frac{\partial G}{\partial n} (r, r') \frac{\partial \phi}{\partial n} = \sum_a \phi_a G_a(r)$$

where $G_a(r) = -\frac{1}{4\pi} \int_{\partial V} \frac{\partial G}{\partial n} (r, r')$ depends on geometry but not on $\phi$. 


From this one can compute $\sigma_b = \int_{\partial V_b} \frac{\partial \phi}{\partial n} \, dS$ and $\Phi_b = \int_{V_b} \frac{1}{\varepsilon} \, dV$.

which gives the $C_{ab}$ in terms of $\int_{\partial V_b} \frac{\partial \phi}{\partial n} \, dS = \text{purely geometric. END of part''}.$

While we are proving things not shown in text nor Jackson: $C_{ab} = C_{ba}$

For this we use "Green's reciprocity": Consider two different charge distributions $\rho_a, \rho_b$ and associated potentials $\phi_a(x), \phi_b(x)$ (for same boundary conditions, including $\phi \to 0 \text{ at } \infty$).

\[
\int_V \left[ \rho_a \nabla \phi_b - \rho_b \nabla \phi_a \right] \, dV = \int_{\partial V} \left[ \rho_a \frac{\partial \phi_b}{\partial n} - \rho_b \frac{\partial \phi_a}{\partial n} \right] \, dS.
\]

For our case, $\rho_a = 0 \in V$, and $\frac{\partial \phi}{\partial n} \equiv 0$. Moreover $\phi_a$ on $\partial V$ is constant:

\[
0 = \int_{\partial V} \rho_a \frac{\partial \phi_a}{\partial n} \, dS = \int_{\partial V} \rho_a (C_{ab} - C_{ba})
\]

The textbook obtains this in a different way by considering point charges and ignoring singular terms. The rest is as in text:

\[
0 = \sum_{a,b} \rho_a \rho_b (C_{ab} - C_{ba})
\]

and arbitraries in $\phi_a, \phi_b \Rightarrow C_{ab} = C_{ba}$.

\[\text{Computations: } \Phi_a = \sum_b C_{ab} \phi_b, \text{ so one may set } \phi = 0 \text{ for all except } c = b.\]

and hence compute $\Phi_a = C_{aba} \phi_b$. This requires solving the boundary value problem $\nabla^2 \phi = 0$ (with $\phi |_{\partial V}$ as explained); $\Phi_a$ is computed from $\Phi_a = \int_{V_a} \frac{1}{\varepsilon} \, dV$.

But only simple geometries can be done analytically.

\[\text{Example.}\]

(a) Single sphere: \[\text{radius } \ell, \quad \Phi(r) = \frac{V}{r} \quad \text{at } \ell \text{ set by } \Phi(r) = \frac{V}{r}, \quad \text{Charge on sphere } q = V a = \frac{\ell^2 - r^2}{2}.\]

Then $\Phi = \int \Phi = (4\pi/2)(\frac{1}{\ell} - \frac{1}{r}) - \frac{V}{r}.$
(iii) Convergent spheres: \( \nabla \phi = 0 \Rightarrow \phi = \frac{k_i}{r} + k_2 \)

\[
S_r \quad \phi = \frac{k_i}{a} + k_2 \quad \phi = \frac{b \phi_i - a \phi_2}{b - a}
\]

\[\Rightarrow k_1 = \frac{a \phi - b \phi_2}{b^2 - a^2} \quad k_2 = \frac{b a \phi_i - a b \phi_2}{b - a} \]

\[\text{Now} \quad \sigma_i = -\frac{1}{4\pi} \frac{\partial \phi}{\partial r} \bigg|_{r} = \frac{1}{4\pi} \frac{k_i}{a} \quad \sigma_2 = \frac{1}{4\pi} \frac{\partial \phi_2}{\partial r} \bigg|_{r} = -\frac{1}{4\pi} \frac{b \phi_i}{b - a} \]

so \( \sigma_1 = k_1 = -\sigma_2 \)

(Note, we knew this all along since \( E = 0 \) in the interior of conductor "1" so that Gauss's law gives charge enclosed in Gaussian surface when \( E = 0 \)).

Compute:

\[Q_1 = C_{11} \phi_1 \quad (\text{set } \phi_2 = 0) \]
\[C_{11} = \frac{k_i}{\phi_1} \quad \frac{1}{\phi_2} \quad (\phi_2 = 0) \]

\[Q_1 = \frac{\sigma_1}{\phi_1} \quad \phi_2 \quad (\text{set } \phi_2 = 0) \]

\[C_{12} = \frac{\sigma_1}{\phi_1} = \frac{k_i}{\phi_2} = -\frac{a b}{b - a} \]

\[C_{21} = C_{12} \]

\[Q_1 = C_{22} \phi_2 \quad (\text{set } \phi_2 = 0) \]

\[C_{22} = \frac{\sigma_2}{\phi_2} = \frac{Q_1}{\phi_2} = -\frac{Q_1}{\phi_2} \]

\[C_{11} = C_{22} \Rightarrow C_1 = C_{22} = \frac{a b}{b - a} \]

Note that there are charges \( \pm Q \), so by definition of expectation \( Q = \langle \mathcal{O} \rangle \) applies.

\[Q_1 = |Q| \quad \phi_2 = \phi_1 \quad = \frac{a b}{b - a} \]

Some additional comments:

(a) Electrostatic energy \( E = \frac{1}{2} \sum_{a} \phi_a \phi_a = \frac{1}{2} \sum_{a \neq b} \phi_a \phi_b \)

\[\phi \text{ with } \phi_a = \sum_{b \neq a} \phi_{b} \quad E = \frac{1}{2} \sum_{a \neq b} \phi_{a} \phi_{b} \quad \text{(when } C' = 1 \text{, } C = 0 \text{ as matrices).} \]

(b) We have found charges when potentials are specified (Diraclet problem).

If charges are specified instead, find \( C_{ab} \) as before, then \( \phi_a = \sum_{b} \phi_{b} \).

(c) One can use \( C_{ab} \) to solve problem with p.d.f and b.c.'s on conducting boundary. Just solve \( \nabla \phi = -\kappa \sigma \) with grounded conductors 1st, and then add to this \( \nabla \phi = 0 \) with appropriate b.c.'s.
For two conductors with potential difference $V$ and with charges $\pm Q$
the "capacitance" $C_a$ (confusion of terminology?) is $Q = CV$.

(Exercise 88.2)

Relation to $C_{ab}$: Use $\phi_a = \frac{1}{b} (C^{-1})_{ab} Q_b$ and $\phi_a = 0$, $Q_b = -Q$.

so $V = \phi_a - \phi_b = (C^{-1})_{11} Q_1 + (C^{-1})_{12} Q_2 - (C^{-1})_{21} Q_1 - (C^{-1})_{22} Q_2$

$= Q \left( (C^{-1})_{11} + (C^{-1})_{22} - 2(C^{-1})_{12} \right)$ (used $C_{12} = C_{21}$).

\[ \Rightarrow \quad \frac{1}{Q} = C_{11} + C_{22} - 2C_{12} \]

To write this in terms of $C_{ab}$,

\[ (C^{-1}) = \frac{1}{\det C} \begin{pmatrix} C_{11} - C_{12} \\ C_{12} - C_{22} \end{pmatrix} \]

so $\frac{1}{Q} = \frac{1}{\det C} (C_{11} + C_{22} + 2C_{12})$

or $Q = \frac{C_{11} C_{22} - C_{12}^2}{C_{11} + C_{22} + 2C_{12}}$

The energy stored in the capacitor is $E = \frac{1}{2} \Sigma_{a,b} (C^{-1})_{ab} Q_a Q_b$

$= \frac{1}{2} Q^2 \left( C_{11} + C_{22} - 2C_{12} \right)$

$= \frac{1}{2} \frac{Q^2}{Q}$

\[ \text{Not added: I just realized $C_{aa}$ is not invertible. Why does the text (Gay) do this?} \]

\[ \text{as in biblical Landau and Lifshitz treat it as such is a mystery. Here is the argument:} \]

We can solve the problem $Q_a = \Sigma_b C_{ab} \phi_b$ for $\phi_a$ using $\phi_b = \phi_a$ arbitrary. This

just amounts to shifting $\phi$ of the previous, $\phi_a = 0$, solution by a constant. This leaves

$Q_a$ unaffected, since it is obtained from a derivative, $\frac{\partial \phi_a}{\partial \phi_b}$.

So $Q_a = \Sigma_b C_{ab} (\phi_b + \phi_a)$ is independent of $\phi_a = \Sigma_b C_{ab} \phi_b = 0 \Rightarrow \det C = 0$

(To see that $\det C = 0$, recall that, considering columns of $M$ as vectors, $\det M \neq 0$

$\Rightarrow$ the vectors are linearly independent. So the columns of $C_{ab}$ are vectors $(\vec{v}^{(a)})_b = C_{ab}$ then

$\Sigma_b C_{ab} = 0 \Rightarrow \Sigma_b \vec{v}^{(a)} = 0$.

2020-06-04 11:08:21
For 4 \times 4 case, \( C_{ab} = C_{ba} \) and \( \sum_b C_{ab} = 0 \) (why?)

\( (C)_{ab} = c \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \) for some \( c > 0 \)

Then \( Q = Q_1 + Q_2 = C_{11} \Phi_1 + C_{12} \Phi_2 = c(\Phi_1 - \Phi_2) \Rightarrow c = \frac{Q}{\Phi_1 - \Phi_2} \) \hspace{1cm} \text{the capacitance}

\textbf{Wow!}
Methods for solving boundary value problems.

1. Solve PDE with separation of variables; special functions: DONE
2. Images
   
3. Green functions (combine the above)
4. Numerical
5. Variational

---

**Method of Images**

By example: point charge with infinite plane conductor:

\[
\phi = \frac{q}{\pi a^2}
\]

Note: if conductor is finite but ends at distance \(L \gg a\), we expect this to be a good approximation.

Consider a problem with charge \(q\), a second "image" charge \(q'\), and no conductor. We seek to find magnitude of \(q\) and location so that:

1. There exists an equipotential \(\phi = \text{constant}\) that is a plane a distance \(a\) from \(q\)
2. \(q'\) is on the other side of this plane

\[
\begin{pmatrix}
\phi \\
\phi'
\end{pmatrix}
= - \begin{pmatrix}
\frac{1}{a} & -
\end{pmatrix}
- \begin{pmatrix}
0 & \text{constant}
\end{pmatrix}
\]

Then \(\phi(x)\) for this problem is a solution to our problem: it satisfies \(\nabla^2 \phi = -q\)

and the b.c. \(\phi = \text{const at plane}\).

In this case the solution is obvious: make \(q' = -q\) a distance \(2a\) from \(q\)

(figure next page)
The points on the mid-planes have potential \( \phi = \frac{q}{r} + \frac{(-q)}{r} = 0 \).

More explicitly, place \( q + \hat{r}_0(0,0,a) \) and \(-q + \hat{r}_0(0,0,-a)\). Then

\[
\phi(r) = q \left( \frac{1}{|r - r_0|} - \frac{1}{|r + r_0|} \right) \tag{8}
\]

Thus

\[
\phi(r) = 0
\]

determines a surface:

\[ |r - r_0| = |r + r_0| \rightarrow x^2 + y^2 + (z-a)^2 = x^2 + y^2 + (z+a)^2 \]

\[ \Rightarrow z = 0 \]

So (8) is a \( \phi(r) \) that gives \( \nabla^2 \phi = -4\pi \rho \) (\( \rho = q \delta(r - r_0) \)) with

\[ \phi(r) = 0 \text{ on } z = 0. \]

One can (see text) compute \( \sigma = \frac{-1}{4\pi} \frac{\partial \phi}{\partial n} \) to find charge distribution on conductor. Clearly, \( \int d\sigma = -q \) (from Gauss' law). One can check this.
One may consider some charges, look for an equipotential of some desired shape, and use the charges on one side as “images.”

Example:

\[ \Phi(r) = \frac{q}{|r - r_1|} + \frac{q'}{|r - r_2|} \]

On \( |F| = R \) (a sphere about origin) we have \( \Phi = 0 \) if

\[ q \frac{|F|}{|F|} = - q' \frac{|F|}{|F|} \]

\[ q \sqrt{R^2 + d^2 - 2Rd \cos \theta} = -q' \sqrt{R^2 + d^2 - 2Rd \cos \theta} \]

where \( \theta \) are

\[ \theta = \theta_1 = \theta_2 = \theta \]

Take, say \( \theta = 0 \). Then

\[ \frac{q}{a - R} = \frac{q'}{a - R} \]

\[ \Rightarrow \quad \frac{R - d}{a - R} = \frac{R + d}{a - R} \quad \Rightarrow \quad R^2 - d^2 = R^2 + d^2 \cos \theta + ad \Rightarrow \quad R^2 = ad \Rightarrow \quad \frac{r}{a} = \frac{R - d}{a} = \frac{R + d}{a} \]

Does this work for general \( \theta \)? Showing \( q \sqrt{r} = -q' \sqrt{r} \):

\[ a^2 \left( R^2 + d^2 - 2Rd \cos \theta \right) = R^2 (R^2 + d^2 - 2Rd \cos \theta) \]

\[ a^2 R^2 + a^2 - 2a^2 \cos \theta \cos \theta = R^2 (R^2 + d^2 - 2Rd \cos \theta) \]

So

\[ \Phi(r) = q \left[ \frac{1}{|r - r_1|} - \frac{R/a}{|r - r_2|} \right] \quad \text{has} \quad \Phi = 0 \quad \text{or} \quad |F| = R \quad \text{and satisfies} \]

\[ \nabla^2 \Phi = -q \frac{q'}{(|r - r_1|) \phi(r - r_2)} \]
Third example: Conducting sphere in uniform external field $E_0$.

Consider a dipole $\vec{d} = d \hat{z}$ plus a field (superposition) $E_0 = E_0 \hat{z}$ so that

$$\phi(\vec{r}) = \frac{d^2 - E_0 z}{r^3} \quad \text{(we have put $\vec{d}$ at the origin)}$$

Then the surface $4\pi a$ of a sphere of radius $a$ has $\phi = 0$ if $\vec{d} = E_0$.

So with our image "charge" being a dipole ($\vec{d}$) we have a conducting sphere of radius $a$ in a field $E_0 = E_0 \hat{z}$ has potential

$$\phi(\vec{r}) = E_0 z \left( \frac{a^2}{r^2} - 1 \right) = -E_0 \hat{z} \left( 1 - \frac{a^2}{r^2} \right) \quad (r > a)$$

The charges have redistributed themselves, $\sigma = -\frac{1}{4\pi} \frac{\partial \phi}{\partial n}$, to create a dipole.

The dipole moment above is $\vec{d} = a \vec{E}_0$.

More generally, the surface $\sigma$ on a conductor placed in an external field $E_0$ produces an induced field that can be expanded in a multiple expansion. The leading term is the dipole (the charge on the conductor is assumed to vanish). The corresponding dipole moment $\vec{d}$ is linear in $E_0$, but in general geometries the linearity means $\vec{P}_i = \alpha_{ij} E_j$. $\alpha_{ij}$ = "polarizability" tensor.

In the case above $\alpha_{ij} = \delta_{ij} a^2$. 
Moreover, the potential energy of the uncharged conductor in an external field, in the dipole approximation, is

$$\mathcal{E} = -\frac{1}{2} \mathbf{P} \cdot \mathbf{E}_0$$

To see this, consider the uncharged conductor in the presence of a point charge $q$ at $\vec{r}$ in a plane with $\mathbf{P}$ at the origin.

For large $R$, the field at the conductor is approximately uniform,

$$\mathbf{E}_0 = -\frac{q\mathbf{r}}{R^3}$$

Now,

$$\mathcal{E} = \frac{1}{8\pi} \int \nabla \cdot \nabla \phi \, dV = \frac{1}{8\pi} \int \nabla \cdot \nabla \phi \, dV = -\frac{1}{8\pi} \int \nabla \cdot \nabla \phi \, dV$$

Assuming fields vanish at infinity, and using $\nabla^2 \phi = -4\pi \rho$

$$\mathcal{E} = \frac{1}{2} \sum \phi \mathbf{a}_k + \frac{1}{2} q \phi(r)$$

a generalization of our previous expression for $\mathcal{E}$ that now includes $q$. Now, we are assuming 1 conductor, with $\phi = 0 \Rightarrow \mathcal{E} = \frac{1}{2} q \phi(r) = \frac{1}{2} q \mathbf{r} \cdot \mathbf{E}_0 = \frac{1}{2} \mathbf{r} \cdot \mathbf{E}_0 = -\frac{1}{2} \mathbf{r} \cdot \mathbf{E}_0$$

**NOTE:**

This derivation, taken from Landau, and is Exercise 8.25 of Garzo (nearly)

subject, a divergence from $\phi$ at $q$ at $q$, so $\mathbf{r}$ (coming from $\int \phi dV$).

This is why the result is negative even if $U = \frac{1}{2} \mathbf{E} > 0$.

I believe this makes this general sounding argument somewhat questionable. In Appendix B of this Unit I compute the total energy of a ganged sphere in $\mathcal{E}$, take away the energy of the non-conductor case. The result is

$$\mathcal{E} = \frac{1}{2} \mathbf{a}^3 \mathbf{E}_0$$
**Variational Method**

Good for analytic approximation, but for precision look at numerical methods.

It is used in other areas of physics — worthwhile taking a look.

Consider the functional

\[
W[\psi] = \int_V \left[ \frac{1}{2\epsilon} \left( \nabla \psi(r) \right)^2 - \psi(r) \rho(r) \right] dr
\]

where \( \psi(r) \) is piecewise smooth, satisfying Dirichlet b.c. on \( \partial V \).

Then \( W \) is minimized by the solution to Poisson, \( \nabla^2 \psi = -\rho(r) \) satisfying the b.c.'s.

**Trivial to show**

\[
\delta W = \int_V \left[ \frac{1}{2\epsilon} \nabla \psi \cdot \nabla \delta \psi - \rho \delta \psi \right] dr
\]

\[
= \int_V \frac{1}{2\epsilon} \left( \delta \psi \nabla \psi \right) - \int_V \delta \psi \left[ \frac{1}{2\epsilon} \nabla^2 \psi + \rho \right] dr
\]

\[= 0 \quad \text{sinc} \delta \psi = 0 \text{ on } \partial V
\]

\[= 0 \quad \text{at extremum} \quad \Rightarrow \quad \frac{1}{2\epsilon} \nabla^2 \psi + \rho = 0 \quad \Rightarrow \quad \text{as advertised}
\]

To see that it is a minimum (not a maximum or saddle point)

Expand \( W[\psi + \delta \psi] \) to order \( \delta \psi \)

\[
\delta^2 W = \int_V \frac{1}{2\epsilon} \left( \nabla \delta \psi \right)^2 > 0 \quad \text{always}
\]

Notes

- To use this, find some functions \( \psi_1, \psi_2, \psi_3, \ldots \) possibly with adjustable parameters, that satisfy the b.c.'s. Then minimize \( W[\psi_0, \psi_1, \psi_2, \ldots] \) w.r.t. \( \psi_0, \psi_1, \psi_2, \ldots \) and adjustable parameters.

- If \( \rho = 0 \) everywhere, then \( W[\psi] = \frac{1}{2\epsilon} \int_V \nabla \psi \cdot \nabla \psi \) — electrostatic energy.

- For 2 conductors, if \( \phi_0 = 0 \), \( \phi_1 = 1 \) \( \Rightarrow \quad W[\psi_{1,2}] = \frac{1}{2\epsilon}(\phi_1 \phi_2)^{1/2} = \frac{1}{2} \epsilon \phi_1 \phi_2 \)
Exercise 9.1

Example: Cylindrical capacitor (circular cross-section):

Some trial functions

(a) \(\alpha (r-a)\)

(b) \(\alpha (r-a) + \beta (r-a)^2\)

We need to satisfy \(\Phi (r=b) = V\).

(a) \(\alpha (b-a) = V\) \(\Rightarrow \alpha = \frac{V}{b-a} \Rightarrow \) no freedom for variation

\[
\frac{1}{\mu C} = \frac{1}{\varepsilon} = \frac{\pi}{\mu} \int_0^b a dr \left[ \frac{1}{b-a} \right]^2 = \frac{1}{\varepsilon} \frac{b^2}{b-a} \Rightarrow \frac{C}{\mu} = \frac{1}{\varepsilon} \frac{b^2}{b-a}
\]

(b) \(\alpha (b-a), \beta (b-a)^2 = V \Rightarrow \alpha = \frac{V}{b-a} - \beta (b-a)\)

So, \(\nabla \Psi = (\frac{\partial \Psi}{\partial r})^2 = (\alpha + 2\beta (r-a))^2\)

\[
W\Phi = \frac{1}{\varepsilon} \int_0^b (\frac{\partial \Psi}{\partial r})^2 = \frac{1}{\varepsilon} \int_0^b \oint r \cdot \nabla \cdot \left[ \alpha + 2\beta (r-a) \right]^2
\]

\[
= \frac{1}{\varepsilon} \int_0^b \left[ \alpha \frac{d}{dr} \left[ \frac{1}{b-a} \right] + \frac{d}{dr} \left[ \frac{1}{b-a} \right] \alpha \right] + 4\beta \frac{d}{dr} \left[ \frac{1}{b-a} \right] + 4\beta^2 \left[ \frac{1}{b-a} \right]^2
\]

\[
= \frac{4\beta}{b-a} \int_0^b \left[ \frac{1}{b-a} \right] + \frac{d}{dr} \left[ \frac{1}{b-a} \right] \int_0^b \left[ \frac{1}{b-a} \right] = \frac{4\beta}{b-a} \int_0^b \left[ \frac{1}{b-a} \right] + \frac{d}{dr} \left[ \frac{1}{b-a} \right]
\]

\[
\Rightarrow \beta = - \frac{V}{b-a} \Rightarrow \alpha = \frac{2bV}{b-a}
\]

\[
\Phi (r) = \frac{V}{b-a} \left[ (r-a) \left[ \frac{2b}{b-a} \right] \right]
\]

\[
\frac{1}{\mu C} = \frac{1}{\varepsilon} \int_0^b \left[ \frac{1}{b-a} \right]^2 = \frac{1}{\varepsilon} \frac{a^2 + \frac{b^2}{4} + \frac{b^2}{4}}{b^2 - a^2}
\]

\[
C = \frac{1}{\varepsilon} \frac{a^2 + \frac{b^2}{4} + \frac{b^2}{4}}{b^2 - a^2}
\]
The exact solution is elementary: use a Gaussian surface

\[ E(r) = \frac{Q_{\text{enc}}}{4\pi \varepsilon_0 r^2} = \frac{4\pi}{4\pi} = 4\pi \varepsilon_0 \]

\[ E(r) = \frac{2\pi}{r^2} \Rightarrow \phi = -2\pi \ln(b/a) \]

\[ Q = C \Delta \phi = C (-2\pi \ln(b/a)) \quad (Q = -2\pi) \]

\[ C = \frac{1}{2\ln(b/a)} \]

One way compare the approximate solutions to the exact one

by, say, plotting \( \frac{C_{\text{approx}}}{C_{\text{exact}}} \) as a function of \( x = \frac{b}{a} \)

For \( x = 1 + \varepsilon \), \( C_{\text{exact}} = \frac{1}{2\ln(1+\varepsilon)} \approx \frac{1}{2\varepsilon} \)

while \( C_{\text{approx}}^{(i)} = \frac{1}{y} \frac{x+1}{x-1} = \frac{1}{2} + \frac{1}{2\varepsilon} \)

and \( C_{\text{approx}}^{(ii)} = \frac{1}{x} \frac{e}{(\varepsilon+1)} = \frac{1}{2\varepsilon} \)

but as \( x \gg 1 \), \( C_{\text{exact}} = \frac{1}{2\ln(x)} \)

while \( C_{\text{approx}}^{(i)} = \frac{1}{2\varepsilon} \) and \( C_{\text{approx}}^{(ii)} = \frac{1}{2\varepsilon} \)
Electrostatics with conductors (16/20)

Force from spherical shell with mass $M$

\[ F = \frac{dF}{d\theta} = \frac{1}{2} \frac{d}{dr} \left( \frac{M}{r^2} \right) \]

\[ \int_0^{2\pi} \int_0^r d\theta \sqrt{r^2 - \rho^2} = \frac{M}{2} \]

\[ V = \frac{1}{2} \int_0^{2\pi} d\theta \sqrt{r^2 - \rho^2} \]

\[ dV = \frac{1}{2} \frac{d}{dr} \left( \frac{M}{r^2} \right) \]

\[ d = \frac{1}{2} \pi r^2 \]

\[ F = \frac{2\pi r^2 V}{r^2} \]

\[ 2\pi r^2 \left( \frac{r^2 + a^2}{r^2} \right) \]

\[ \frac{d}{dr} \left( \frac{r^2 + a^2}{r^2} \right) = \frac{2a}{r^2} \]

\[ F = \frac{2\pi r^2 V}{r^2} \]

\[ \int \frac{dV}{r^2} = \frac{1}{2} \frac{d}{dr} \left( \frac{M}{r^2} \right) \]

\[ \frac{d}{dr} \left( \frac{M}{r^2} \right) = \frac{2a}{r^2} \]

\[ \int \frac{dV}{r^2} = \frac{1}{2} \frac{d}{dr} \left( \frac{M}{r^2} \right) \]

\[ F = \frac{2\pi r^2 V}{r^2} \]

\[ 2\pi r^2 \left( \frac{r^2 + a^2}{r^2} \right) \]

\[ \frac{d}{dr} \left( \frac{r^2 + a^2}{r^2} \right) = \frac{2a}{r^2} \]

\[ F = \frac{2\pi r^2 V}{r^2} \]

\[ \int \frac{dV}{r^2} = \frac{1}{2} \frac{d}{dr} \left( \frac{M}{r^2} \right) \]

\[ \frac{d}{dr} \left( \frac{M}{r^2} \right) = \frac{2a}{r^2} \]

\[ \int \frac{dV}{r^2} = \frac{1}{2} \frac{d}{dr} \left( \frac{M}{r^2} \right) \]

\[ F = \frac{2\pi r^2 V}{r^2} \]

\[ 2\pi r^2 \left( \frac{r^2 + a^2}{r^2} \right) \]

\[ \frac{d}{dr} \left( \frac{r^2 + a^2}{r^2} \right) = \frac{2a}{r^2} \]

\[ F = \frac{2\pi r^2 V}{r^2} \]

\[ \int \frac{dV}{r^2} = \frac{1}{2} \frac{d}{dr} \left( \frac{M}{r^2} \right) \]

\[ \frac{d}{dr} \left( \frac{M}{r^2} \right) = \frac{2a}{r^2} \]

\[ \int \frac{dV}{r^2} = \frac{1}{2} \frac{d}{dr} \left( \frac{M}{r^2} \right) \]

\[ F = \frac{2\pi r^2 V}{r^2} \]

\[ 2\pi r^2 \left( \frac{r^2 + a^2}{r^2} \right) \]

\[ \frac{d}{dr} \left( \frac{r^2 + a^2}{r^2} \right) = \frac{2a}{r^2} \]

\[ F = \frac{2\pi r^2 V}{r^2} \]

\[ \int \frac{dV}{r^2} = \frac{1}{2} \frac{d}{dr} \left( \frac{M}{r^2} \right) \]

\[ \frac{d}{dr} \left( \frac{M}{r^2} \right) = \frac{2a}{r^2} \]

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\[ F = \frac{2\pi r^2 V}{r^2} \]

\[ 2\pi r^2 \left( \frac{r^2 + a^2}{r^2} \right) \]

\[ \frac{d}{dr} \left( \frac{r^2 + a^2}{r^2} \right) = \frac{2a}{r^2} \]

\[ F = \frac{2\pi r^2 V}{r^2} \]

\[ \int \frac{dV}{r^2} = \frac{1}{2} \frac{d}{dr} \left( \frac{M}{r^2} \right) \]

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\[ \frac{d}{dr} \left( \frac{r^2 + a^2}{r^2} \right) = \frac{2a}{r^2} \]

\[ F = \frac{2\pi r^2 V}{r^2} \]

\[ \int \frac{dV}{r^2} = \frac{1}{2} \frac{d}{dr} \left( \frac{M}{r^2} \right) \]

\[ \frac{d}{dr} \left( \frac{M}{r^2} \right) = \frac{2a}{r^2} \]

\[ \int \frac{dV}{r^2} = \frac{1}{2} \frac{d}{dr} \left( \frac{M}{r^2} \right) \]

\[ F = \frac{2\pi r^2 V}{r^2} \]

\[ 2\pi r^2 \left( \frac{r^2 + a^2}{r^2} \right) \]

\[ \frac{d}{dr} \left( \frac{r^2 + a^2}{r^2} \right) = \frac{2a}{r^2} \]

\[ F = \frac{2\pi r^2 V}{r^2} \]

\[ \int \frac{dV}{r^2} = \frac{1}{2} \frac{d}{dr} \left( \frac{M}{r^2} \right) \]

\[ \frac{d}{dr} \left( \frac{M}{r^2} \right) = \frac{2a}{r^2} \]

\[ \int \frac{dV}{r^2} = \frac{1}{2} \frac{d}{dr} \left( \frac{M}{r^2} \right) \]

\[ F = \frac{2\pi r^2 V}{r^2} \]
\[ F = \frac{\pi \rho e^2}{2\varepsilon} \left[ \frac{(r-a)^2}{r^3} \right] + \frac{1}{2\varepsilon} \left[ \frac{(a+\eta)^2}{(r-a)^2} - \frac{(a-\eta)^2}{(r-a)^2} \right] \]

For \( r = a \)

\[ F_{1r} = \frac{\pi e^2}{2\varepsilon} \left[ \frac{1}{r^2} \right] \left[ \frac{(r-a)^2}{r^3} \right] = \frac{\pi e^2}{2\varepsilon} \left[ \frac{2a - 2a}{r^3} \right] = 0 \]

\[ \Phi = \frac{q_0}{r} \quad F = \frac{q_0}{r^2} \quad \rho = 2 + \eta \quad \sigma \to \sigma (l + \eta) \quad (\text{anchor} \ 1 \ \text{anchor}) \]

\[ F = \frac{\pi e^2}{2\varepsilon} \left[ \frac{(r-a)^2}{r^3} \right] \left[ \frac{1}{l + \eta} \right] \left[ (r-a)^{l+\eta} - (r-a)^{l+\eta} \right] = \frac{1}{l + \eta} \left[ (l + \eta)^{l+\eta} - (l + \eta)^{l+\eta} \right] \]

\[ \Phi = \frac{q_0}{r} \quad F = \frac{\pi e^2}{2\varepsilon} \left[ \frac{1}{l + \eta} \right] \left[ (l + \eta)^{l+\eta} - (l + \eta)^{l+\eta} \right] = \frac{1}{l + \eta} \left[ 2a + \eta \left( 2a - r \frac{a + \eta}{l + \eta} - 2 a \frac{a + \eta}{l + \eta} \right) \right] \]

\[ \Phi = \frac{2 \pi e^2}{l + \eta} \left[ (l + \eta)^{l+\eta} - (l + \eta)^{l+\eta} \right] = \frac{2 \pi e^2}{l + \eta} \left[ 2a + \eta \left( 2a - r \frac{a + \eta}{l + \eta} - 2 a \frac{a + \eta}{l + \eta} \right) \right] \]

\[ \Phi = \frac{2 \pi e^2}{l + \eta} \left[ 2a + \eta \left( 2a - r \frac{a + \eta}{l + \eta} - 2 a \frac{a + \eta}{l + \eta} \right) \right] \]

\[ \Phi = \frac{2 \pi e^2}{l + \eta} \left[ 2a + \eta \left( 2a - r \frac{a + \eta}{l + \eta} - 2 a \frac{a + \eta}{l + \eta} \right) \right] \]

\[ \Phi = \frac{2 \pi e^2}{l + \eta} \left[ 2a + \eta \left( 2a - r \frac{a + \eta}{l + \eta} - 2 a \frac{a + \eta}{l + \eta} \right) \right] \]

Set \[ \Phi_{\text{inner}} = \phi_{\text{outer}} \]

\[ \Phi(r = a) = \frac{2 \pi e^2}{l + \eta} \left[ 2a + \eta \left( 2a - r \frac{a + \eta}{l + \eta} - 2 a \frac{a + \eta}{l + \eta} \right) \right] \]
**Appendix B: Energy of conducting sphere is \( E_0 \) relative to no sphere.**

The energy density with conductor relative to point without it is

\[
U = \frac{1}{8\pi} \left( \vec{E}_0^2 - \vec{E}_{\text{ind}}^2 \right) = \frac{1}{6\pi} \vec{E}_0 \cdot \left( 2\vec{E}_0 + \vec{E}_{\text{ind}} \right)
\]

Integrating over all space

\[
\int d^3 r \vec{E}_0 \cdot \vec{E}_{\text{ind}} = \int d^3 r \vec{E}_{\text{ind}} = 0
\]

Since \( \vec{E}_0 = \frac{3\rho \phi r^2 - r^2 \vec{\rho}}{r^5} \)

\[\int \vec{E}_0 = 0 \quad (\text{see spherical symmetry argument below})\]

We are left with

\[
\frac{1}{8\pi} \int d^3 r \vec{E}_{\text{ind}}^2
\]

Now

\[
\vec{E}_{\text{ind}}^2 = \frac{1}{r^4} \left[ \left( \vec{\rho} \phi r \right)^2 \left( 9 - 3\phi^2 \right) + r^2 \vec{\phi}^2 \right] = \frac{1}{r^4} \left( 3 \left( \vec{\rho} \phi \right)^2 + r^2 \vec{\phi}^2 \right)
\]

Now the volume \( V \) is the space exterior to \( \vec{r} = a \), so it is spherically symmetric around \( \vec{r} = 0 \); so we get

\[
\int_V r^2 f(r) \, d\vec{r}_i \cdot \vec{J}_i = \int_V r^2 f(r) \, \frac{1}{2} \vec{J}_i \cdot \vec{J}_i
\]

Using this,

\[
\frac{1}{8\pi} \int_V \vec{E}_{\text{ind}}^2 = \frac{1}{8\pi} \int_V \frac{1}{r^4} \left( 3 \left( \vec{\rho} \phi \right)^2 + r^2 \vec{\phi}^2 \right) = \frac{\rho^2}{4\pi} \int_0^\infty r^2 f(r) \, dr = \frac{\rho^2}{4\pi}
\]

In subtracting \( E_0^2 \) from \( (\vec{E}_0 + \vec{E}_{\text{ind}})^2 \) we must also subtract \( \int \) inside the ball (which vanishes for conductor):

\[
\int_{\vec{r} < a} d^3 r \frac{1}{2} \vec{\rho}^2 = \frac{4\pi}{2} \left( \frac{1}{3} \vec{\rho} \cdot \vec{E}_0 \right) = \frac{1}{6} \vec{\rho} \cdot \vec{E}_0^2
\]

\[
= \Delta \vec{E} = \left( \frac{1}{3} \hat{\vec{r}} - \vec{\rho} \right) \vec{E}_0 = \frac{1}{6} \left( \hat{\vec{r}} \cdot \vec{E}_0 \right) - \vec{\rho} \cdot \vec{E}_0
\]

This disagrees with textbook calculation. Likely a typo.
Furthermore...

The dipole model $\mathbf{D}$, as a charge $q$ at $z = s$ with $q = 0$ at $s = 0$ and $s = \frac{a}{2}$:

$\Phi(r) = \frac{q}{r} \left[ \frac{1}{r^2 - s^2} - \frac{1}{r^2 - \frac{a^2}{4}} \right]$

On surface:

$\sigma(\mathbf{r}) = \frac{1}{\mu_0} \frac{\partial \Phi}{\partial r} \bigg|_{r = a} = \frac{q}{\mu_0} \left[ \frac{a - s \cos \theta}{(a^2 + s^2 - 2as \cos \theta)^{3/2}} - \frac{\frac{a}{2} \sin \theta}{(a^2 + \frac{a^2}{4} - 2as \cos \theta)^{3/2}} \right]$

$= \frac{q}{\mu_0} \left[ \frac{a - s \cos \theta}{(a^2 + s^2 - 2as \cos \theta)^{3/2}} - \frac{s^2}{a^2 - (s^2 - 2as \cos \theta)^{3/2}} \right]$

Potential energy:

$\Phi(\mathbf{r}) \mathbf{q} \bigg|_{r = a} = \int \Phi(\mathbf{r}) \mathbf{q} \cdot d\mathbf{r} = \frac{q^2}{\mu_0} \left[ \frac{a^2 - s^2}{a^2 + s^2 - 2as \cos \theta} \right]$

$\text{Interference term}$

$= \frac{1}{2} \frac{q^2 a (s^2 - a^2)}{\mu_0 a} \left[ \frac{1}{(a^2 - s^2)^{1/2}} - \frac{1}{(s^2 - a^2)^{1/2}} \right]$

$= \frac{1}{2} \frac{q^2 a (s^2 - a^2)}{\mu_0 a} \frac{4as}{(s^2 - a^2)^{5/2}} = \frac{q^2 a}{s - a}$

Note that this is the same as for the homogeneous charge:

$\frac{q q'}{s - a/s} = \frac{q^2 a}{s - a/s}$

with $q = E_0 s^2 / \varepsilon$ and $E = -E_0 s^2 q / (\varepsilon - q)$.

This is not $-\frac{1}{2} \mathbf{p} \cdot \mathbf{E}$, so exercise 8.9.5, or apply in previous exercise.

It diverges. Set $\varepsilon = 0$. It is nongyric.

The self-interaction term $\int d^2 \mathbf{r} \alpha(\mathbf{r}) \mathbf{q}(\mathbf{r}) = \frac{q^2 (s^2 - a^2)}{\mu_0 a} \int d^2 \mathbf{r} \alpha(\mathbf{r}) \left( \frac{1}{(a^2 + s^2 - 2as \cos \theta)^{3/2}} \right)^{3/2}$

seems convergent and positive, and should be added. No time to compute right now.
Another approach: This also be $\phi = 0$ on $|r| = a$ and $E = \text{const}$ at $r > a$. $\phi = \phi_0$.

$\begin{align*}
E_0 &= \frac{q}{s^3} \\
E &= \frac{2q}{s} \left(1 - \frac{r^2}{s^2}\right) + \frac{q}{s} \left(1 + \frac{r^2}{s^2}\right) \\
&= \frac{q}{s} \left[1 - \frac{2}{s^2} \left(1 + \frac{r^2}{s^2}\right) \right]
\end{align*}$

$\begin{align*}
\phi &= -\frac{1}{4\pi} \frac{E_0}{S} \sum a^2 \left[1 + a^2 \frac{1}{s^2} \right] + \frac{1}{2\pi} \frac{1}{s} \\
&= -\frac{1}{4\pi} E_0 \frac{S}{a^3} \left[1 + \frac{a^2}{s^2} \right] + \frac{1}{2\pi} \frac{1}{s}
\end{align*}$

Check sign: is $\vec{\rho} = \vec{a} \vec{E}_0$ or $-\vec{a} \vec{E}_0$?

$\begin{align*}
\vec{E}_{\text{ind}} &= -\nabla \phi = -\nabla \left[ -\frac{1}{4\pi} \frac{E_0}{S} \sum a^2 \left(1 + \frac{a^2}{s^2}\right) \right] = -E_0 \frac{1}{s^2} \sum a^2 \left[ \frac{-2\pi x}{r^3}, \frac{-2\pi y}{r^3}, \frac{-2\pi z}{r^3} \right] \\
&= \frac{2\pi \vec{E}_0 \cdot \vec{r}}{r^3} \text{ for } \vec{r} \neq \vec{0} \\
\end{align*}$
**Electrostatics with Dielectrics (Garg: Chap 15)**

**Basics** (from previous chapters):

\[
\nabla \cdot \mathbf{D} = \rho_f \quad (\mathbf{E} = \text{free}) \quad \nabla \times \mathbf{E} = 0
\]

where \( \mathbf{D} = \mathbf{E} + \varepsilon_0 \mathbf{P} \)

We'll use \( \mathbf{D} = \varepsilon \mathbf{E} \) (not a law, not general, good enough for now)

Then, equivalently

\[
\varepsilon \mathbf{E} = \mathbf{E} + \varepsilon_0 \mathbf{P} \quad \text{or} \quad \mathbf{P} = \frac{\varepsilon - 1}{\varepsilon} \mathbf{E} = \chi_0 \mathbf{E} \quad \chi_0 = \frac{\varepsilon - 1}{\varepsilon} \quad \text{"electric susceptibility"}
\]

**Boundary Value Problem with Dielectrics:**

Since \( \nabla \times \mathbf{E} = 0 \) \( \Rightarrow \mathbf{E} = -\nabla \phi \) still

Then \( \nabla \cdot \mathbf{D} = \varepsilon \nabla \cdot \mathbf{E} \) (assumed \( \mathbf{E} = \) uniform in medium) \( \Rightarrow \nabla^2 \phi = \frac{-\nabla \cdot \mathbf{P}}{\varepsilon} \)

At interfaces:

\[
\phi \text{ is continuous (so } \mathbf{E} = -\nabla \phi \text{ is } \text{known}) \quad \phi_1 = \phi_2, \quad \mathbf{n} \cdot (\mathbf{A} - \mathbf{D}) = \mathbf{n} \cdot \mathbf{A}_1 = \varepsilon_1 \sigma_0 \quad (\mathbf{n} = \text{normal})
\]

\[
\Rightarrow \quad \varepsilon_1 \frac{\partial \phi_1}{\partial n} = \varepsilon_2 \frac{\partial \phi_2}{\partial n}
\]

Note \( \mathbf{E}_1 = \mathbf{E}_2 \) (largest), from \( \nabla \times \mathbf{E} = 0 \).

Follow \( \mathbf{E} \) continuously \( \phi_1 = \phi_2 \)

(Note \( \nabla \cdot \mathbf{P} = 0 \) does not imply \( \frac{\partial \phi_1}{\partial n} = \frac{\partial \phi_2}{\partial n} \)
Example:
Find $E$ (or $\varphi$) everywhere.

At the expense of just doing busy work, we do $\frac{1}{2}$ example. This one has a solution using method of images. In contrast to analogous conductor case — where we knew $E=0$ in the conductor, so only need to determine $E$ outside — where the free charge is located — here we want $E$ (or, equivalently $\varphi$) both in regions 1 and 2 of the figure. So now we need

(a) image charge in 2 plus $q$ in 1 to give $\varphi$ in 1. Call image $q_2$.
(b) image charge in 1, say $q_1$, to give $\varphi$ in 2

By symmetry place all in same axis, 1 to plane interface:

$$\varphi_1(r) = \frac{1}{\varepsilon_1} \left( \frac{q}{|r-a|} + \frac{q_1}{|r-s_i|} \right)$$

$$\varphi_2(r) = \frac{1}{\varepsilon_2} \frac{q_1}{|r-s_i|}$$

The factors of $\frac{1}{\varepsilon}$ are so that $\int \nabla \cdot B = 4\pi \rho$, $\frac{B}{\varepsilon} = \vec{E}$, $E = \psi$,$\nabla \psi = -4\pi \rho E$).

Continuity at interface $\Rightarrow \varphi_1(r) = \varphi_2(r)$ on $P = (x,y,0)$.

No surface charge: $\frac{\partial \varphi_1}{\partial z} = \frac{\partial \varphi_2}{\partial z}$ on $P = (x,y,0)$.
\[
\frac{q}{\sqrt{x^2+y^2+a^2}} + \frac{q_1}{\sqrt{x^2+y^2+\varepsilon_1 b^4}} = \frac{\varepsilon_0 q}{\sqrt{x^2+y^2+\varepsilon_1 b^4}}
\]

and

\[
\frac{a q}{(x^2+y^2+a^2)^{3/2}} - \frac{S_2 q_2}{(x^2+y^2+S_2 b^2)^{3/2}} = \frac{S_1 q_1}{(x^2+y^2+S_1 b^2)^{3/2}}
\]

To satisfy for arbitrary \( \rho^2 = x^2 + y^2 \Rightarrow S_2 = S_1 = a^2 \Rightarrow q_1 = q_2 = a \) since we choose \( S_2 > 0, S_1 > 0 \).

Then \( q + q_2 = \frac{\varepsilon_1 q}{\varepsilon_1 b^2} \) and \( q - q_2 = q_1 \)

\[
q_1 = \frac{a^2}{4 \varepsilon_0} q, \quad q = \frac{a^2}{\varepsilon_1 b^2} q
\]

Now one can compare \( E, D, \mathbf{p} \). . .

Dielectric sphere in uniform external electric field

(Not unlike a conductor)

Spheres suggest use spherical coordinates: center at center of sphere, \( \hat{r} = \hat{E}_0 \).

Axisymmetrical symmetry \( \Rightarrow \gamma_{2m} \) with \( m = 0 \) only \( \Rightarrow P_{0}^1 \).

Inside sphere: \( \Phi_{m} (r) = \sum_{l=0}^{\infty} A_{l} r^{l} P_{l} (\cos \theta) \) (no \( \frac{1}{r^{l+1}} \) terms because \( r = 0 \) included in our region)

Outside sphere: \( \Phi_{m} (r) = -E_{0} \cos \theta - \sum_{l=0}^{\infty} B_{l} r^{l+1} P_{l} (\cos \theta) \) external applied \( E_0 \) on hat

Conditions:

\[
\Phi_{m} = \Phi_{m} (r = 0) = \frac{A_{l} a^2}{3} \text{ except } l = 1: \quad A_{1} a = -E_{0} a + \frac{B_{l}}{a^2}
\]

\[
\delta \Phi_{m} - \partial \Phi_{m} \over \partial r \bigg\vert_{r = 0} \Rightarrow \varepsilon_{0} A_{l} a^{l+1} = - (l+1) B_{l} a^{l+1} \text{ except } l = 1: \quad \varepsilon_{0} A_{1} = -E_{0} - 2 B_{l} a^{2}
\]

\[
\text{except } l = 2: \quad A_{2} a^{2} = 0 \quad \Rightarrow \quad A_{2} = -E_{0} a^{2}
\]

(Notice 
\[
A_{1} + 2a^{3} B_{l} = -E_{0} \quad \Rightarrow \quad B_{l} = \frac{E_{0} - E_{0}}{a^{3}}
\]

radius: a
\[ \Phi_m(r) = -\frac{\beta}{r^{e+2}} E_0 \cos \theta \]

Hence

\[ \Phi_{\text{out}} = -E_0 \cos \theta + \frac{\varepsilon - 1}{\varepsilon + 2} E_0 \left( \frac{\beta}{r} \right)^2 \cos \theta \]

Note \( \beta = \frac{\varepsilon - 1}{\varepsilon + 2} \) \( E_0 \) \( \frac{1}{r^2} \left( \frac{\varepsilon - 1}{\varepsilon + 2} E_0 \right) = \frac{\beta}{r} \)

\[ \text{Depolarization: If } \vec{E}_m = \vec{E}_0 + \vec{E}_d \text{ “depolarization”} \]

\[ = \text{ we get } \vec{E}_d = \left( \frac{3 \varepsilon - 1}{\varepsilon + 2} \right) \vec{E}_0 = \frac{\varepsilon - 1}{\varepsilon + 2} \vec{E}_0 = \frac{4 \beta}{3} \]

\[ \text{“depolarization coefficient”} \]

\( \vec{E}_d \) is the field produced by surface charges from aligning dipoles.

In our example we can check this: use \( \vec{\sigma}_m = \vec{p} \vec{n} \) on the sphere to compute \( \vec{E}_d \) directly:

\( \vec{E}_d = \int \vec{\sigma}_m \hat{r} \) and then the \( \vec{E}_d = -\nabla \Phi \).

The result gives \( \vec{E}_d \) as above.

(The calculation uses \( \frac{1}{r^2} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell + 1} Y_{\ell m}(\hat{r}) Y_{\ell m}(\hat{r}) \hat{r} + r^2 = a > r \)
and the integral with \( \vec{\sigma}(\vec{r}) = \vec{p} \vec{n} = \rho \vec{n} \) picks up only \( \ell = 1, m = 0 \) term, so

\( \sigma(\vec{r}) = \rho_0 \vec{n} \cos \theta \) \)
Thermodynamics with dielectrics

For 1st law: work.

\[ \delta W = \sum_a \varphi_a \delta Q_a \]
\[ = \sum_a \frac{1}{\epsilon_0} \int_{\sigma_a} \nabla \phi \cdot \vec{n} \, dS \]
\[ \text{with } \nabla \phi \text{ pointing away from conductor} \]
\[ = -\frac{1}{\epsilon_0} \int_{V} \nabla (\phi \, \delta \vec{D}) \, dV \]
(by Gauss's theorem)

Use \( \nabla \cdot \vec{D} = 0 \) (\( \rho_{\text{free}} = 0 \)) and \( \vec{E} = -\nabla \phi \)

\[ \delta W = \frac{1}{\epsilon_0} \int_{V} \vec{E} \cdot \delta \vec{D} \, dV \]

Thermodynamic potentials:

let \( S \) = entropy, \( T \) = temperature, \( U \) = internal energy

\[ \delta U = \frac{\delta T}{\text{heat!}} + \frac{1}{\epsilon_0} \int_{V} \vec{E} \cdot \delta \vec{D} \, dV \]
\[ U = U(S, \vec{D}) \]

Now, \( F = F(T, \vec{D}) \) "free energy" with \( F = U - TS \)

\[ \delta F = -S \delta T + \frac{1}{\epsilon_0} \int_{V} \vec{E} \cdot \delta \vec{D} \, dV \]

\[ \overline{U} = \overline{U}(S, \vec{E}) \text{ with } \overline{U} = U - \frac{1}{\epsilon_0} \int_{V} \vec{E} \cdot \delta \vec{D} \]

and \( \overline{F} = \overline{F}(T, \vec{E}) \text{ with } \overline{F} = \overline{U} - TS \)

Now, \( \int \vec{E} \cdot \delta \vec{D} \) is \( U - \overline{U} \) has interesting interpretation.
\[
\frac{1}{\mu} \int \vec{E} \cdot \vec{D} \, dV = -\frac{1}{\epsilon_0} \int \vec{V} \cdot \vec{D} \, dV = -\frac{1}{\mu} \int \vec{V} \, dV \quad \text{(since } \rho_m = 0) \\
= -\frac{1}{\mu} \int \phi \hat{n} \cdot \vec{D} \, dS \quad \text{this points out at } V \text{ as in figure} \\
= \sum_a \phi_a Q_a
\]

\[2x\text{ the energy in the conductors}\]

\[(\delta F)_T = (\delta U)_S = \sum_a \phi_a \delta Q_a\]

ie. "at fixed \( F \)"

And \( \vec{F} = F - \sum_a \phi_a \vec{Q}_a \Rightarrow (\delta F)_T = (\delta U)_S = -\sum_a \delta \phi \, Q_a \)

\[
\begin{pmatrix}
T & \phi_a \\
S & \phi_a \\
S & \phi_a
\end{pmatrix}
\]

fixed, system relaxes to minimum of \( \begin{pmatrix} F \\ \vec{F} \\ U \end{pmatrix} \) in equilibrium

Free energy in linear media: \( F = \sum_i \phi_i E_i \)

\[\vec{E} \cdot \delta \vec{D} = \frac{1}{\epsilon_0} \vec{E} \cdot \delta \vec{D} = \delta \vec{E} \cdot \vec{D} \quad \text{so } \delta U = \int \vec{E} \cdot \delta \vec{D} \quad \text{can be integrated}\]

If \( F_0 = \) free energy at zero field \( \Rightarrow \vec{F} - F_0 = \int \frac{1}{\epsilon_0} \vec{E} \cdot \vec{D} \, dV = \frac{1}{\epsilon_0} \sum_a \phi_a Q_a \)

Similarly \( \vec{F} - F_0 = -\frac{1}{\epsilon_0} \sum_a \phi_a Q_a \)

The difference in sign is the familiar effect that the plates of a

capacitor held at fixed charge attract \( (F = F(T, Q)) \) decreases towards equilibrium

for \( Q \) decreasing \( \Rightarrow \) move conductors closer) but capacitors held at fixed potential repel \( (\vec{F} = \vec{F}(T, Q_a)) \) decreases for \( Q \) increasing, but larger \( Q \) requires

moving plates apart to keep \( Q \) fixed).
What about the dielectric? We used $\vec{D}$ throughout and assumed linearity ($\vec{D} = \varepsilon_0 \vec{E}$), but it is hidden in $\sum \phi_a \hat{Q}_a$ which depends on free charge $\hat{Q}_a$ — but $\phi_a$'s actually will depend on $\varepsilon$.

Example: see text for details, here only rough:

\[
\begin{align*}
\text{Cap plates (dimension left in page)} \\
\text{\text{oil (e)} density p}
\end{align*}
\]

\[\int E_x \, dx = \left[ (h-h_0) L_0 \right] E_x \left( e_0 \right) + \left[ h L_0 \right] E_x \left( e_0 \right) = \left( \frac{V}{a} \right) \tilde{E}_a \left[ h \left( e_0 - 1 \right) + \text{constant} \right] \quad (h-\text{independent})
\]

Minimize (w.r.t. $h$):

\[
\frac{\tilde{V}}{L_0} = \frac{1}{2} pg h^2 - \frac{1}{80} \left( \frac{V}{a} \right)^2 \left( e_0 - 1 \right) h
\]

\[
h_a^* = \frac{1}{2pg} \left( \frac{V}{a} \right)^2 \left( e_0 - 1 \right)
\]

\[\text{or } \frac{1}{2} \text{ from } \int_0^h dx \tilde{E}_a L_0 dx .\]
If \( \varepsilon \) depends on \( T \) and on its volume \( \varepsilon = \varepsilon(T, V) \) (depends on \( V \) because it may change if compressed, it if under pressure), then we can write (assuming \( \vec{D} = \varepsilon \vec{E} \))

\[
\tilde{F}^\varepsilon(V, T, \varepsilon) = \tilde{F}(V, T, 0) - \frac{1}{q_0} V_0 \varepsilon^2 \varepsilon(V, T)
\]

Assumed \( \varepsilon \) uniform and only on volume \( V_0 \).

Since \( \delta \tilde{F} = -S \delta T - \frac{1}{q_0} \int \vec{D} : \vec{E} \Rightarrow S = -\frac{\partial \tilde{F}}{\partial T} \)

\( \vec{U} = \tilde{F} + ST \)

\( \Rightarrow S(V, T, \varepsilon) = S(V, T, 0) + \frac{1}{q_0} V_0 \varepsilon^2 \frac{\partial \varepsilon}{\partial T} \)

and \( \tilde{U}(V, T, \varepsilon) = \tilde{U}(V, T, 0) - \frac{1}{q_0} V_0 \varepsilon^2 \left( \varepsilon - T \frac{\partial \varepsilon}{\partial T} \right) \)
Exercise 97.2 (electrostriction)

$G$ depends on $\rho$ (for some materials)

Apply uniform $E$ to volume $V$ of dielectric; in such cases what is the change in volume $\Delta V$ of material?

$$ G = F + \rho V \quad \text{Gibbs free energy} $$

$\tilde{C}$ i.e. work $E$ as variable.

As above $d\tilde{C} = V d\rho + \cdots \quad V = \frac{\partial \tilde{C}}{\partial \rho}$

$$ V(p, T, E) = V(p, T, 0) - \frac{1}{\gamma_0} V_0 e^{\gamma} \left[ \frac{\partial E}{\partial \rho} \right]_T $$

$$ \Rightarrow \frac{V(p, T, E) - V(p, T, 0)}{V_0} = \frac{\Delta V}{V} = -\frac{1}{\gamma_0} e^{\gamma} \left[ \frac{\partial E}{\partial \rho} \right]_T $$
Models of $\varepsilon$

(Rarefied) gases. First look at collections of molecules such
are so far apart (small number density $n$) that the
force from some molecules on any one molecule is
negligible compared to the applied field (for numbers
quantifying this see text).

1. Non-polar molecules/atoms (like He, H2)

If $\vec{P} = n\vec{E}$ for one molecule $\Rightarrow \vec{P} = n\vec{E}$

$\Rightarrow \chi_e = n\alpha \quad \varepsilon = 1 + 4\pi\alpha n$

At STP, $n = 2.7 \times 10^9$ cm$^{-3}$. And $\alpha \approx 1$ A$^3$

(For H one can use AM b to calculate, $\alpha = \frac{9}{2} a_0^3$. $a_0 = 1.198$ pm (radius))

and $d_H = 0.7 \times 10^{-5}$ cm$^3$ = 0.07 A$^3$).

Then $n\alpha \approx 10^{-5}$ and $\varepsilon = 1 \approx 10^{-4}$

2. Polar molecules (like H$_2$O)

These can be polarized too, so have an $d\vec{E}$ contribution
as above. But also have a permanent $d_0$ dipole moment

Thermal fluctuations about alignment

with $\vec{E}$. Need thermal average $\langle \vec{d} \rangle$.

Assume $\vec{E}$ uniform (on scale of interest). $U(\theta) = -\vec{d}_0 \cdot \vec{E} = -d_0 \varepsilon \cos \theta$

$\langle \vec{d} \rangle = \int d\Omega (d_0 \varepsilon \cos \theta \vec{E}) e^{i d_0 \varepsilon \cos \theta /kT} d(3\theta) d\phi = d_0 \varepsilon \cos \theta$ expand in powers

$\int d\Omega e^{i d_0 \varepsilon \cos \theta /kT}$

so linear order

since $\int d\Omega \cos \theta = 0$
\[ \langle \sigma \rangle = \hat{E} \frac{\int dV \, d_0 \, \cos \theta \left( \frac{d_0 E_0 \cos \theta}{e_0} \right)}{3k_B T} = d_0^2 \frac{E^2}{3k_B T} \]

The rest is as above: \[ \chi = n \left( \alpha + \frac{d_0^2}{3k_B T} \right) \]

\[ \varepsilon = 1 + 4\pi n \left( \alpha + \frac{d_0^2}{3k_B T} \right) \]

Plot vs \( \frac{1}{T} \) to get both \( \alpha \) and \( d_0 \).

Typical numbers:

Steam at 400K:

\[ n = 10^{19} \text{ cm}^{-3}, \quad d_0^2 = 2.1 \times 10^{-23} \text{ cm}^3 \]

\[ \varepsilon - 1 = 4\pi n \left( 7.1 \times 10^7 \right) \sim 2 \times 10^{-3} \]

\[ \varepsilon \approx 2 \times 10^{-18} \text{ e.s.u. cm} \]

Back to rarefied approximation.

Dipole field \( E_{\text{dip}} \sim \frac{d}{r^3} \). With density \( n \), typical distance

\[ \sim n^{-1/3} \Rightarrow E_{\text{dip}} \sim \frac{d}{(n^{1/3})} = n d \sim n d E \]

\[ \text{So} \quad \frac{E_{\text{dip}}}{E} = n \alpha \quad \text{for} \quad n \left( \alpha + \frac{d_0^2}{3k_B T} \right) \quad \text{polarizable case} \]

\[ \text{So, approximation is good provided} \quad \frac{\varepsilon - 1}{4\pi n} \ll 1. \]
Due to dielectrics.

If \( \epsilon \neq 1 \) the field of nearby molecules on a molecule cannot be neglected. (compare to the ambient \( \mathbf{E} \)). We need to go back to the microscopic description in terms of \( \mathbf{E} \).

Now, we break the effect of \( \mathbf{E} \) into two pieces

\[
\mathbf{E} = \mathbf{E}_{\text{near}} + \mathbf{E}_{\text{far}}
\]

where \( \mathbf{E}_{\text{near}} \) is from a ball of radius \( a \) around the molecule of interest, and \( \mathbf{E}_{\text{far}} \) from outside the ball. The ball is large enough that the effect of molecules outside the ball can be averaged as the field due to a polarizable material drawn which has been removed, plus the applied field:

\[
\mathbf{E}_{\text{near}} = \mathbf{E}_{\text{far}}
\]

Here is the picture:

\[
\begin{array}{c}
\text{ball} \\
= \\
+ \\
\text{ball}
\end{array}
\]

Yet \( a \) is small enough that the field \( \mathbf{E}_{\text{near}} \) is uniform in it.

If \( \mathbf{E} \) is the macroscopic field at the center of the ball we can write:

\[
\mathbf{E} = \mathbf{E}_{\text{near}} + \mathbf{E}_{\text{far}}
\]

where \( \mathbf{E}_{\text{near}} \) is the field we would have if the ball is in an applied uniform external field \( \mathbf{E}_{\text{far}} \).
But from the example in pp 3-4 of this note, the field in the ball is
\[ \mathbf{E} = \mathbf{E}_0 + \mathbf{E}_d \] where \( \mathbf{E}_0 \) is the applied uniform field and \( \mathbf{E}_d \) is the depolarization field
\[ \mathbf{E}_d = -\frac{\rho}{2\varepsilon_0}. \] In our case \( \mathbf{E}_0 = \mathbf{E}_{\text{far}} \) and \( \mathbf{E} = \mathbf{E}_{\text{near}} \) so
\[ \mathbf{E} = \mathbf{E}_{\text{far}} + \mathbf{E}_{\text{near}} = \mathbf{E}_0 - \frac{\rho}{2\varepsilon_0}. \]
\[ \Rightarrow \mathbf{E}_{\text{far}} = \mathbf{E}_0 - \frac{\rho}{2\varepsilon_0}. \]

We still have to account for \( \mathbf{E}_{\text{near}} \). This depends on the specific arrangement of molecules inside the ball.

Suppose the molecules are electric dipoles all of same magnitude and all aligned, and placed in a cubic lattice.

\[ \mathbf{E}_{\text{near}} = \sum \frac{3 (\mathbf{r}_{i} \cdot \mathbf{d}) \mathbf{r}_{i} - \mathbf{r}_{i} \mathbf{d}}{r_{i}^7} \]

where \( \mathbf{r}_{i} \) are the locations of the vertices on the cubic lattice centered at the molecule of interest. This vanishes. To see this in a pedestrian way, take \( \mathbf{r} = b (n, m, k) \)

and consider \( \sum 3 \mathbf{r}_{i} \mathbf{g}_{i} - \mathbf{r}_{i} \mathbf{g}_{i}^2 \):

\[ \sum_{n, m, k} \frac{3 n^2 - (n^2 + m^2 + k^2)}{(n^2 + m^2 + k^2)^{3/2}} = \frac{3}{b^3} \sum_{n, m, k} \frac{2 n^2 - m^2 - k^2}{(n^2 + m^2 + k^2)^{3/2}} \]

\[ n^2 + m^2 + k^2 = 1 = \frac{b}{2} \sum_{n, m, k} [(2n^2 + (2m^2 + (2k^2)] = 0 \quad (8 \text{ if } n = \pm 1 \text{ or } m = \pm 1 \text{ or } k = \pm 1) \]

\[ n^2 + m^2 + k^2 = 2 = \frac{b}{6} \sum_{n, m, k} [(2m^2 + (2n^2 + (2k^2)] > 0 \quad \text{from here on, ignore multiple city).} \]

\[ n^2 + m^2 + k^2 = 3 = \frac{b}{24} \sum_{n, m, k} [(2m^2 + (2n^2 + (2k^2)] = 0 \quad \text{from here on, ignore multiple city}. \]

and so on.

For \( x \neq 0 \)
\[ \frac{1}{b^3} \sum_{n, m, k} \frac{3n^2 m (a^2 - n^2 - \rho)}{(n^2 + m^2 + k^2)^{5/2}} = 0 \quad \text{for } m = \pm |n| \text{ pairs}. \]
Whether $\mathcal{E}_{\text{rec}}$ vanishes depends on the specific lattice.

For a liquid with dipoles at random locations $\langle \mathcal{E}_{\text{rec}} \rangle = 0$ too. This follows from

$$\langle \mathcal{E}_i \mathcal{E}_j \rangle = \frac{1}{3} \delta_{ij} r^2$$

on average.

Assuming $\mathcal{E}_{\text{rec}}$ vanishes, then $\mathcal{E} = \mathcal{E}_{\text{ext}} + \mathcal{E}_{\text{for}} = \mathcal{E}_{\text{for}} = \mathcal{E}_{\text{for}} = \mathcal{E} + \frac{4\pi}{3} \rho$

Using $\rho = \frac{E-1}{4\pi} \mathcal{E}$ then

$$\mathcal{E} = \left(1 + \frac{1}{3}(E-1)\right) \mathcal{E} = \mathcal{E} + \frac{E+2}{3} \mathcal{E}$$

To finish the calculation take $\chi = $ molecular polarization.

$n = $ number density, so $\rho = n \alpha = n \alpha \frac{E+2}{3} \mathcal{E}$.

This gives $\chi = n \alpha \frac{E+2}{3}$ and therefore

$$\epsilon = 1 + \chi n \alpha \frac{E+2}{3}$$

Or $\epsilon \left(1 - \frac{\chi n \alpha}{3} \right) = 1 + \frac{\chi n \alpha}{3} \Rightarrow \epsilon = \frac{1 + \frac{8\chi n \alpha}{3}}{1 - \frac{\chi n \alpha}{3}}$

Or

$$\epsilon = 1 + \frac{\chi n \alpha}{1 - \frac{\chi n \alpha}{3}}$$

"Clausius-Mosotti"

One can determine $\alpha$ from a dilute gas of a material and then compute $\epsilon$ for a liquid (dense) of the same material. It works pretty well. See Table 15.3. For example
For $Cs_2$, $\varepsilon_{\text{Clausius-Mossotti}} = 2.35$ vs $\varepsilon_{\text{exp}} = 2.64$ ($n = 10^{22} \text{cm}^{-3}$).

The CM formula fails as $\frac{n}{\varepsilon_0} \to 1$. For polar materials this is the case. But this is far beyond the scope of our study here.
Frequency Dependent Response of Materials

Preamble: General treatment of response functions, or "The generalized susceptibility"
(Taken from Landau & Lifshitz, Stat. Phys. 1, Sec 123).

Let \( s(t) \) describe the state of a system, on which a "force" \( f(t) \) acts through a "susceptibility" \( \chi(t) \). In our applications we have cases where this terminology is very appropriate, e.g., for \( s(t) \) could be \( E(t) \) and \( f(t) \) the electric field \( E(t) \) (literally a force).

We will use Fourier transforms and their inverses for all quantities:

\[
\begin{align*}
\tilde{s}(\omega) &= \int_{-\infty}^{\infty} s(t) e^{-i\omega t} dt, \\
\tilde{\chi}(\omega) &= \int_{-\infty}^{\infty} \chi(t)e^{-i\omega t} dt,
\end{align*}
\]

Now \( s(t) = \int_{-\infty}^{\infty} \tilde{s}(\omega) \tilde{\chi}(\omega)e^{i\omega t} d\omega \).

Note that the integral goes up to \( t' = t \) and no further because causality dictates that the force \( f(t') \) does not affect \( s(t) \) for times \( t' > t \).

Actively, \( s(t) = \int_{-\infty}^{\infty} \chi(t-t')f(t') dt' \).

With \( \chi(t-t') = 0 \) for \( t-t' < 0 \), i.e., \( \chi(t) = 0 \) for \( t < 0 \).
Now
\[
\int_0^\infty dt' \mathcal{F}(t') \chi(t-t') = \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} e^{i\omega t} \tilde{f}(\omega) \tilde{\chi}(\omega) e^{i \omega_1 t' t'} \tilde{f}(\omega) \tilde{\chi}(\omega)
\]
\[
= \int \frac{d\omega}{2\pi} \int \frac{d\omega_1}{2\pi} \tilde{f}(\omega) \tilde{\chi}(\omega) e^{i\omega t} \underbrace{\int dt' e^{i(\omega - \omega_1)t'}}_{2\pi \delta(\omega - \omega_1)}
\]
\[
= \int \frac{d\omega}{2\pi} e^{i\omega t} \tilde{f}(\omega) \tilde{\chi}(\omega)
\]

So that
\[
\tilde{\chi}(\omega) = \tilde{\chi}(\omega) \tilde{f}(\omega)
\]

Examples:
For conductors \( \tilde{f}(\omega) = \sigma(\omega) \tilde{E}(\omega) \), for dielectrics \( \tilde{D}(\omega) = \epsilon(\omega) \tilde{E}(\omega) \).

We want to study properties of \( \tilde{\chi}(\omega) \). Let \( \tilde{\chi}(\omega) = \tilde{\chi}(\omega) + i \tilde{\chi}_i(\omega) \)

Note that \( \tilde{\chi}_i = R \tilde{\chi} \) and \( \tilde{\chi}_i = I \tilde{\chi} \). Since \( \chi(t) = 0 \) for \( t < 0 \) we have
\[
\tilde{\chi}(\omega) = \int_0^\infty \hat{\xi}(\omega) \chi(t) dt
\]
Moreover, \( \chi(t) \) is real. Therefore

1. \( \tilde{\chi}^*(\omega) = \tilde{\chi}(-\omega) \)

That is \( \tilde{\chi}_1(\omega) = \tilde{\chi}(\omega) \) and \( \tilde{\chi}_i(\omega) = -\tilde{\chi}_i(\omega) \) (even, odd functions).

For \( \chi(t) \) having support over some interval of size \( T \)
we expect \( \tilde{\chi}(\omega) \) to have support over \( \Delta \omega \sim \frac{1}{T} \), so \( \tilde{\chi}(\omega) \to 0 \) as \( \omega \to \infty \). Some care is required for \( T \to \infty \) (as \( \chi(t) = 1 \)) and for \( T \to 0 \) (as \( \chi(t) = \delta(t-t_1) \)) but we'll assume \( \lim_{\omega \to \infty} \tilde{\chi}(\omega) = 0 \).
2. \( \omega \chi_2(\omega) > 0 \)

That is, \( \chi_2(\omega) > 0 \) for \( \omega > 0 \).

The proof relies on the 2nd Law of Thermodynamics and the interpretation of \( f \) as a generalized force and \( s \) as a generalized displacement. In the absence of \( f(t) \), the evolution of \( s \) is determined by a Hamiltonian \( H_0 \), and the effect of \( f \) is described as a perturbation \( H' = -sf(t) \). The sign is such that \( \dot{\rho} = -\frac{\partial H}{\partial s} = f \).

Now, facts on body, and changes to the state of the body are accompanied by dissipation (heat lost in the process). Then

\[
\frac{dE}{dt} = \frac{\partial H}{\partial t} - s \frac{df(t)}{dt}
\]

Now, for any two functions

\[
\int_{-\infty}^{\infty} dt \ a(t)b(t) = \int \frac{d\omega}{2\pi} \tilde{a}(\omega) \tilde{b}(-\omega) = \frac{i}{2} \int \frac{d\omega}{2\pi} [\tilde{a}(\omega)\tilde{b}(-\omega) + \tilde{a}(-\omega)\tilde{b}(\omega)]
\]

so \( \Delta E = \int \frac{dE}{dt} dt = \frac{-i}{2} \int \frac{d\omega}{2\pi} \left[ \bar{s}(\omega) \left[ \bar{a}(\omega) \tilde{b}(-\omega) + \bar{a}(-\omega) \tilde{b}(\omega) \right] + \bar{s}(\omega) \left[ \bar{a}(\omega) \tilde{b}(-\omega) + \bar{a}(-\omega) \tilde{b}(\omega) \right] \right] \)

now use \( \bar{s} = \tilde{X} \tilde{f} \)

\[
= \frac{i}{2} \int \frac{d\omega}{2\pi} \left[ \bar{X}(\omega) \left[ \bar{a}(\omega) \tilde{f}(\omega) + \bar{a}(\omega) \tilde{f}(\omega) \right] \right]
\]

\[
= \frac{i}{2} \int \frac{d\omega}{2\pi} \left[ 2 \bar{a} \bar{X}(\omega) \omega |\tilde{f}(\omega)|^2 \right]
\]

Now, \( \tilde{f}(\omega) \) is arbitrary and \( \Delta E > 0 \) \( \Rightarrow \omega \chi_2(\omega) > 0 \).
Analytic continuation: extend definition of \( \hat{\chi}(\omega) \) to complex argument, \( \omega = \omega_0 + j \omega_1 \).

3. \( \hat{\chi}(\omega) \) is analytic for \( \text{Im}(\omega) > 0 \).

Because
\[
\hat{\chi}(\omega) = \int_0^\infty dt \, e^{j\omega t} \, e^{-\omega t} \chi(t)
\]
and the integral converges provided \( \omega_0 > 0 \) (since \( \hat{\chi}(\omega) \)
for real \( \omega \) is assumed to exist for some range of \( \omega \), we
need not worry about the integral not converging because
\[
\chi(t) \sim e^{\lambda t},
\]
Moreover
\[
\frac{d^n \hat{\chi}(\omega)}{d\omega^n} = j^n \int_0^\infty dt \, e^{j\omega t} \, e^{-\omega t} \chi(t)
\]
also converges (\( e^{-\omega t} t^n \to 0 \) as \( t \to \infty \) for any \( n \)).

Note that this is a consequence of causality
(we used \( \chi(t) = 0 \) for \( t < 0 \)).

1'. \( \hat{\chi}(-\omega^*) = \hat{\chi}(\omega) \)

Is the generalization of (1) to complex argument.

Then \( \hat{\chi}_1(-\omega_0 + j \omega_0) = \chi_1(\omega_0 + j \omega_0) \) and \( \hat{\chi}_2(-\omega_0 + j \omega_0) = -\chi_2(\omega_0 + j \omega_0) \)
In particular, on the imaginary axis \( \chi_2(i \omega_0) = 0 \) \( \Rightarrow \chi'(i \omega_0) \) is real.
5. For $\omega > 0$ (upper half plane), $\chi \neq 0$, except on $\omega = 0$ (imaginary axis). For $\omega > 0$, $\tilde{X}(\omega)$ is monotonically decreasing from $X_0 = \tilde{X}(\omega_0)$ to $\tilde{X}(\omega \to \infty) = 0$.

Therefore $\tilde{X}(\omega)$ has no zeroes in upper half plane.

Proof: From complex analysis
$$\oint_{C} \frac{f(z)}{z} \, dz = 2\pi i \cdot \text{Res}(f)$$

$N_{z(p)} =$ number of zeroes (poles) of $f(z)$ in region inferior to $C$.

Consider $I = \frac{1}{2\pi i} \oint_{C} \frac{d\tilde{X}(\omega)}{d\omega} \frac{1}{\tilde{X}(\omega) - \chi}$ where $\chi$ is real and

the integral is over $C$.

Now, for upper half plane, $\chi$ is analytic $\Rightarrow$ so is $\frac{d\chi}{d\omega}$. So in the

statement about complex analysis above, $f = \chi(\omega) - \chi$ is analytic

($N_{p} = 0$) and therefore $I =$ number of zeroes of $\chi(\omega) - \chi =$ number

of times $\chi(\omega)$ takes on the real value $\chi$.

Now compute $I$: change variables:

$$I = \frac{1}{2\pi i} \oint_{C} \frac{d\chi}{\chi - \chi}$$

Let's figure out $C'$.
Since \( \gamma \) goes through \( \nu = 0 \) and \( \nu = \pm \infty \), we start with those \( \chi(\nu) = \chi_0 \) and \( \chi(\nu) = 0 \)

Actually, all of the semi-circle of \( \gamma \) maps to 0, so we are left with

Now \( \chi_0 > 0 \) for \( \nu > 0 \) and \( \chi_0 < 0 \) for \( \nu < 0 \) by (2).

So

\[ \chi_0 (0) \]

and \( \chi_0(-\nu) = \chi_0(\nu) \)

The point is \( \gamma \) crosses the real axis only at 0 and \( \chi_0 \). So, \( I = 1 \) for all values of \( \chi \) for \( 0 < \chi < \chi_0 \) and \( I = 0 \) otherwise.

To complete the argument (1) since \( \chi \) is real on the positive imaginary axis, and it is analytic and goes from \( \chi_0 \) to 0 on the axis, it must take on every value in the interval \( (0, \chi_0) \) along the axis. But it takes on each value only once. It must have \( \chi_2 \neq 0 \) everywhere.
else on the upper half plane \( \tilde{X} \neq 0 \) except at \( \pm \infty \).

(iii) Since it takes on every value in \((0, \infty)\) only once, \(X(i\omega)\) cannot have a local minimum or maximum along the line; it is monotonic.

(iv) Since \(\tilde{X} \neq 0\) everywhere (on upper half plane) except on the imaginary axis, and there \(\tilde{X}(i\omega) \neq 0\) except at \(\omega = \pm \infty\), we have \(\tilde{X}(\omega) \neq 0\) (except at \(\pm \infty\)).

G. Kramers - Kronig Relation

Consider

\[
\frac{1}{2\pi i} \oint_C \frac{\tilde{X}(z)}{z-\omega} \, \text{d}z
\]

for \(C\):

Since \(\tilde{X}(z)\) is analytic for \(\text{Im}(z) > 0\) and \(\omega\) is outside \(C\), there are no poles inside \(C\); by Cauchy's Theorem the integral vanishes.

The integral over the small semicircle is, with \(z = \omega + \epsilon e^{i\theta}\)

\[
\lim_{\epsilon \to 0} \int_{\pi}^{0} e^{i\phi} \, \text{d}\phi \frac{\tilde{X}(\omega + \epsilon e^{i\phi})}{\epsilon e^{i\phi}} = -i\pi \tilde{X}(\omega)
\]

The integral over the real axis is the principal value integral. So

\[
0 = -i\pi \tilde{X}(\omega) + P \int_{-\infty}^{\infty} \frac{\tilde{X}(\omega)}{\omega-\omega} \, \text{d}\omega
\]

or

\[
\tilde{X}(\omega) = -i \frac{P}{\pi} \int_{-\infty}^{\infty} \frac{\tilde{X}(\omega)}{\omega-\omega} \, \text{d}\omega
\]
Taking \( \text{Im} \) of both sides of this equation:

\[
\hat{\chi}_2(\omega) = -\frac{i}{\pi} P \int_{-\infty}^{\infty} d\omega' \frac{\hat{\chi}_1(\omega')}{\omega' - \omega}
\]

Kramers-Kronig relations

\[
\hat{\chi}_1(\omega) = \frac{i}{\pi} P \int_{-\infty}^{\infty} d\omega' \frac{\hat{\chi}_2(\omega')}{\omega' - \omega}
\]

Then, \( \hat{\chi}_1(\omega) \) completely fixes \( \hat{\chi}_2(\omega) \), and vice versa.

Many additional results follow.

**Exercises:**

(i) Show \( \hat{\chi}_1(\omega) = \frac{2}{\pi} P \int_{0}^{\infty} d\omega' \frac{\omega' \hat{\chi}_2(\omega')}{\omega'^2 - \omega^2} \)

(ii) By considering

\[
\int_{0}^{\infty} d\omega \frac{2 \hat{\chi}(\omega)}{\omega^2 + \omega^2}
\]

for real \( \omega \), along a contour

\[
\text{show} \quad \int_{-\infty}^{\infty} d\omega \frac{\omega \hat{\chi}(\omega)}{\omega^2 + \omega^2} = i \pi \hat{\chi}(i\omega)
\]

Use this to show \( \hat{\chi}(i\omega) = \frac{2}{\pi} \int_{0}^{\infty} d\omega \frac{\omega \hat{\chi}_2(\omega)}{\omega^2 + \omega^2} \)

(iii) Use the previous result (integrate over \( \omega \)) to show

\[
\int_{0}^{\infty} d\omega \hat{\chi}(i\omega) = \int_{0}^{\infty} \hat{\chi}_2(\omega) d\omega
\]

To derive these formulae (including Kramers-Kronig) we only used the fact that \( \hat{\chi}(\omega) \) is regular in the upper half-plane. If in addition we know \( \hat{\chi}(\omega) = \hat{\chi}(-\omega) \) so \( \hat{\chi}(i\omega) = \hat{\chi}_1(i\omega) \) we have

\[
\int_{0}^{\infty} d\omega \hat{\chi}_1(i\omega) = \int_{0}^{\infty} \hat{\chi}_2(\omega) d\omega
\]
Frequency dependent conductivity

Ohm's law: \[ \mathcal{J}(\omega) = \sigma(\omega) \mathcal{E}(\omega) \]

where, e.g. \[ \mathcal{J}(t) = \int_{-\infty}^{\infty} \mathcal{E}(\omega) \sigma(\omega) e^{i\omega t} d\omega \]

Aside on Fourier Transform of a product.

Let \[ a(t) = \int_{-\infty}^{\infty} e^{i\omega t} a(\omega) \] and \[ b(t) = \int_{-\infty}^{\infty} e^{i\omega t} b(\omega) \]

where \[ a(\omega) = \int_{-\infty}^{\infty} dt e^{-i\omega t} a(t) \]

Then, the WFT of the product is

\[ \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} a(\omega) b(\omega) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \int_{-\infty}^{\infty} dt e^{i\omega t} a(t) \int_{-\infty}^{\infty} dt e^{i\omega t} b(t) \]

\[ = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt_1 a(t) b(t_1) \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega(t+t_1-t)} \]

\[ = \delta(t+t_1-t) \]

\[ = \int_{-\infty}^{\infty} dt, a(t_1) b(t-t_1) \]

Ohm's law in time domain

\[ \mathcal{J}(t) = \int_{-\infty}^{\infty} dt' \sigma(t-t') \mathcal{E}(t') \]

The response function \( \sigma(t-t') \) must vanish for \( t'>t \) since \( \mathcal{E}(t') \) cannot influence the current \( \mathcal{J}(t) \) at time \( t \), per time \( t > t' \). This follows from causality.

So \( \sigma(t) = 0 \) for \( t < 0 \).
Drude Model:

Electrons move with average velocity $\overline{\nu(t)}$.

They accelerate ($F = \frac{e}{m} \overline{E}$) due to electric field $\overline{E}$.

Electrons bounce off fixed atoms.

Simple model: probabilistic

* Each electron has probability per unit time $\frac{1}{\tau}$ of colliding

* After collision, velocity is randomized

$$\overline{\nu}(t+\Delta t) - \overline{\nu}(t) = \frac{\Delta t}{\tau} \overline{\nu}(t) + \frac{q \overline{E}(t) \Delta t}{m}$$

(directions average to zero)

$$= -\frac{\Delta t}{\tau} \overline{\nu}(t) + \frac{q \overline{E}(t) \Delta t}{m}$$

(Openshot uses $q = -e$)

(..."relaxation" or "collision" time).

$$\frac{\partial \overline{\nu}}{\partial t} = -\frac{1}{\tau} \overline{\nu} + \frac{n \overline{E}}{m}$$

with $n =$ number density

$$\overline{\nu}(t) = n q \overline{\nu}(0) \Rightarrow \frac{\partial \overline{\nu}}{\partial t} = -\frac{1}{\tau} \overline{\nu} + \frac{n q E}{m}$$

or, after Fourier transform

$$-i\omega \overline{\nu} = -\frac{1}{\tau} \overline{\nu} + \frac{n q E}{m}$$

Solving for $\overline{\nu}$

$$\overline{\nu}(\omega) = \frac{n q E}{\frac{1}{\tau} - i\omega} = \frac{n q E}{m} \frac{1}{1 - i\omega \tau}$$

Notes:

* $\tau \sim 10^{-14}$ sec is typical. So for frequencies $\omega \ll \frac{1}{\tau} = 10^9 \text{Hz}$, $\overline{\nu}(\omega) \approx \overline{\nu} = \frac{n q E}{m}$

Using $n \sim 10^{22} \text{ cm}^{-3}$ and $q, m$ for electron $\Rightarrow \overline{\nu} \sim 10^9 \text{ sec}^{-1}$ or $\frac{1}{\tau} \sim 10^4 \text{ ohm cm}$
• Opposite limit: \( \omega \gg \tau^{-1} \Rightarrow \omega^2 \approx j \frac{\alpha}{m \omega} \) (A)

is purely imaginary, and \( n \approx \frac{1}{\omega} \).

This is as if there were no collisions \( \Rightarrow \) response is inertial \( (\text{is from } E = mf) \)

and \( \delta \to 0 \) as \( \omega \to \infty \) means

\[
\frac{dv}{dt} = \frac{qE}{m} e^{j \omega t} \Rightarrow v = \frac{qE}{m j \omega} \to 0 \text{ as } \omega \to \infty \text{ i.e. electrons can't keep up with E.}
\]

(A) is purely from applied \( E \), so very general \( (\text{independent of model of collision of electrons}) \). Will use this!

Aside

If magnetic field is present, add to force a \( \frac{q}{c} \vec{B}(t) \times \vec{A}(t) \) form. Then

after multiplies by \( \frac{n q}{c} \frac{q}{m} \vec{B}(t) \times \vec{A}(t) \).

The P.T. of a product

\[
\int dt \vec{A}(t) \cdot \vec{B}(t) e^{j \omega t} = \int \frac{d\omega}{2\pi} \int \frac{d\omega}{2\pi} \vec{A}(\omega) \cdot \vec{B}(\omega) \int dt e^{j (\omega - \omega') t} = 2 \pi \delta(\omega - \omega')
\]

So

\[
-2i \omega \vec{F}(\omega) = \frac{1}{c} \vec{B}(\omega) + \frac{n q^2}{m} \vec{E}(\omega) + \frac{q}{c} \int \frac{d\omega'}{2\pi} \vec{F}(\omega') \times \vec{B}(\omega' - \omega)
\]

Yikes! See Exercise 11.4.1 for static case.
General properties of $\tilde{\sigma}(\omega)$. (Garg sec 121 - we'll go back to 120 later)

Let's write $\tilde{\sigma} = \tilde{\sigma}_1 + \tilde{\sigma}_2$ ($\tilde{\sigma}_1, \tilde{\sigma}_2$ are real).

Then (much of this follows from the general susceptibility notes above).

1. $\tilde{\sigma}^*(\omega) = \tilde{\sigma}(-\omega)$

This is because $\tilde{E}$ and $\tilde{J}$ are real. For any real $f(t)$ its FT has $\tilde{f}(\omega) = \tilde{f}(-\omega)$ -- we have seen this. And $\tilde{E} = \tilde{\sigma} \tilde{J}$.

2. $\tilde{\sigma}_1(\omega) > 0$. (This is slightly different than for $x$ above):

   \[
   \text{Power dissipated } = P_{\text{diss}} = \tilde{\sigma}_1 \tilde{E}^2 > 0 \text{ by 2nd Law of Thermodynamics.}
   \]

   \[
   \begin{align*}
   \int dt \tilde{J}(t) \tilde{E}(t) &= \int_{\omega} \tilde{J}(\omega) \tilde{E}(\omega) = \int_{\omega} \frac{d\omega}{2\pi} \left( \tilde{\sigma}_1(\omega) \tilde{E}(\omega) + \tilde{\sigma}_2(\omega) \tilde{E}(\omega) \right) \\
   &= \int_{\omega} \frac{d\omega}{2\pi} \text{Re}(\omega) |\tilde{E}(\omega)|^2
   \end{align*}
   \]

   **This must be positive for arbitrary $\tilde{E}$ $\Rightarrow$ $\tilde{\sigma}_1(\omega) > 0$.**

3. $\tilde{\sigma}(\omega)$ is analytic for $\text{Im}(\omega) > 0$.

(Note that this depends on our definition of FT $\tilde{f}(\omega) = \int_{t_0} f(t) e^{-i\omega t} dt$)

We have seen that causality $\Rightarrow$ $\tilde{\sigma}(t) = 0$ for $t < 0$.

So

\[
\begin{align*}
\tilde{\sigma}(\omega) &= \int_{-\infty}^{\infty} e^{i\omega t} \sigma(t) \, dt \equiv \int_{0}^{\infty} e^{i\omega t} \sigma(t) - e^{-i\omega t} \sigma(t) \\
\sigma(t) &= \int_{-\infty}^{\infty} e^{-i\omega t} \tilde{\sigma}(\omega) \, d\omega
\end{align*}
\]

For $\text{Im}(\omega) > 0$ the integral converges provided $\sigma(t)$ does not grow exponentially.
We may safely assume $\sigma(t)$ does not grow exponentially, recall

$$\tilde{f}(t) = \int_{-\infty}^{t} dt' \sigma(t - t') \tilde{E}(t')$$

and we do not expect $\tilde{f}(t)$ to depend on $\tilde{E}(t')$ as $t' \to -\infty$ as $e^{-t'}$.

Moreover, we can safely take derivatives, as in

$$\frac{d^n \tilde{\sigma}}{d\omega^n} = (-i)^n \int_{-\infty}^{\infty} dt \ t^n e^{i\omega t} \sigma(t)$$

since this is still convergent for $\text{Im}(\omega) > 0$.

$\Rightarrow \tilde{\sigma}(\omega)$ is analytic in $\text{Im}(\omega) > 0$.

Note also that $\sigma(\omega) \to 0$ as $\text{Im} \omega \to \infty$.

This (analyticity in upper half plane plus vanishing at $\infty$) means:

1. $\tilde{\sigma}$ satisfies Kramers-Kronig relations

Consider

$$\oint \d\zeta \tilde{\sigma}(\zeta) = 0 \quad (\text{Cauchy: } \tilde{\sigma}(\zeta) \text{ is analytic in region bounded by } C).$$
On semicircle $|z| = R \to \infty$ the integral vanishes because
$\sigma \to 0$, so $|\frac{\sigma}{z-\omega}| \to 0$ faster than $\frac{1}{|z|}$.

The small semicircle gives

\[
\lim_{\gamma \to 0} \int_{\gamma} e^{i\phi} \frac{\sigma(\omega + e^{i\phi})}{e^{i\phi}} = -i\pi \sigma(\omega)
\]

The rest is the principal value of the integral on the real line, so

\[
P \int_{-\infty}^{\infty} dx \frac{\sigma(x)}{x-\omega} = -i\pi \sigma(\omega) = 0
\]

Separating into real and imaginary parts, and using $x=\omega$ so that the dummy variable reminds us it refers to frequency:

\[
\bar{\sigma}_2(\omega) = -\frac{i}{\pi} P \int_{-\infty}^{\infty} d\omega' \frac{\sigma(\omega)}{\omega - \omega'}
\]

Kramers-Kronig relations

\[
\bar{\sigma}_1(\omega) = \frac{i}{\pi} P \int_{-\infty}^{\infty} d\omega' \frac{\sigma(\omega)}{\omega - \omega'}
\]

If you know $\bar{\sigma}_1$ (or $\bar{\sigma}_2$) you can compute $\bar{\sigma}_2$ (or $\bar{\sigma}_1$).

5. $\bar{\sigma}(\omega) \neq 0$ in upper half-plane.

This was done for $\bar{\sigma}(\omega)$ above, and won't repeat here.
6. $f$-sum rule

From Kramers- Kronig we have

$$\tilde{\sigma}_2(\omega) = -\frac{2\omega}{\pi} P \int_0^\infty d\omega' \frac{\tilde{\sigma}_1(\omega')}{\omega'^2 - \omega^2}$$

As $\omega \to \infty$ this is

$$\tilde{\sigma}_2(\omega) = \frac{2}{\pi \omega} \int_0^\infty d\omega' \tilde{\sigma}_1(\omega')$$

But from Drude's model, $\tilde{\sigma}(\omega) = i \frac{n q^2}{m \omega}$ as $\omega \to \infty$, and we explained this is model independent. Comparing

$$\frac{n q^2}{m \omega} = \frac{2}{\pi \omega} \int_0^\infty d\omega' \tilde{\sigma}_1(\omega')$$

or

$$\int_0^\infty d\omega' \tilde{\sigma}_1(\omega') = \frac{7\pi q^2}{2 m} \quad "f-sum\ rule"$$
Dielectric response function and Garg's "propensity"

The distinction between free and bound, and particularly \( \Phi_{\text{free}} \) and \( \Phi_{\text{bound}} \) (and we will see \( \Phi_{\text{free}} \) too) gets blurred with harmonic fields.

Both free and bound charges exhibit oscillatory motion. At very high frequency they are indistinguishable.

We will later consider a microscopic model of the response of bound electrons. But let's try to 1st capture the ambiguity in free vs bound described above, in a macroscopic description. From Ampere's law

(Here I deviate from Garg slightly: his breaking of \( \vec{J} = \vec{J}_\text{ext} + \vec{J}_\text{int} \)
- external and internal to the material - is not sense, since these are local quantities, i.e., \( \vec{J} = \vec{J}(x, t) \), \( \vec{D} = \vec{D}(x, t) \)

\[ \nabla \times \vec{H} - \frac{1}{c} \frac{\partial \vec{D}}{\partial t} = \frac{4\pi}{c} \vec{J} \]

In \( \omega \)-domain:

\[ \nabla \times \vec{H} + \frac{i\omega}{c} \vec{D} = \frac{4\pi}{c} \vec{J} \Rightarrow \nabla \times \vec{H} + \frac{i\omega}{c} \vec{E} = \frac{4\pi}{c} \vec{E} \]
We can rewrite this as
\[
\nabla \cdot \vec{\mathbf{E}} + \frac{i}{\varepsilon} \omega (\sigma + \frac{4\pi}{\omega} \varepsilon) \vec{\mathbf{E}} = 0 \quad \text{or} \quad \nabla \cdot \vec{\mathbf{E}} = \frac{4\pi}{\varepsilon} \left( \sigma - i \frac{\omega \varepsilon}{\omega} \right) \vec{\mathbf{E}}
\]

sort of effective permittivity or kind of effective conductivity.

We also have \(\nabla \cdot \vec{\mathbf{D}} = 4\pi p\). From continuity \(\frac{\partial p}{\partial t} = -\nabla \cdot \vec{\mathbf{J}}\),

we have \(-i \omega \vec{\mathbf{J}} = -\nabla \cdot \vec{\mathbf{J}}\). Using \(\vec{\mathbf{D}} = \varepsilon \vec{\mathbf{E}}\) and \(\vec{\mathbf{J}} = \sigma \vec{\mathbf{E}}\),

\[
\nabla \cdot \vec{\mathbf{D}} = 4\pi p \Rightarrow \nabla \cdot (\varepsilon \vec{\mathbf{E}}) = -\frac{4\pi}{\omega} \nabla \cdot (\sigma \vec{\mathbf{E}})
\]

\[
= -\nabla \left[ \varepsilon + \frac{4\pi}{\omega} \sigma \right] \vec{\mathbf{E}} = 0
\]

Although you will find some textbooks that state that there is an ambiguity in whether we combine \(\varepsilon\) and \(\sigma\) into permittivity or conductivity, the interpretation of Gauss's law suggests an effective permittivity is a better choice.

Garg invents the term (I have not seen it used elsewhere)

"electric propensity" for

\[
\tilde{\mathbf{\varepsilon}}(\omega) = \varepsilon(\omega) + \frac{4\pi i \sigma(\omega)}{\omega}
\]

\(\tilde{\mathbf{\varepsilon}}\) is often used for \(\varepsilon(\omega)\).

Much of the literature calls it dielectric constant, or complex dielectric constant, or AC dielectric constant. None of these names capture the facts that \(\varepsilon(\omega)\) not a constant, \(i\) not purely dielectric and \(i\) not the same as \(\varepsilon(\omega)\) (even though this symbol is often used for \(\varepsilon(\omega)\)).
We'll stick with Garg.

Aside: if there are additional currents not subject to Ohm's law (e.g., superconducting current) then add to right hand side:

\[ \nabla \times \mathbf{H} + \frac{i}{c} \frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{e} \mathbf{J}' \]
\[ \nabla \cdot \mathbf{E} = 4\pi \rho' \]

Garg also defines \( \mathbf{Z} = \mathbf{E}/\mathbf{H} \) so that in t-domain,

\[ \nabla \times \mathbf{E} - \frac{1}{\epsilon} \frac{\partial \mathbf{Z}}{\partial t} = \frac{4\pi}{e} \mathbf{J}' \]
\[ \nabla \cdot \mathbf{Z} = 4\pi \rho' \]

Beware that in most of the literature \( \mathbf{Z} = \mathbf{E}/\mathbf{E} \) is \( \mathbf{D} = \varepsilon \mathbf{E} \)

Best is to understand what you are doing; then you don't get confused with symbols.
Electromagnetic energy in material media

We saw that $\sigma \neq 0$ in the upper half $\sigma$-plane, and in particular $\sigma_1 > 0$ on real axis. This was a result that followed from the 2nd law, that energy is dissipated in the material body.

Now, the microscopic theory tells us exactly where the energy goes:

$$\mathbf{-\nabla} \cdot \mathbf{\Sigma} = \mathbf{f} \cdot \mathbf{E} + \frac{\partial \mathbf{u}}{\partial t}$$

as was shown in 203A, where $\mathbf{\Sigma} = \frac{\mu_0}{4\pi} \mathbf{e} \times \mathbf{b}$ is the (microscopic version of) Poynting vector giving the energy flux and $\mathbf{u} = \frac{1}{8\pi} (\mathbf{e}^2 + \mathbf{b}^2) \mu$ (microscopic energy density). The question is what replaces $\mathbf{\Sigma}$ in heat dissipated in the presence of dielectrics.

The answer is

$$\mathbf{-\nabla} \cdot \mathbf{\Sigma} = \mathbf{f}_{\text{free}} \cdot \mathbf{E} + \frac{d\mathbf{u}}{dt} + \mathbf{Q}$$

where (i) Fields are assumed quasimonochromatic
(ii) $X(t)$ means $X(t)$ is averaged over the period of the quasimonochromatic fields
\( \bar{U} \) has no interpretation of internal energy and \( \dot{Q} \) is\n
\[
\frac{d}{dt}(\text{heat}) \quad \text{with}
\]

\[
\bar{U} = \frac{1}{8\pi} \left[ \frac{d}{d\omega}(\omega \varepsilon(\omega)) \overline{E}^2 + \frac{d}{d\omega}(\omega \mu(\omega)) \overline{H}^2 \right]
\]

and

\[
\dot{Q} = \frac{1}{4\pi} \left[ \omega \varepsilon(\omega) \overline{E}^2 + \omega \mu(\omega) \overline{H}^2 \right]
\]

The rest of this section is just computation deriving this result (plus a definition of terms, e.g. "quasimonochromatic").

Consider quasimonochromatic field \( \bar{E}(t) \). That is \( \bar{E}(\omega) \) has frequency centered on \( \omega_0 \) with small dispersion.

We want to show that

\[
\bar{E}(t) = \bar{a}(t) e^{i\omega_0 t} + \text{c.c.}
\]

To show this, consider

\[
\bar{E}(t) = \int_{-\infty}^{\infty} \bar{E}(\omega) e^{-i\omega t} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \bar{E}(\omega) + \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} \bar{E}^*(\omega)
\]

Now write \( \bar{E}(\omega_0 + \omega) = \bar{a}(\omega) \) so that \( (\omega = \omega_0 + \alpha \) above\)

\[
\bar{E}(t) = e^{-i\omega_0 t} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \bar{a}(\omega) e^{-i\omega t} + \text{c.c.}
\]

Now, assume that \( \bar{a}(\omega) \) is localized about some frequencies well above zero. Then we can approximately replace the lower limit by \(-\infty\):

\[
\bar{E}(t) = \bar{a}(t) e^{i\omega_0 t} + \text{c.c.}
\]

which is the desired result.
\( \hat{a}(t) \) varies little over a period \( \frac{2\pi}{\omega_0} \). So if we average \( \overline{E(t)} \) over \( t > \frac{2\pi}{\omega} \) the \( e^{i\omega t} \) terms do not contribute

\[
\overline{E^2(t)} = 2 \overline{\hat{a}(t) \hat{a}^*(t)}
\]

The average is over \( t > \frac{2\pi}{\omega} \) but small over the typical time over which \( \hat{a}(t) \) varies.

Likewise for other fields, like \( \overline{H(t)} \).

Now, we have shown

\[
\nabla \cdot \left[ -\frac{1}{\mu_0} (\mathbf{E} \times \mathbf{H}) \right] = \mathbf{j} \cdot \mathbf{E} + \frac{1}{\mu_0} \left( \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \right)
\]

Integrate over volume, for some \( V \), \( \int dV \mathbf{j} \cdot \mathbf{E} \) is work done by \( \mathbf{E} \) on free charges. \( \int dV \nabla \cdot \left[ -\frac{1}{\mu_0} (\mathbf{E} \times \mathbf{H}) \right] = -\int_{V_0} dV \nabla \cdot \left[ -\frac{1}{\mu_0} (\mathbf{E} \times \mathbf{H}) \right] \) is men energy/time flowing into \( V_0 \) so \( \mathbf{S} = \frac{\partial}{\partial t} \mathbf{E} \times \mathbf{H} = \text{energy flux} / \text{vol} \).

The last term must be the change in internal energy plus heat produced. Consider \( \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} \). Write \( \mathbf{E} \) and \( \mathbf{D} \) in terms of \( \mathbf{E}(\omega) \)

and \( \mathbf{D}(\omega) = \mathbf{E}(\omega) \mathbf{E}(\omega) \).

\[
\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} = \int \frac{d\omega}{2\pi} \int \frac{d\omega'}{2\pi} \mathbf{E}(\omega) \mathbf{E}(\omega') e^{i(\omega-t \omega')} \mathbf{E}(\omega) \mathbf{E}(\omega') e^{-i\omega t} = \int \frac{d\omega}{2\pi} \int \frac{d\omega'}{2\pi} \mathbf{E}(\omega) \mathbf{E}(\omega') e^{i\omega t} \mathbf{E}(\omega) \mathbf{E}(\omega') e^{-i\omega t}
\]

so adding these:
\[ 2 \mathbf{E} \cdot \mathbf{D} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \, e^{i\omega t - i\omega t} \mathbf{E}(\omega) \cdot \mathbf{E}(\omega') \, i \left[ (\omega - \omega') E^{*}(\omega') - \omega E^{*}(\omega) \right] \]

\[
= \int_{0}^{\infty} \frac{d\omega}{2\pi} \left[ e^{i\omega t} \mathbf{E}(\omega) \cdot \mathbf{E}(\omega) + e^{-i\omega t} \mathbf{E}(\omega) \cdot \mathbf{E}(\omega) \right] \]

and so on, so that we only integrate over positive frequencies. Since we will want to average over quasi-statical fields, drop terms \( \mathbf{E}(\omega) \cdot \mathbf{E}(\omega) \) or \( \mathbf{E}(\omega)^* \cdot \mathbf{E}(\omega)^* \).

Next average over period \( T = \frac{2\pi}{\omega_0} \). Use \( \omega = \omega_0 t \), and substitute \( \omega_0 \) for \( \omega \).

\[
2 \mathbf{E} \cdot \mathbf{D} \approx 2 \int_{0}^{\infty} \frac{d\omega}{2\pi} \int_{0}^{\infty} \frac{d\omega}{2\pi} \, i \left[ (\omega - \omega_0) E^{*}(\omega_0) - \omega_0 E^{*}(\omega_0) \right] \mathbf{E}(\omega_0) \cdot \mathbf{E}(\omega_0) \, e^{iKc_0} \delta(\omega - \omega_0) \]

We separate \( E(\omega) \) into real and imaginary parts since physically we expect \( \varepsilon_2 = \text{Im} \varepsilon \) to be associated to heat (dissipation) while \( \varepsilon_1 = \text{Re} \varepsilon \) ought to be related to internal energy for each of this we expand in \( \omega = \omega - \omega_0 \) and retain leading terms:

\[
\text{Re} \left[ (\omega - \omega_0) E(\omega) \right] = (\omega + d_1) E(\omega) - (\omega_0 + d_0) E_1(\omega_0 + d_0) = (d_1 - d_0) \frac{d}{d\omega} (\omega E_1(\omega_0)) \]

\[
\text{Im} \left[ \text{idem} \right] = - (\omega_0 + d_0) E_2(\omega_0 + d_0) - (\omega_0 + d_0) E_2(\omega_0 + d_0) = - 2 \omega_0 E_2(\omega_0) \]
Write this in time domain (and drop "ω" in ω(t)):

\[
\overline{E} \cdot \overline{D} = \frac{1}{2} \frac{d}{d\omega} \overline{E}(\omega) \frac{d}{dt} \overline{E}^2(t) + \omega \overline{E}(\omega) \overline{E}^2(t)
\]

and

\[
\text{this x } \frac{1}{4\pi} \text{ is } \frac{d}{dt} \overline{U}
\]

\[
\text{this x } \frac{1}{4\pi} \text{ is } \overline{Q}
\]

as advertised.

What remains is justifying the interpretation of the two terms as above. Note that if you proceed slowly and adiabatically in polarizing the medium, the mechanical work done (which should go fully into internal energy) is,

\[
\frac{1}{80} \frac{d}{d\omega} \overline{E}^2.
\]

But \( \frac{1}{80} \frac{d}{d\omega} \overline{E}^2 \) gives this in the quasistatic approximation. So we interpret the first term as \( \frac{d}{dt} \overline{U} \)

and infer the 2nd is heat that shows up when the process of polarizing the medium is not adiabatic.

Beware of the limits of applicability: we assumed linearity, quasi-monochromatic fields, retained leading terms in Taylor expansion, ...
Electronic Response Model of Drude, Kramer, and Lorentz.

Each atom/molecule as a polarizable unit.

Model the atom as a charge (electron) bound by a harmonic force, with dissipation and under and applied electric field force:

\[ m \ddot{r} + m \omega_0^2 r = q \vec{E}(t) \quad (q = e - e^+) \]

If the “atom” is neutral and has no permanent dipole moment then we need a charge \(-q\) (i.e. e) at the center \( r = 0 \). The dipole moment is then \( \vec{d}(t) = q \vec{r}(t) \).

In Fourier space (we have solved this eq. several times before)

\[ -\omega^2 \vec{r} - i \omega \dot{\vec{r}} + \omega_0^2 \vec{r} = \frac{q}{m} \vec{E} \]

or

\[ \vec{r} = -\frac{q}{m} \vec{E} \frac{i}{\omega^2 - \omega_0^2 + i \omega \gamma} \]

With \( n = \text{number density of bound electrons}/\text{volume} \)

Polarization vector \( \vec{P} = n q \vec{r} = -\frac{n q^2}{m} \vec{E} \frac{1}{\omega^2 - \omega_0^2 + i \omega \gamma} \)

\[ \Rightarrow \chi_e = -\frac{n q^2}{m} \frac{1}{\omega^2 - \omega_0^2 + i \omega \gamma} \]
\[\tilde{\varepsilon}(\omega) = \frac{1 + \frac{4\pi n^2 q}{m}}{\varepsilon_0} \frac{1}{\omega^2 - \omega_0^2 + i\omega}\]

This is for rarefied media. For dense media, a Clausius-Mosotti model treatment gives

\[\frac{\tilde{\varepsilon}(\omega) - 1}{\varepsilon(\omega) + 2} = \frac{4\pi n^2 q}{3}\tilde{\chi}_e(\omega)\]

or \(\tilde{\varepsilon} (1 - \frac{4\pi n^2 q}{3}\tilde{\chi}_e) = 1 + \frac{4\pi n^2 q}{3}\tilde{\chi}_e, \quad \tilde{\varepsilon} = 1 + \frac{4\pi n^2 q}{3}\tilde{\chi}_e\)

or \(\varepsilon = 1 + \frac{4\pi n^2 q}{m} \frac{(-1)}{\omega - \omega_0 + i\omega'} \frac{1}{\omega - (\omega_0 - \frac{4\pi n^2 q}{3}) + i\omega'}\)

\[= 1 + \frac{4\pi n^2 q}{m} \frac{1}{\omega_i - \omega_i - i\omega'}\]

where \(\omega_i = \omega_0 - \frac{4\pi n^2 q}{3}\frac{1}{m}\)

**Improvement:** Many resonant frequencies of electrons in real atoms, given by

\[\hbar\omega_i = \varepsilon_i - \varepsilon_0\]

\(\hbar\) energy level, \(\varepsilon_0\) ground state energy
and introduce $f_i = \text{oscillator strength}$

$= \text{amplitude of dipole moment when oscillating between i-th state and ground state, } w_i$

$\sum f_i = Z = \text{number of electrons in atom}$

Then, the improved model is

$$\tilde{\varepsilon}(\omega) = 1 + \frac{4\pi n q^2}{m Z} \sum f_i \frac{\omega_i^2}{(\omega_i^2 - \omega^2 - i\omega\gamma_i)}$$

($\gamma_i = \text{damping of response at frequency } \omega_i$).

Note:
This is a rough model. Do not attach too literal a meaning to constants like $n, Z,

Let's plot $\tilde{\varepsilon}_1(\omega)$ and $\tilde{\varepsilon}_2(\omega)$

$$\tilde{\varepsilon}_1(\omega) = \text{Re}(\tilde{\varepsilon}(\omega)) = 1 + \frac{4\pi n q^2}{m Z} \sum f_i \frac{\omega_i^2 - \omega^2}{(\omega_i^2 - \omega^2)^2 + \omega_i^2 \gamma_i^2}$$

$$\tilde{\varepsilon}_2(\omega) = \text{Im}(\tilde{\varepsilon}(\omega)) = \frac{4\pi n q^2}{m Z} \sum f_i \frac{\omega_i \gamma_i}{(\omega_i^2 - \omega^2)^2 + \omega_i^2 \gamma_i^2}$$

Near resonance $i$,

$$\tilde{\varepsilon}_1(\omega) \approx \frac{4\pi n q^2 f_i (\omega_i^2 - \omega^2)}{m \omega_i^4 \gamma_i^2}$$

$$\tilde{\varepsilon}_2(\omega) \approx \frac{4\pi n q^2 f_i (\omega_i \gamma_i)}{m (\omega_i^2 - \omega^2)^2 + \omega_i^2 \gamma_i^2}$$
So

\[ \varepsilon'_1(\omega) \]

\[ \varepsilon'_2(\omega) \]

\[ \frac{d\varepsilon'_1}{d\omega} \sim \text{"anomalous dispersion"} \text{ (see later, below)} \]

Notes:

1. If \( \omega_i = 0 \) the contribution of this resonance to

\[ \varepsilon'(\omega) = \frac{4\pi n_i q^2}{m} \left( \frac{1}{\omega^2 - \omega_i^2 + i\omega \delta} \right) \]

which looks like conductivity in Drude's model: and it is!

\( \omega_i = 0 \) means no restoring force \( \Rightarrow \) free electrons.

Recall the Drude model has

\[ \tilde{\sigma}(\omega) = \frac{N_i q^2 \omega}{m} \frac{1}{1 - \omega^2 / \omega_i^2} \] with \( N_i = n_i \text{ free electrons}. \)

And "propensity" is

\[ \tilde{\sigma}(\omega) = \tilde{\varepsilon'}(\omega) + \frac{4\pi n_i q^2}{m} \frac{1}{\omega^2 - \omega_i^2 + i\omega \delta} \]

So \( \tilde{\sigma} \) in Drude's model has

\[ \tilde{\sigma}(\omega) - 0(\omega) = \frac{4\pi n_i q^2}{m} \frac{1}{\omega^2 - \omega_i^2 + i\omega \delta} \]

\[ = -\frac{4\pi n_i q^2}{m} \frac{1}{\omega(\omega + i\xi)} \]
which matches the above with

\[ n_f = \frac{n_f}{2} \quad \text{and} \quad \tau = \tau_i \]

Nice to get a unified treatment in one simple model.

a. The static case \( \tilde{\varepsilon}(0) = 1 + i \frac{\mu \gamma q^2}{\omega^2} \sum \frac{f_i}{\omega_i} \)

The large frequency limit

\[ \tilde{\varepsilon}(\omega) = 1 - i \frac{\mu \gamma q^2}{m \omega^2} \]

\((\omega \tilde{\varepsilon}(\omega) \approx \text{Re})\), where \( \frac{1}{2} \sum f_i \tau_i = 1 \) was used. Again, as in the case of \( \tilde{\sigma} \), the \( \omega \to \infty \) behavior is model independent (does not depend on \( f_i, \tau_i, \omega_i \)), since at high frequency electrons are "paralyzed" (in the words of Garg).

We may write \( \tilde{\varepsilon}(\omega) = 1 - \frac{\omega_p^2}{\omega^2} \)

where \( \omega_p^2 = \frac{\mu \gamma q^2}{m} \) is the "plasma" frequency of the medium.
Addendum: Wave propagation in dispersive medium.

In PHYS203A we discussed wave propagation in dispersive media briefly. We took

$$\omega(k) = \boldsymbol{v}_k = \frac{k}{\sqrt{\varepsilon(\omega)\mu(\omega)}}$$

and then found that $$\frac{\omega}{v_k} = \frac{\varepsilon}{\varepsilon_m}$$ is the phase velocity, while $$\frac{\partial \omega}{\partial k}$$ gives the group velocity.

With $$\frac{1}{\varepsilon} \varepsilon(\omega) \varepsilon_m = \left[ \frac{d}{d\omega} \sqrt{\varepsilon(\omega)\mu(\omega)} \right]^{-1}$$

and taking $$\mu = 1$$, we have $$\frac{\omega}{v_g} = \sqrt{\varepsilon} + \frac{\omega}{2\sqrt{\varepsilon}} \frac{d\varepsilon}{d\omega}$$

Ignoring (for now) the imaginary part of $$\varepsilon$$, we see that in the region of anomalous dispersion ($$\frac{d\varepsilon}{d\omega} < 0$$) $$\frac{\omega}{v_g}$$ increases. Worse $$\varepsilon_1 < 0$$ so even if one neglects $$\frac{\omega}{2\sqrt{\varepsilon}}$$, the index of refraction $$\tilde{n}(\omega) = \sqrt{\varepsilon}$$ is purely imaginary so neither $$\tilde{v}_p$$ nor $$\tilde{v}_g$$ are well defined.

Sticking to the $$\mu = 1$$ case, generally

$$\tilde{n} = \tilde{n}_1 + i \tilde{n}_2 = \sqrt{\varepsilon_1 + i\varepsilon_2} \Rightarrow \tilde{n}^2 - \tilde{n}^2_0 = \tilde{\varepsilon}_1, \quad 2\tilde{n}_1 \tilde{n}_2 = \tilde{\varepsilon}_2$$

(Solve: $$\tilde{n}_1^2 - \tilde{\varepsilon}_1 - i\frac{\varepsilon^2_2}{\tilde{n}_1} = 0 \Rightarrow \tilde{n}_1 = \frac{1}{2}(\tilde{\varepsilon}_1 + \sqrt{\tilde{\varepsilon}_1^2 + \varepsilon^2_2}) = \frac{\tilde{\varepsilon}_1 + \sqrt{\tilde{\varepsilon}_1^2 + \varepsilon^2_2}}{2}$$)
Recall wave equation is:
\[
(V \times E - j \omega \vec{B} = 0, \quad V \times \vec{B} + j \omega \varepsilon(\omega) \vec{E} = 0)
\]
\[
(V^2 + \omega^2 \varepsilon(\omega)) \vec{E} = 0
\]

So \( e^{i(kz - \omega t)} \) for plane wave we have
\[
k^2 = \frac{\omega^2}{c^2} \varepsilon(\omega)
\]
\[
k = \frac{\omega}{c} \sqrt{\varepsilon} = \frac{\omega}{c} (\varepsilon_1 + i \varepsilon_2)
\]

So the region of anomalous dispersion, which coincides with non-negligible \( \varepsilon_2 \) and therefore \( \varepsilon_2 \), one has
\[
E_L \propto e^{i \omega \left( \frac{\varepsilon_1}{c} - t \right)} e^{-\frac{\varepsilon_2}{c} z}
\]

and
\[
\text{Intensity} \sim |E_L|^2 \propto e^{-2\delta / c} \quad \delta^{-1} = 2 \frac{\varepsilon_2}{c}
\]

where \( \delta \) is the penetration length.
Sec 13.4: To make sense of velocity of propagation particularly in the region of anomalous dispersion one may define “energy velocity” $\bar{v}$:

$$ \bar{S} = \bar{v} \bar{U} $$

with Poynting and internal energy defined as previously. Since we are interested in velocity of propagation, but not in attenuation along the wave we ignore absorption (set $\tilde{\alpha} = 0$ so $E^* = 0$). With $\tilde{\alpha} = 0$, $\tilde{S}$ is along $\tilde{E}$. Moreover $E' E^* = \tilde{E} \tilde{H}$ so

$$ \bar{S} = \frac{c}{16\pi} (\tilde{E} \tilde{H}^* + c.c.) = \frac{c}{16\pi} \left( \sqrt{E'} E' + \sqrt{H'} H' \right) $$

Now $\bar{U}$ was determined earlier, $\bar{U} = \frac{1}{8\pi} \left[ \frac{d}{d\omega} (\omega \tilde{E}^2) \tilde{E}^2 + \frac{\tilde{E}^*}{E} \right]$ so

$$ \bar{S} = \frac{c}{16\pi} \sqrt{\tilde{E}'} \tilde{E}^2, \quad \bar{U} = \frac{1}{16\pi} \left[ \frac{d}{d\omega} (\omega \tilde{E}^2) \tilde{E}^2 + \frac{\tilde{E}^*}{E} \right] \tilde{E}^2 \left( \text{He addition} \right) $$

or

$$ \frac{c}{\bar{v}} = \frac{d}{d\omega} \left( \omega \sqrt{\tilde{E}'} \right) = \frac{d}{d\omega} (\tilde{h}, \omega) = \frac{c}{V_g} $$

the group velocity.
In the region of anomalous dispersion one cannot neglect absorption. The above treatment fails. But for the harmonically-bound charges model one can compute explicitly.

See details in textbook. It shows $\frac{V_f}{c} \leq 1$.

(End Addendum)
Note on \( \tilde{M}(\omega) \)

For frequencies \( \omega > \omega_0 \) where \( \omega_0 \) is no higher than optical, but possibly lower, it makes no physical sense to distinguish between \( \tilde{H} \) and \( \tilde{B} \).

Recall
\[
\tilde{H} = u x \tilde{M} \quad \text{and} \quad \tilde{B}_{\text{bound}} = c \nabla \times \tilde{M} + \frac{\partial \tilde{P}}{\partial t}
\]

Under what condition is the 1st term bigger than 2nd?

Estimate
\[
|c \nabla \times \tilde{M}| \sim c \frac{1}{\text{length of variation}} \quad \chi_m B \sim \chi_{cb} \frac{B}{d}
\]

Also for \( \frac{\partial \tilde{P}}{\partial t} \), use the induced \( \tilde{E} \) field (it is magnetic; response we care about)
\[
\nabla \times \tilde{E} + \frac{1}{c} \frac{\partial \tilde{B}}{\partial t} \sim \tilde{E} \sim 1 \omega B/c
\]
and \( \tilde{P} \sim \chi E \sim E \quad |\frac{\partial \tilde{E}}{\partial t}| \sim 1 \omega B/c \)

So for \( |\frac{\partial \tilde{P}}{\partial t}| \ll c |\nabla \times \tilde{M}| \) we need \( d \omega B / c \ll \chi_m B / \omega \)

\[
\Rightarrow \quad l > \chi_m \frac{c^2}{\omega^2}
\]

Moreover, the dimensions of the body over which variations are considered, \( l \), should be much larger than atomic, \( l \gg a \).
Need to know some rough scaling of $\chi_m$.

For a diamagnetic material (sec. 102)

model atom as bound electron in circular orbit

- $m_m = \frac{1}{2}\hbar q \omega r^2$ (moment).

- $F=m_a : m_a \omega^2 r = \frac{2q^2}{r^2} + \frac{q\omega B}{c}$

- Central applied $B$ ($E = \frac{q}{c} \nabla \times B$)

- If $\omega_0 > F_0, B=0$ then $\omega_0^2 = \frac{2q^2}{mr^2}$

and $\omega^2 = \omega_0^2 + \omega_0 \omega_c$

where $\omega_0 \omega_c = \frac{q_0 B}{mc}$

but we work at small $B$ so set $\omega = \omega_0$ in Rits

$\Rightarrow \omega_c = \frac{qB}{mc}$ is the Larmor frequency.

So the change in $\chi_m$ due to the applied field is

$$\Delta \chi_m = \frac{2q^2\omega^2}{2c} \Delta \omega = \frac{q^2}{2c} \left(\frac{\omega_0}{2}\right)^2 = \frac{q^2}{4mc^2} \omega_0 \omega_c$$

Magnetization: $M = n \Delta \chi_m = \frac{q^2 n^2 r^2 B}{4mc^2} \Rightarrow \chi_m = \frac{q^2 n r^2}{4mc^2}$

Now $\frac{q^2}{m} = \omega_0 r^3$ and $r^3 n \leq 1$ so $\chi_m \approx \left(\frac{\omega_0 r^3}{c}\right)^2$
Added comment: there are two problems with the above (correct) argument

(i) we are taking \( r = \text{constant} \), but this is not guaranteed.

(ii) \( \vec{B} \) does no work but our higher \( \omega = \omega_0 + \omega_1 \) state has higher energy: \( \vec{B} \) as taken (1 to plane of orbit) does no torque, so \( \vec{L} = m \vec{r} \times \vec{v} = \text{constant} \), so \( mr^2 \omega = \text{constant} \), also not consistent with \( r = \text{constant} \).

The solution to this is that since \( \vec{B} \) increases, \( \vec{B} = \vec{B}(t) \) is not constant → \( \vec{E} \cdot \vec{B} = -\frac{1}{c} \frac{\partial B}{\partial t} \) \( \Rightarrow \vec{E} \) is indeed, does work and produces torque \( \Rightarrow r \) remains constant.

Let's check: increase \( \partial \vec{B} = \dot{B} \frac{\partial \vec{B}}{\partial t} \). Assume we current produced by circling electron produces a magnetic field \( \vec{B}_o \) in direction opposite \( \vec{B} \)

\[
\begin{align*}
\vec{B} & = \text{opposite the } e^+ \text{ current} \\
\end{align*}
\]

Then \( \vec{B} \cdot \vec{B}_o \) is along \( \vec{B}_o \) (by Gauss' law): \( \int \vec{E} \cdot dl = -\frac{1}{c} \int \frac{\partial \vec{B}}{\partial t} \cdot dl \)

\[ E 2\pi r = \frac{1}{c} \frac{\partial \vec{B}}{\partial t} \cdot \vec{B}_o \] Note that \( E 2\pi r = \text{work done on } e^- \)

and \( N = g F r = \frac{\phi}{2\pi} \frac{\partial \vec{B}}{\partial t} \) = torque on \( e^- \) \( \Rightarrow \vec{B}_o = N \dot{t} \).

The initial trajectory (on)

\[
\begin{align*}
E &= \frac{1}{2} m (\omega_0 r_0)^2 - \frac{2e^2}{r_0} \\
L &= m \omega_0 r_0^2
\end{align*}
\]
The final form has:

\[ \varepsilon_{\text{f}} = \frac{1}{2} m (\omega_n)^2 - \frac{q^2}{r^2} = \varepsilon_0 + \frac{q^2}{2} r_0^2 \frac{\Delta \varepsilon}{\Delta z} \]

\[ L = m^\prime r^2 \omega \]

\[ \omega_0 = \omega_0 (r_0^2 + 2 r_0 \Delta r) (\omega_n + \Delta \omega) = L_0 + \frac{2 m r_0 \omega_0 \sigma r + \frac{m^r}{r}}{\omega_0} \Delta \omega \]

\[ = L_0 + \frac{2 m r_0 \omega_0 \sigma r + \frac{m^r}{r}}{\omega_0} \Delta \omega \]

The Euler equation gives \( (F = m \omega) \)

\[ m \omega_r = g \frac{E}{c} = \frac{q}{c} \frac{d}{dt} B \Rightarrow \omega = \omega_n + \frac{q B}{2mc} \]

\[ \text{and} \]

\[ m \omega^2 r = \frac{2 \omega_n^2}{r} + \frac{q \omega_n B}{c} \]

\[ \text{with, this already gives} \]

\[ \Delta \omega = \frac{1}{2} \omega_n = \frac{g B}{2mc} \]

\[ m \omega (\omega_n - \omega_n) r = \frac{2 \omega_n^2}{r} \]

\[ \Rightarrow m (\omega_n - \frac{1}{2} \omega_n) (\omega_n + \frac{1}{2} \omega_n) r = \frac{2 \omega_n^2}{r} \]

Somewhere is wrong? Is the statement true only to linear order?

\[ \text{Also,} \quad \frac{d L}{dt} = N = g \varepsilon r = \frac{q r^2}{2c} \frac{d B}{dt} \]

\[ L = m^r \omega \]

\[ \frac{d L}{dt} = 2 m r \omega + m r^2 \omega = 2 m r \omega + \frac{r^2 q B}{2c} = \frac{q r^2}{2c} \frac{d B}{dt} \Rightarrow \omega = 0. \]

Clearly \( r = r_0 + 0 (\Delta \omega^2) \) is good enough for our purpose (linear response), but would be nice to figure out:

**END ADDENDUM**
Returning to the question of when the frequency response becomes relevant we had
\[ a^2 \ll \ell^2 \ll \chi_n \frac{c^2}{\omega^2} \] and now we know \( \chi_n \ll \left( \frac{\omega a}{c} \right)^2 \)

\( a = r = \text{atomic size} = \text{atomic separation} \); \( \omega \ll \) because we used \( n a^2 = \frac{1}{2} \), but for rare media \( n a^2 \ll 1 \).

\[ \text{Hence} \]
\[ \omega \ll \omega_0 \sim \text{optical frequencies} \]

is the condition for \( \tilde{u}(\omega) \) to make sense physically.

For optical frequencies and above (and possibly starting even below that, or due string of "\( \ll \)" and assumptions above shows) we may as well use \( \mu = 1 \) and keep track of \( c \hat{\nabla} \hat{\mu} + \frac{\partial p}{\partial t} \)

(dominated by \( \frac{\partial p}{\partial t} \)) through \( \hat{E}(\omega) \).

\[ \text{Note added: why keep } r \text{ fixed when } B \neq 0? \] Don't come up with a paradox and:
\[ E_0 = E_{\text{tot}}, \quad E_0 = \frac{1}{2} m v_0^2 - \frac{k_0}{r} = \frac{1}{2} m v_0^2 - \frac{2 q^2}{\epsilon r}, \quad \omega_0 = m v_0 / r = \frac{q B}{\epsilon} \]

\( m v_0 \hat{r} = m v_0 \hat{r}_0 \) and
\[ \frac{v^2}{2m} - \frac{2q^2}{\epsilon r} = \frac{v^2}{2m} - \frac{2q^2}{\epsilon r} + \frac{q \mu B}{2} \]

\( 2v, 2 \text{ unknown} \implies \)
**Quasistatic phenomena in conductors**

**Quasistatic Fields**

in time dependent field study $a \ll \lambda$ - wavelength of $\mathbf{B}$ or $\mathbf{E}$ field

$\omega a \ll c \quad \text{eg } \omega < \frac{c}{100 a} \text{ for } a = 1 \text{ cm}$.

Recall (Drude's model) $\tilde{\sigma}(\omega) = \sigma_0 \frac{1}{1 - \omega^2 \tau^2} \quad \left( \sigma_0 = \frac{n e^2}{m} \right)$

So for $\omega \ll \frac{1}{\tau}$, $\tilde{\sigma}(\omega) \approx \sigma_0 = \text{constant (independent of } \omega)$. For good conductors $\tau^{-1} \gg \frac{a}{c}$ (unless $a$ is tiny) so we will be in the regime where we can take $\tilde{\sigma}(\omega) = \sigma_0$.

Moreover, for good conductors $\sigma_0 \sim 10^8 \text{ Hz } \Rightarrow \omega \sigma_0 \ll c$.

The problem we want to solve is: put a conductor in an external time dependent magnetic field, $\mathbf{B}(t)$. What are the fields (both magnetic and electric) inside the conductor? Is there a resulting electric field outside the conductor? How is $\mathbf{A}$ modified outside the conductor? What currents are produced in the conductor?

That $a \ll \lambda$ = working in "near zone", so there are no retardation effects to worry about.
Typical situation: conductor placed inside coil generating $\vec{B}(t)$. Also conductor moving into (possibly constant) field $\vec{B}_0$.

Simplification of Maxwell's macroscopic equations:

\[
\nabla \cdot \vec{B} = 0 \quad \text{and} \quad \nabla \times \vec{E} + \frac{1}{c^2} \frac{\partial \vec{B}}{\partial t} = 0 \quad \text{stay the same}
\]

We want to use Faraday's law to give us $\vec{E}$ from $\vec{B}$. Since $\omega$ is small, we expect $|\vec{E}| \sim |\vec{B}|$.

Now $|\vec{B}| \sim |\vec{E}| \sim |\vec{H}|$ so $\frac{\partial \vec{B}}{\partial t} \sim \omega \vec{B} \sim \omega ^2 \vec{B}$ can be neglected in Ampere's law:

\[
\nabla \times \vec{H} - \frac{1}{c^2} \frac{\partial \vec{D}}{\partial t} = \frac{q}{\varepsilon} \vec{j} \quad \Rightarrow \quad \nabla \times \vec{H} = \frac{\mu_0}{\varepsilon} \vec{j}
\]

Note also that

\[
\frac{1}{c^2} \frac{\partial \vec{D}}{\partial t} \sim \frac{\omega}{c} \vec{E} = \frac{\omega}{c} \vec{E} \varepsilon \vec{j} \ll \frac{\mu_0}{\varepsilon} \vec{j}
\]

Using Ohm's law

\[
\Rightarrow \quad \nabla \times \vec{H} = \frac{\mu_0}{\varepsilon} \vec{j} \quad \Rightarrow \quad \nabla \times \vec{H} = \frac{\mu_0}{\varepsilon} \alpha \vec{E}
\]

Now $\nabla \cdot (\mu_0 \alpha \vec{E}) = \nabla \cdot (\nabla \times \vec{H}) = 0 \quad \Rightarrow \quad \nabla \cdot \vec{E} = 0$

and with $\nabla \cdot \vec{D} = \rho$ \quad $\Rightarrow$ \quad $\rho = 0$

\Rightarrow No free charges in bulk of conductor, just as in electrostatics.
This is not a surprise: we are taking $\omega = 0$ in Maxwell equations for conductors.

**Summary:**
\[
\nabla \cdot \mathbf{B} = 0 \quad \nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0} \\
\nabla \times \mathbf{H} = \frac{\mu_0}{c} \mathbf{E} \quad \nabla \cdot \mathbf{E} = 0
\]

Take $\nabla \times (\nabla \times \mathbf{E}) = -\nabla \mathbf{H}$
\[
\begin{align*}
\omega &= \nabla \times (\frac{\mu_0}{c} \frac{\partial \mathbf{E}}{\partial t}) = \frac{\mu_0}{c} \left( -\frac{1}{c} \nabla \times \mathbf{B} \right) = -\frac{\mu_0 \sigma_0}{c} \frac{\partial \mathbf{H}}{\partial t} \\
\Rightarrow \quad \nabla \times \left( -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \right) &= -\frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \nabla \cdot \mathbf{H} = -\frac{\mu_0 \sigma_0}{c} \frac{\partial \mathbf{E}}{\partial t}
\end{align*}
\]

Alternatively
\[
\nabla \times (\nabla \times \mathbf{E}) = -\nabla \mathbf{H}
\]

\[
\begin{align*}
\nabla \mathbf{E} &= \frac{\mu_0 \sigma_0}{c} \frac{\partial \mathbf{E}}{\partial t} \\
\nabla \mathbf{H} &= -\frac{\mu_0 \sigma_0}{c} \frac{\partial \mathbf{E}}{\partial t}
\end{align*}
\]

**Discussion:**

Each component of $\mathbf{H}$ and $\mathbf{E}$ satisfies the diffusion or heat conduction equation,
\[
\nabla^2 \psi = \kappa \frac{\partial \psi}{\partial t}
\]

If $\psi(x, t) = \psi(x, t)$ only then $\frac{\partial^2 \psi}{\partial x^2} = \kappa \frac{\partial \psi}{\partial t}$

To solve this let $\psi(x, t) = \int \frac{dk}{2\pi} \hat{\psi}(k, t) e^{ikx} \Rightarrow -k^2 \hat{\psi} = \kappa \frac{\partial \hat{\psi}}{\partial t} \Rightarrow \hat{\psi} = \hat{\psi}_0 e^{-\frac{k^2}{\kappa} t}$

\[
\begin{align*}
\Rightarrow \psi(x, t) &= \int \frac{dk}{2\pi} \hat{\psi}_0 e^{ikx} e^{-\frac{k^2}{\kappa} t} \\
&= \hat{\psi}_0 e^{-\frac{\kappa x^2}{4\kappa t}} \int_{-\infty}^{\infty} \frac{e^{-\frac{k^2}{4\kappa} t}}{2\pi} dk = \hat{\psi}_0 \left( \frac{\sqrt{\pi\kappa}}{2} \right) e^{-\frac{x^2}{4\kappa t}} \quad \text{[I have absorbed a constant into $\hat{\psi}_0$.]}
\end{align*}
\]

Check:
\[
\frac{\partial^2 \psi}{\partial t^2} = \psi \frac{\kappa}{2\pi} \left( -\frac{\kappa}{8\kappa} e^{-\frac{\kappa x^2}{4\kappa t}} \right) = \psi_0 \frac{\kappa}{4\pi} \left( -\frac{\kappa}{8\kappa} + \frac{\kappa}{4\kappa} \right) e^{-\frac{\kappa x^2}{4\kappa t}}
\]

\[
\kappa \frac{\partial \psi}{\partial t} = \kappa \psi_0 \left( -\frac{\kappa}{8\kappa} + \frac{\kappa}{4\kappa} \right) e^{-\frac{\kappa x^2}{4\kappa t}} = \psi_0 \frac{\kappa}{4\pi} \left( -\frac{\kappa}{8\kappa} + \frac{\kappa}{4\kappa} \right) e^{-\frac{\kappa x^2}{4\kappa t}}
\]

2020-06-04 11:05:15
3D case: using $\psi (r,t) = X(x) Y(y) Z(z)$

$$\frac{1}{x} x'' + \frac{1}{y} y'' + \frac{1}{z} z'' = K \left( \frac{1}{x} \dot{X} + \frac{1}{y} \dot{Y} + \frac{1}{z} \dot{Z} \right)$$

and

$$\frac{1}{x} x'' = K \left( \frac{1}{x} \dot{X} + f_x (t) \right) \text{ et al.} \quad \text{with} \quad f_x (t) + f_y (t) + f_z (t) = 0$$

For example, if $f_x (t) = 0$, we have the taut string case

$$\psi = \psi_0 \frac{1}{E \varepsilon_0} \exp \frac{-r^2}{4 \alpha^2}$$

These well known solutions are appropriate for diffusion: as $t \to 0+$

$$\psi (r,t) \to \delta (r) \quad \text{and} \quad \psi (r,t) \to \delta (r)$$

with a clear interpretation: put a pointlike "drop" of fluid and it diffuses out, with distance $E$.

In the cases we study the problem is different. Imagine starting with a field $\psi_0 (r)$ at $t = 0$ (say an electric field that is turned off). What happens next? To this end solve the eigenvalue problem

$$\nabla^2 \psi_n (r) = - \chi_n \psi_n (r) \quad n = 1, 2, \ldots$$

Then $\psi (r,t) = \sum \psi_n (r) e^{-i \omega_n t}$ solves $\nabla^2 \psi = K \frac{\partial \psi}{\partial t}$

and we can choose $\psi_0 (r)$ so that $\psi (r,t) = \sum \psi_n (r)$

(As usual, with eigenfunctions $(\psi_n, \psi_m) = 0$ if $n \neq m$ so one can orthonormalize the solutions so $\psi_n = (\psi_n, \psi_0)$).

The important point is that $\psi_n$ dies exponentially within a time $\tau \sim k \chi_n^{-1}$ (assuming $\chi_n < \chi_0$).

Since we expect $\tau \sim \alpha^2 (t)$, the typical decay time is $\tau \sim e^{-\frac{a^2}{\alpha^2}} = \frac{\alpha^2}{a^2}$

which for $a = 3 \mu m$ and $\alpha^2 = 10^5$ sec$^{-1}$ gives $\tau \sim 10^{-3}$ sec.
Boundary conditions: (to solve problem fully)

Assume boundaries are between conductor and vacuum.

\[ \nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0 \implies \mathbf{E}_{\text{in}} = \mathbf{E}_{\text{out}} \]

\[ \nabla \times \mathbf{H} = \mu_0 \sigma_0 \mathbf{E} \implies \mathbf{H}_{\text{in}} = \mathbf{H}_{\text{out}} \]

\[ \nabla \cdot \mathbf{B} = 0 \implies \mathbf{B}_{\text{out}} = \mathbf{B}_{\text{in}} \]

\[ \text{Note that with } m = 1 \text{ this means } \mathbf{H}_{\text{in}} = \mathbf{H}_{\text{out}} \text{ (or } \mathbf{B}_{\text{in}} = \mathbf{B}_{\text{out}}) \]

left with \( E_n \) ? L. L. says: \( \nabla \cdot \mathbf{J} = 0 \) and \( \mathbf{J}_{\text{out}} = 0 \Rightarrow J_{\text{in}} = 0 \)

and since \( E_{\text{in}} = \sigma \mathbf{J}_{\text{in}} = E_{\text{in}} = 0 \).

Digression: Gaug sets a more refined version. From the previous unit, we had

\[ \nabla \cdot \mathbf{E} = \rho' \quad \text{where} \quad \mathbf{E} = \mathbf{\tilde{E}} \quad \text{and} \quad \mathbf{\tilde{E}} = \mathbf{\tilde{E}} + i \frac{\omega_0 \mathbf{\tilde{E}}}{c} \]

and \( \rho' \) are charges from currents not subject to Ohm’s law (ie, not included in \( E = \sigma \mathbf{J} \)). From this \( \mathbf{E}_{\text{out}} = \mathbf{\tilde{E}}_{\text{in}} \) (for \( \rho' = 0 \)).

From this we recover \( E_{\text{in},n} = 0 \) (that is \( E_{\text{in},n} \sim \frac{\omega_0}{c} E_{\text{out}} \to 0 \) in the approx).

If \( \Sigma = \text{surface charge density} \) (use \( \Sigma \) rather than \( \sigma \) to avoid confusion with conductivity), then \( \mathbf{Z}_{\text{in}} - \mathbf{Z}_{\text{out}} = \frac{i \omega_0}{c} \mathbf{\Sigma} \) (sign from \( \mathbf{H} = \text{rightward pointing} \)).

Then, using \( E_{,n} = -\frac{\omega_0}{c} E_{,n} \), \( \mathbf{E}_{\text{out}} = \mathbf{E}_{\text{out}} (1 - i \frac{\omega_0}{c}) = \mathbf{\Sigma} \Rightarrow \mathbf{E}_{\text{out}} = (\mathbf{\Sigma} + i \frac{\omega_0}{c}) \mathbf{\Sigma} \)

while \( E_{\text{in},n} = -i \frac{\omega_0}{c} E_{\text{in},n} \Rightarrow E_{\text{in},n} = -i \frac{\omega_0}{c} \mathbf{\Sigma} \)
So where are we going with all this?

Put a conductor in an external quasistatic field \((\mathbf{E}, \mathbf{H})\).

From the diffusion/heat transfer equation we expect the fields will not penetrate the conductor much. For \(\mathbf{E}\) it is clear, much like in electrostatic case, charge at surface will screen. But now the charge is spread over some “skin depth” \(d\) fixed by diffusion equation.

For magnetic field to be screened we need a current on the surface down to depth \(d\). Since \(\mathbf{J} = \sigma \mathbf{E}\) and \(\mathbf{J} = 0\) at boundary, we will have an \(\mathbf{E}_{\text{in}}\), but then \(\mathbf{E}_{\text{out}} = \mathbf{E}_{\text{in}}\), so also outside.

So we want to understand the skin depth and these currents called eddy currents.
Once we look at these in general terms, we can look at specific cases. Garg shows two geometries of conductors in time-varying (harmonic) external $\mathbf{H}$: cylinders and sphere.

Qualitatively:

\[ \text{With } \mathbf{E}_{\text{tot}} \parallel \mathbf{E} \text{ which is } \perp \mathbf{H} \]
We also want to understand energy conservation.

We see (above pic’s)

\[ \nabla \cdot \mathbf{E} = \sigma \mathbf{E} \]

There is energy flow into the conductor. Where does it go?

There is also energy dissipation, from \( \nabla \cdot \mathbf{E} = \sigma \mathbf{E} \) in the conductor.

The energy flux flows in = energy dissipated.
Plane conductor

While $a \to \infty$ is outside the regime we are studying, we can look at a plane conductor as a local approximation of a large but finite size conductor.

Take $a = \infty$. Set boundary of conductor on $xy$ plane, and conductor on $z > 0$.

Assume $\mathbf{B} (= \mathbf{H}) = B_x \hat{x}$ (along plane; $e^{-j\omega t}$ dependence implicit) for $z = 0^-$.

$\Rightarrow \mathbf{B} = B_x \hat{x}$ for $z = 0^+$ because $\mathbf{B}_n = \mathbf{B}_t$.

The diffusion equation is

$$\nabla^2 \mathbf{B} = -\frac{j \mu_0 \sigma_0 \omega}{c^2} \mathbf{B}$$

For $B_y$, with $B_z = 0$ at $z = 0$ gives $B_y = 0$.

For $B_x$, we look for a solution $B_x(t)$ depends on $z$ only, $B_x = B_x(z) e^{ikz}$.

$$\Rightarrow k^2 = \frac{j \mu_0 \sigma_0 \omega}{c^2} = k = \pm \sqrt{\frac{\mu_0 \sigma_0 \omega}{c^2}}$$

$\omega \mu_0 \sigma_0 \left(e^{j\phi_t} / 
\left(\phi_t\right) = e^{j\phi_t} / 
\left(\phi_t\right) = \frac{1}{\epsilon_0} (1 + i) \quad \text{if } k = \pm (1 - i) \sqrt{\frac{\mu_0 \sigma_0 \omega}{c^2}}$.

This gives $e^{\pm i(1 - i) \sqrt{\mu_0 \sigma_0 \omega / c^2}}$: The $+$ sign solution gives $B_x$ increasing with $z$, which is unphysical. So keep only $+$ sign:

$$B_x = B_x e^{-(1 - i) \sqrt{\frac{\mu_0 \sigma_0 \omega}{c^2}}} = B_x e^{-(1 - i) \frac{2 \delta}{c}}$$

where $\delta = \frac{\sqrt{\mu_0 \sigma_0 \omega}}{c}$ is the "skin depth".

For $\omega \epsilon > 300 \mu$, $\delta(60 \text{ kHz}) \approx 8.5 \text{ mm}$, $\delta(100 \text{ MHz}) = 7 \text{ mm}$.
From $\nabla \times \mathbf{H} = -\frac{\mu_0 \sigma_0}{c} \mathbf{E}$ we can compute $\mathbf{E}$:

$$\varepsilon_{ijk} \frac{\partial}{\partial x_k} B_i = \varepsilon_{ijk} \frac{2 B_k}{c} = \delta_j \left( \frac{\partial}{\partial x} \left( B_0 e^{-(1-i) \omega t} \right) \right) = \delta_j \left[ -\left( 1-i \right) \frac{i}{\delta} B_0 e^{-(1-i) \omega t} \right]$$

$$\Rightarrow \mathbf{E} = -\frac{c B_0}{\mu_0 \omega \delta} (1-i) \mathbf{\hat{y}} e^{-(1-i) \omega t} = E_y \mathbf{\hat{y}} e^{-(1-i) \omega t}$$

where $E_y = -\frac{c B_0}{\mu_0 \omega \delta}$

Writing the fields as real part of and restoring $\omega$-dependence we discover phase shift:

$$\mathbf{B} = B_0 \mathbf{\hat{x}} e^{-2i\gamma} e^{\omega t} \left( \frac{3}{8} - \frac{c}{\omega} \right)$$

$$\mathbf{E} = -\left( c E_y \mathbf{\hat{y}} e^{2i\gamma} \cos \left( \frac{3}{8} - \omega t - \frac{c}{\omega} \right) \right)$$

(The phase shift is from $1-i = \sqrt{2}e^{-i\pi/4}$.)

In addition $\mathbf{H} = \mathbf{\sigma}_0 \mathbf{E}$ is now determined. Note that for $\delta \ll a$ the current is confined to the "surface" of the conductor, and can be modeled by a surface current density $\mathbf{K} = \int_{\partial \Omega} \mathbf{J} = \sigma_0 \mathbf{E}_0 \mathbf{\hat{y}}_0 \left( \frac{5}{12} \right) = -\frac{c B_0}{4 \pi} \mathbf{\hat{y}}_0$

In $K = -\frac{c B_0}{4 \pi}$ there is no $\sigma_0$. $K$ is there to shield $\mathbf{B}$:

In the naive approach one has, from Ampere's law

$$\int_{\partial \Omega} \mathbf{J} \cdot \nabla \times \mathbf{B} = \frac{\mu_0}{c} \int \mathbf{J} \cdot d\mathbf{S}$$

out

$$\int_{\partial \Omega} \mathbf{J} \cdot d\mathbf{S} = (B_{ox} - B_{ox}) I = \frac{\mu_0}{c} \mathbf{K}$$

$$\Rightarrow B_{ox} - B_{ox} = \frac{\mu_0}{c} \mathbf{K}$$

We have $K = -\frac{c B_0}{4 \pi}$ so it must be that $B_{ox} = 0$. In this naive approximation $B_{ox}$ corresponds to the "true" $B_{ox}$ at $\delta \gg \delta$, hence vanishingly small.
Define "surface impedance" \( Z_s : \overline{E} = Z_s \overline{K} \) 

So here in the present case \( Z_s = \frac{(1-i)}{\sigma_0} \).

Note also that, as expected \( |\overline{E}|/|\overline{0}| << 1 \):

\[
\begin{align*}
\frac{|\overline{E}|}{|\overline{0}|} = \frac{\xi}{\eta_0 \sigma_0} \frac{\sqrt{\xi}}{\eta_0 \sigma_0} = \frac{\xi}{\eta_0 \sigma_0} \frac{\sqrt{\eta_0 \sigma_0}}{\eta_0 \sigma_0} = \frac{\omega}{\eta_0 \sigma_0} << 1
\end{align*}
\]

**Energetics:** Compute \( \overline{S}_{out} / \overline{E}_{in} = \overline{E}_{tot} \) gives \( E \) outside conductor 

\( (E_{n,\text{out}} = \nabla E_{n,\text{in}} = 0) \). Then

\[
\begin{align*}
\overline{S} &= \frac{\varepsilon_0}{\eta_0} (\overline{E} \times \overline{B}) = \frac{\varepsilon_0}{\eta_0} \left( E_0 B_0 \hat{y} \times \hat{x} \right) = \frac{\varepsilon_0}{\eta_0} E_0 \left( -\frac{\eta_0 \sigma_0 d E_0}{c(1-\lambda)} \right) (-\hat{z}) \\
&= \frac{\sigma_0 \delta}{1-\lambda} E_0^2 \hat{z}
\end{align*}
\]

**Brief review of averaging over time:** Complex fields act on reality

\[
\frac{1}{T} (a e^{i\omega t} + a^* e^{-i\omega t})
\]

Then \( \overline{ab} = \frac{1}{T} \int_0^T \frac{1}{T} (a e^{i\omega t} + c.c.) (b e^{-i\omega t} + c.c.) \)

\[
= \frac{1}{T} (ab^* + a^*b) = \frac{1}{T} \text{Re}(ab^*)
\]

Time average \( \overline{S} : \)

\[
\overline{S} = \frac{1}{T} \text{Re} \left( \frac{\sigma_0 \delta}{1-\lambda} E_0 \xi \hat{z} \right) = \frac{1}{T} \frac{\sigma_0 \delta}{1-\lambda} E_0^2 \hat{z}
\]
Let's compare with the energy dissipated. Work done per unit volume per unit time: $\mathbf{J} \cdot \mathbf{E}$. Time averaged: $\frac{1}{2} \text{Re}(\mathbf{J} \cdot \mathbf{E}^*)$.

\[ \frac{d\mathcal{W}}{dA} = \int_0^\infty \frac{1}{2} \text{Re}(\mathbf{J} \mathbf{E}^*) \, dA \]

$\text{work done in volume: } \frac{1}{2} \text{Re}(\mathbf{J} \cdot \mathbf{E}^*)$  

$\text{work done / unit surface area}$

$\frac{d\mathcal{W}}{dA} = \int_0^\infty \frac{1}{2} \text{Re}(\mathbf{J} \mathbf{E}^*) \, dA$

$\mathbf{E}^* = E_o \mathbf{y} e^{i(-\omega)\sqrt{s}}$  

$\frac{1}{2} E_o \mathbf{y} \int_0^\infty e^{-2\sqrt{s}} \, ds$

$= \frac{1}{4} E_o \delta(E^2)$

Same as $|\mathbf{E}|^2$. Energy flows in = energy dissipated.  

Note we can also work

\[ Z_s = \frac{\sigma_s}{\omega_0} = \frac{\sigma_s}{2} \text{(in)} \]

So

\[ \frac{d\mathcal{W}}{dA} = \frac{1}{2} \text{Re}(\mathbf{E}^*) |E_o|^2 = \frac{1}{2} \text{Re}(\mathbf{E} \cdot \mathbf{E}^*) = \frac{1}{2} \text{Re}(\mathbf{E} \cdot \mathbf{E}^*) \]

\[ \text{Corresponding } |\mathbf{E}|^2 = \frac{\sigma_s}{\omega_0} \frac{1}{2} \mathbf{E} \cdot \mathbf{E} = \frac{1}{4} 2 \left( \frac{E_o}{q} \right)^2 \frac{1}{\sigma_s} \text{Re}(\mathbf{E} \cdot \mathbf{E}^*) = \frac{1}{4} \left( \frac{E_o}{q} \right)^2 \text{Re}(\mathbf{E} \cdot \mathbf{E}^*) \]

$\mathbf{E}$ outside conductor? (ie for $z < 0$).

As we said in the introduction, it is given by Faraday's law

\[ \nabla \times \mathbf{E} - \frac{1}{\mu_0} \frac{d\mathbf{B}}{dt} = \mathbf{0} \]

with b.c. $\mathbf{E}(z=0^-) = E_o \mathbf{y}$
By symmetry $\vec{E} = \hat{y} E_y z$ only, and recall $\vec{B} = \hat{x} B_0$.

So $(\vec{\nabla} \times \vec{E})_x = -\frac{\partial E_y}{\partial x} = \frac{i \omega}{c} B_0$

$\Rightarrow E_y = E_0 - \frac{i \omega}{c} B_0 e^2 \quad (\text{disagree with sign in rear}).$

$\lambda \omega = \frac{c}{2\pi}$

$E_y(z) \approx E_0 - \frac{i \omega}{c} (\lambda) B_0$

So $E_y(z)$ seems to increase without bounds as $z \to -\infty$. But this is not so; we are assuming distance scales are $\ll \lambda$. Thus the solution is a good approximation only close to the conductor. For example, if conductor is inside a shield, $z \to -\infty$ will take us outside this region. Moreover, as $|E|$ increases, it cannot be neglected in Ampere’s law.

$\delta(\omega) = \frac{C}{\sqrt{2\pi}} \sim \frac{1}{\lambda} \to \infty \text{ as } \omega \to 0.$ It would appear that in static case $E^2$ penetrates the whole conductor? But wait

$|\vec{E}|/|\vec{B}| = \sqrt{\frac{\omega}{\eta_0}} \to 0 \text{ as } \omega \to 0.$ So there is no field.
\[ \mathbf{H} = H_0 \hat{z} \]

Now suppose the external applied field is perpendicular to the surface \( z = 0 \), \( \mathbf{H}_0 = H_0 \hat{z} \)

\[ H_x, \ H_y \to \mathbf{H}_0 \]

We want to find \( \mathbf{H} \) inside. But importantly, \( \mathbf{H} \) outside is not \( \mathbf{H}_0 \). Still as a 1st guess we can assume \( \mathbf{H} = \mathbf{H}_0 \) at \( z = 0 \) - (just outside) \( \Rightarrow \mathbf{H}_m = \mathbf{H}_0 \) means \( \mathbf{H}(z = 0+) = H_0 \hat{z} \). Let’s use the diffusion equation to find \( \mathbf{H} \) (\( \nabla \times \mathbf{B} = \mathbf{B} \) since we are using \( \mu = 1 \)): but this is the same as before, except for \( H_x = B_x \) instead of \( B_x \): 

\[ H_x = H_0 \hat{z} e^{-(x^2+y^2)/\delta} \]

The problem here is \( \nabla \cdot \mathbf{B} = \mu \nabla \times \mathbf{H} = 2\pi B_x \neq 0 \) in violation of \( \nabla \cdot \mathbf{B} = 0 \).

To understand what happens we cannot continue to take an infinite plane approximation to the finite size body, of size \( a \ll \delta \). A field \( \mathbf{H}_m \) must be produced that is parallel to the surface (Gary calls it \( B_z \) because it is perpendicular to \( H_0 \) - I find his nomenclature confusing). With this one can satisfy \( \nabla \cdot \mathbf{B} = 0 \) since \( \nabla \cdot \mathbf{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_x}{\partial y} \).

Note that \( \mathbf{B} \) is then confined to a region of depth \( \delta \) in the conductor.
Let's assume $\delta < a$ (the opposite limit $\delta \gg a$ is basically that of $a = 0$, i.e., magnetostatics). Then, in order to shield the bulk of the conductor from $\vec{B}$ we need a current $J$ in the skin. What breaks the symmetry in the xy plane if $\vec{H}_0 \neq H_0 \hat{z}$, i.e., is $H_0$ along $\hat{x}$ or $\hat{y}$? The answer is the finite size, as is easily seen from the picture:

And, of course that means here is an $E$ field ($E = \frac{1}{\epsilon} \vec{D}$). Note that $H_0$ changes on scale of curvature, which itself is the scale of the size of the body, while $H_0$ changes over scale $\delta$.

Since $\nabla \cdot \vec{B} = 0$ we have $H_0 = \frac{\mu}{\delta} A_0$, or $H_0 = \frac{\mu}{\delta} H_0 \Rightarrow H_0$: The $H$ field dominates and again $H_{\text{in}} = H_{\text{out}}$ means now that close to the body $\vec{H}$ is nothing like the uniform applied $\vec{H}_0$ — on surfaces that are not parallel to $\vec{E}_0$. 
One can see this explicitly in analytic solutions of the cylinder and sphere problems (those shown in p. 7 of these notes), but we will not go through those calculations. The general principle is what we are after, and what is enough to figure out generally what happens in other geometries, e.g.,

\[ \vec{H} \]

\[ \vec{H}_0 \]