Maxwell equations in media (Garg: Chap 13, Sec 81).

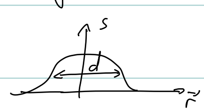
Model media as made up of charges that are fixed (as in molecules, which may not be neutral as they may have deficit or excess of electrons) plus free charges (as in conduction electrons in conductors, or added charges in insulators).

Denote by $\vec{e}$ & $\vec{b}$ the microscopic fields, ie, the fundamental fields. These change over atomic distance scales. So we have

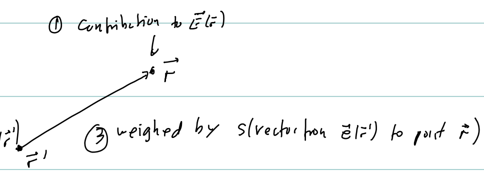
$$\vec{\nabla}\cdot\vec{e} = 4\pi\rho_{micro} \quad \nabla\times\vec{e} + \frac{1}{c}\frac{\partial\vec{b}}{\partial t} = 0 \quad \vec{\nabla}\times\vec{b} - \frac{1}{c}\frac{\partial\vec{e}}{\partial t} = \frac{4\pi}{c}\vec{j}_{micro} \quad \vec{\nabla}\cdot\vec{b} = 0$$

Now, smooth these out over "macroscopic" distances (where "macroscopic" depends on context, but can be as short as $\sim \times 10$ atomic distance, say). To this end use a smoothing (averaging) function $s(\vec{r})$: we want

$$\int d^3r\, s(\vec{r}) = 1$$

and  with $d \gg$ typical fast variation of $\vec{e}$ & $\vec{b}$, on "atomic" scale

Then let $\vec{E}(\vec{r},t) = \int d^3r'\, s(\vec{r}-\vec{r}\,')\, \vec{e}(\vec{r}\,',t)$



and $\vec{B}(\vec{r},t) = \int d^3r'\, s(\vec{r}-\vec{r}\,')\, \vec{b}(\vec{r}\,',t)$

Now $\partial_i E_j = \int d^3r'\, \partial_i s(\vec{r}-\vec{r}\,')\, e_j(\vec{r}\,',t) = \int d^3r' (-\partial_i' s) e_j = \int d^3r'\, s\, \partial_i' e_j$ (int. by parts + s has local support)

So all diff ops go through and:

$$\vec{\nabla}\cdot\vec{E} = 4\pi\langle\rho\rangle_{micro} \quad \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0 \quad \vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\langle\vec{j}\rangle_{micro} \quad \vec{\nabla}\cdot\vec{B} = 0$$

where $\langle(\cdots)\rangle = \int d^3r'\, s(\vec{r}-\vec{r}\,')(\cdots)$ as above.

Now task is to break $\langle\rho\rangle$ & $\langle\vec{j}\rangle$ into pieces in a useful/convenient way.

Text discusses nuances of physics at atomic scales (requires QM). This is one of those cases that "it's not relevant, except when it is". So ignore for now and deal with QM as needed.

(Sec 82): ρ and $\vec{P}$.

ρ_{micro} → charges bound to atoms/molecules

ρ_{micro} → charges not bound to atoms/molecules → conduction e's in metals; externally supplied charge in insulators; other ...

Bound charge: molecules are neutral; effect at distances large compared to size ⟶

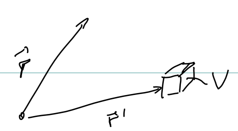
multipole expansion ⟶ usually dipole $\vec{d}$ suffices. BOTH for its field and its response to applied fields.

let $\vec{P}$ = dipole moment/volume, a local quantity (ie $\vec{P} = \vec{P}(\vec{r})$)

in this δV, $\vec{P} = \dfrac{\sum_{\text{molecules}} \vec{d}}{\delta V}$: δV is large enough to smooth out atomic scale fluctuations, yet small enough that multipole expansion makes sense.

Alternatively, text says $\vec{P} = n d$ $\quad n = \#$ density

Now $\vec{P} \Rightarrow$ charge distribution. Recall $\phi(\vec{r}) = \dfrac{\vec{d}\cdot\vec{r}}{r^3}$ from dipole at origin.



$$\Rightarrow \quad \phi(\vec{r}) = \int_V (d^3r' \vec{P}(\vec{r}')) \cdot \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

Use $\vec{\nabla}' \dfrac{1}{|\vec{r}-\vec{r}'|} = \dfrac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$, integrate by parts, keep surface term:

$$\Rightarrow \quad \phi(\vec{r}) = \int_{\partial V} \frac{\vec{P}(\vec{r}')\cdot\hat{n}'}{|\vec{r}-\vec{r}'|} d^2s - \int_V \frac{\vec{\nabla}'\cdot\vec{P}(\vec{r}')}{|\vec{r}-\vec{r}'|} d^3r'$$

$\Downarrow$ $\vec{P}\cdot\hat{n} = \sigma_{pol}$ $\qquad$ $\Downarrow$ $\vec{\nabla}\cdot\vec{P} = -\rho_{pol}$

σ_{pol} is cancelled by that of adjacent volume  . Not so for surface of material.

cancelling σ_{pol} because $\hat{n}_1 = -\hat{n}_2$

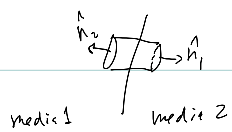
So break $\langle \rho_{micro} \rangle$ into $\rho_{pol} + \rho_{free}$

$$\Rightarrow \quad \vec{\nabla}\cdot\vec{E} = 4\pi(\rho_{pol} + \rho_{free}) = 4\pi(-\vec{\nabla}\cdot\vec{P} + \rho_{free})$$

$$\Rightarrow \quad \vec{\nabla}\cdot(\underbrace{\vec{E} + 4\pi\vec{P}}_{\text{"}\vec{D}\text{" electric displacement}}) = 4\pi\rho_{free} \quad \Rightarrow \quad \boxed{\vec{\nabla}\cdot\vec{D} = 4\pi\rho_{free}}$$

with this definition then also

$$\vec{E}_2\cdot\hat{n}_1 + \vec{E}_1\cdot\hat{n}_2 = (\vec{E}_2 - \vec{E}_1)\cdot\hat{n}_{21} = 4\pi(\sigma_{pol} + \sigma_{free}) = 4\pi(\vec{P}_1\cdot\hat{n}_1 + \vec{P}_2\cdot\hat{n}_2 + \sigma_{free})$$



$(\hat{n}_1 = -\hat{n}_2 \equiv \hat{n}_{21})$

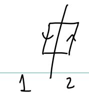
$$\Rightarrow \left[(\vec{E}_2 + 4\pi\vec{P}_2) - (\vec{E}_1 + 4\pi\vec{P}_1)\right]\cdot\hat{n}_{21} = 4\pi\sigma_{free}$$

$$\boxed{(\vec{D}_2 - \vec{D}_1)\cdot\hat{n}_{21} = 4\pi\sigma_{free}}$$

(We used this last quarter in wave propagation in media).

To complete boundary conditions at interfaces, $\vec{\nabla}\times\vec{E} = 0$

$$\Rightarrow \oint \vec{E}\cdot d\vec{l} = 0 \quad \Rightarrow$$



$\vec{E}_{1t} = \vec{E}_{2t}$ "t" = tangential.

Notes:

- $\vec{D}$ is not sourced by ρ_{free}: in addition to $\vec{\nabla}\cdot\vec{D} = 4\pi\rho_{free}$ we have $\vec{\nabla}\times\vec{D} = \vec{\nabla}\times(\vec{E} + 4\pi\vec{P}) = 4\pi\,\vec{\nabla}\times\vec{P}$. That is, we have to look at $\vec{D}$, $\vec{E}$ and some way of determining $\vec{P}$ to get the whole picture.
- Dimensions & Units: in Gaussian things are simple and make sense $\vec{D}$, $\vec{E}$ and $\vec{P}$ have same units. In SI $\vec{D} = \epsilon_0\vec{E} + \vec{P}$, so $\vec{D}$ has unit of $\vec{P}$ $(= [Q][L]^{-2})$, different than $\vec{E}$.

  See text for translation (or work it out).

(Sec 83): Macroscopic current density

Break into components:

$$\langle \vec{J}_{micro} \rangle = \vec{J}_{free} + \vec{J}_{pol} + \vec{J}_{conv} + \vec{J}_{mag}$$

$\vec{J}_{free}$ is from motion of ρ_{free} so $\frac{\partial \rho_{free}}{\partial t} + \vec{\nabla}\cdot\vec{J}_{free} = 0$

$\vec{J}_{pol}$ is from t-dependence on $\vec{P}$: $\rho_{pol} = -\vec{\nabla}\cdot\vec{P} \Rightarrow \frac{\partial \rho_{pol}}{\partial t} = -\vec{\nabla}\cdot\frac{\partial \vec{P}}{\partial t} \Rightarrow \vec{J}_{pol} = \frac{\partial \vec{P}}{\partial t}$

The convection current $\vec{J}_{conv}$ (gases & liquids only), charges carried by overall motion of fluid. Artificial breakdown, but useful. If $\vec{v}$ is fluid velocity field

$$\vec{J}_{conv} = (\rho_{free} + \rho_{pol})\,\vec{v}$$

Then one has to make sure of no double counting so this is subtracted from $\vec{J}_{free}$ & $\vec{J}_{pol}$ which then measure current in a comoving fluid element.

<u>Most interesting</u>: $\vec{J}_{mag}$ magnetization current.

$\vec{M}(\vec{r})$ = magnetic dipole moment / volume.

Let's recall field due to magnetic dipole: for this need:

Brief review of magnetostatics: recall $\partial^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) = \frac{4\pi}{c} J_\nu$. Spatial components steady state: $-\vec{\nabla}^2\vec{A} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) = \frac{4\pi}{c}\vec{J}$, choose $\vec{\nabla}\cdot\vec{A} = 0$ gauge, and

$$\vec{A} = \frac{1}{c}\int \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3r'$$

Magnetostatics: $\frac{\partial \rho}{\partial t} = 0 \Rightarrow \vec{\nabla}\cdot\vec{J} = 0$.

Multipole expansion:

$$\vec{A}(\vec{r}) = \frac{1}{c}\int d^3r' \; \vec{J}(\vec{r}') \left[\frac{1}{|r|} + \frac{\vec{r}'\cdot\vec{r}}{|\vec{r}|^3} + \cdots \right]$$

Trick: With $\vec{J}$ localized, given two functions $f(\vec{r})$ & $g(\vec{r})$ we have

$$\int d^3r \; \vec{\nabla}(fg\vec{J}) = 0 \quad \Rightarrow \quad \int d^3r \left(f\vec{J}\cdot\vec{\nabla}g + g\vec{J}\cdot\vec{\nabla}f \right) = 0 \quad (\text{used } \vec{\nabla}\cdot\vec{J} = 0).$$

With $f = 1$, $g = r_i$ so $\vec{\nabla}f = 0$, $\nabla_j g = \delta_{ij}$ we have

$$\int \vec{J}\, d^3r = 0 \quad \Rightarrow \text{monopole term vanishes.}$$

Next $f = x_i$, $g = x_j$ $\Rightarrow \int d^3r \left(r_i J_j + r_j J_i \right) = 0$

For next (dipole) term need

$$\int d^3r' \, J_i(\vec{r}')\, r_j' = \int d^3r' \left[\frac{1}{2}\underbrace{\left(J_i r_j' + J_j r_i' \right)}_{0 \text{ (above)}} + \frac{1}{2}\left(J_i r_j' - J_j r_i' \right) \right]$$

$$\Rightarrow A_i(\vec{r}) = \frac{1}{2c} \frac{r_j}{|r|^3} \int d^3r' \left(J_i r_j' - J_j r_i' \right)$$

Now $J_i r_j' - J_j r_i' = \epsilon_{ijk}\epsilon_{kmn} J_m r_n' = \epsilon_{ijk}(\vec{J}\times\vec{r}')_k$

$$\vec{A}(\vec{r}) = \frac{\vec{m}\times\vec{r}}{|r|^3} \quad \text{with } \vec{m} = \frac{1}{2c}\int d^3r' \; \vec{r}'\times\vec{J}(\vec{r}') \quad \text{the magnetic dipole of } \vec{J}.$$

$\vec{B} = \vec{\nabla}\times\vec{A}$ was an assignment (earlier in course 203A)

Gives $\vec{B}$ in terms of $\vec{m}$ just as $\vec{E}$ in terms of $\vec{d}$.

Back to $\vec{J}_{mag}$: from $\vec{A} = \frac{\vec{m}\times\vec{r}}{r^3}$ we have

$$\vec{A}(\vec{r}) = \int_V d^3r' \, \frac{\vec{M}(\vec{r}') \times (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

Now repeat steps we did for ϕ: $\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = \vec{\nabla}' \frac{1}{|\vec{r}-\vec{r}'|}$, integrate by parts:

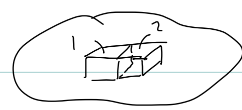
$$\vec{A}(\vec{r}) = \int_S d^2s \, \frac{\vec{M}(\vec{r}')\times\hat{n}}{|\vec{r}-\vec{r}'|} + \int_V d^3r' \, \frac{\vec{\nabla}'\times\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

The 2nd term, cu , i.e. $\vec{A}(\vec{r}) = \frac{1}{c}\int d^3r' \, \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}$ $\Rightarrow$ $\boxed{\vec{J}_{mag}(\vec{r}) = c\,\vec{\nabla}\times\vec{M}(\vec{r})}$

The 1st term is similar but for a surface current density $\vec{K}$:

$$\boxed{\vec{K}_{mag} = c\,\vec{M}\times\hat{n}}$$

Again, in the interior of the material adjacent volume elements give cancelling contributions



$\hat{n}_1 = -\hat{n}_2 \Rightarrow \vec{K}_{mag\,1} + \vec{K}_{mag\,2} = 0$.

But not so for bounding surface.

More generally, $\vec{M}$ should include $\vec{m}$'s from intrinsic magnetic dipole moments from particle spin.

Now $\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}(\vec{J}_{free} + \underbrace{\vec{J}_{pol}}_{\frac{\partial\vec{P}}{\partial t}} + \vec{J}_{conv} + \underbrace{\vec{J}_{mag}}_{c\,\vec{\nabla}\times\vec{M}})$

$$\Rightarrow \vec{\nabla}\times(\vec{B}-4\pi\vec{M}) - \frac{1}{c}\frac{\partial}{\partial t}(\vec{E}+4\pi\vec{P}) = \frac{4\pi}{c}(\vec{J}_{free}+\vec{J}_{conv})$$

Define $\vec{H} = \vec{B} - 4\pi\vec{M}$ $\Rightarrow$ $\boxed{\vec{\nabla}\times\vec{H} - \frac{1}{c}\frac{\partial\vec{D}}{\partial t} = \frac{4\pi}{c}(\vec{J}_{free}+\vec{J}_{conv})}$

For Garg, $\vec{H}$ = "magnetizing field", $\vec{B}$ = "magnetic field" which I like.

Interface ① ② $\hat{n}_2$ $\hat{n}_1 = \hat{n}_{21} = -\hat{n}_1$

$$\vec{B}_{1n} = \vec{B}_{2n} \quad (\text{from } \vec{\nabla}\cdot\vec{B} = 0)$$

$$(\vec{H}_2 - \vec{H}_1)\times\hat{n}_{21} = \frac{4\pi}{c}\vec{K}_{free}$$

(We have given up convection currents here).

For Dimensions ($\vec{H}$ & $\vec{M}$ same as $\vec{B}$ same as $\vec{E}$) and units (including translation to SI) see Garg.

Constitutive Relations (Garg Sec 84).

To solve the macroscopic Maxwell equations we need additional relations (giving eg, $\vec{P}$ in terms of $\vec{E}$ or $\vec{D}$, and $\vec{M}$ in terms of $\vec{B}$ or $\vec{H}$). We also need to know something about separating current/charge into free part.

Conductors: Ohm's law $\quad \vec{J} = \sigma\vec{E}$

σ = "conductivity"

Not a "law". Fails in semiconductors, or at large fields in conductors. Often frequency dependent, so in Fourier space $\vec{J}(\omega) = \sigma(\omega)\,\vec{E}(\omega)$

Dielectrics: For insulators

$$\vec{D} = \epsilon \vec{E}$$

ϵ = "dielectric constant"

Only at small fields. Sometimes need different ϵ in different directions $D_i = \epsilon_{ij} E_j$ (non-isotropic materials)

Also frequency dependent

$$\vec{D}(\omega) = \epsilon(\omega) \vec{E}(\omega)$$

Permeability: $\vec{B} = \mu \vec{H}$

Not for ferromagnets, nor superconductors.

For ferromagnets, complicated functional relation.

For superconductors $\vec{B} = 0$ in bulk ("Meissner" effect).

(type I superconductors, and Meissner phase of type II).

Energetics (Garg 85)

Issues:

- Is $E^2+B^2 = e^2+b^2$?

No! There is a lot of E&M energy in binding charges to form molecules, in making the structure of a solid, and so on. None of this is captured by $\vec{E}^2+\vec{B}^2$ not even when compared to averages $\langle e^2+b^2\rangle$ because these are averages over positive definite quantities.

- So there is some internal energy that is not in $\vec{E}$ & $\vec{B}$.
- Dissipation: lose energy, need time averages over small enough times to consider internal energy

So calculate work on free charges. That on bound charges goes into internal energy or lost to heat.

$$\frac{\text{work on free charges}}{\text{time}} = \vec{J}_{free}\cdot\vec{E} = \frac{c}{4\pi}\left(\vec{\nabla}\times\vec{H} - \frac{1}{c}\frac{\partial\vec{D}}{\partial t}\right)\cdot\vec{E}$$

Now

$$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = \epsilon_{ijk}\,\partial_i(E_jH_k) = \epsilon_{ijk}(\partial_i E_j)H_k + \epsilon_{ijk}E_j\,\partial_i H_k$$

$$= \vec{H}\cdot\vec{\nabla}\times\vec{E} - \vec{E}\cdot(\vec{\nabla}\times\vec{H})$$

$$= \vec{H}\cdot\left(-\frac{1}{c}\frac{\partial\vec{B}}{\partial t}\right) - \vec{E}\cdot(\vec{\nabla}\times\vec{H})$$

$$\Rightarrow \boxed{-\vec{\nabla}\cdot\left[\frac{c}{4\pi}\vec{E}\times\vec{H}\right] = \vec{J}_{free}\cdot\vec{E} + \frac{1}{4\pi}\left(\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right)}$$

$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{H}$ is the macroscopic version of Poynting vector.

Recall $\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S} = \frac{\text{work}}{\text{time}}$ microscopically, but here

we have $\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}$ instead of $(2\times)\ \frac{\partial}{\partial t}(e^2+b^2)$.

To get beyond this we need constitutive relations.

Then the last term is

$$\frac{1}{8\pi}\frac{\partial}{\partial t}\left(\epsilon E^2 + \mu H^2\right)$$

But this has limited use/validity.

- $\frac{1}{4\pi}\left(\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right)$ includes both energy increase plus power that goes into heat. Cannot generally disentangle. $\Rightarrow$ not $\frac{\partial}{\partial t}$ of a single quantity.

- For validity: ϵ & μ are time independent, really meaning, frequency independent $\epsilon = \epsilon(\omega) \approx$ constant, $\mu = \mu(\omega) \approx$ constant

- For validity, must be in linear regime so the simple constitutive relations apply.

Keep this in mind!