Electrostatics. Spherical Harmonics. Multipole Expansion

Electrostatics: \( \frac{\partial E}{\partial t} = 0, \quad B = 0, \quad \nabla \cdot E = 0, \quad \nabla \times E = 0 \) in Maxwell's Eqs:

\[ \nabla \times E = 0 \quad \nabla \cdot E = \rho_f \]

\[ E = -\nabla \phi \quad (\phi = \phi^0, \quad \rho_f = 0) \quad \Rightarrow \quad \nabla^2 \phi = -4\pi \rho_f \] Poisson Equation

We have seen the solution in terms of Green functions:

\[ \phi(x) = \phi_{\text{hom}}(x) + \int G(x-x') \rho(x') \, dx' \]

where \( \nabla^2 G(x) = -4\pi \delta^3(x) \), \( \nabla^2 \phi_{\text{hom}} = 0 \)

and had determined \( G(x) \) by Fourier transform. We can also infer \( G(x) \)

from our knowledge of Coulomb's law:

\[ \phi(x) = \frac{q}{|x|} \quad \text{is for} \quad \rho(x) = q \delta^3(x) \rightarrow \nabla^2 \phi = -4\pi q \delta^3(x) \]

\[ \Rightarrow \quad G(x) = \frac{1}{|x|} \]

So

\[ \phi(x) = \phi_{\text{hom}}(x) + \int \frac{\rho(x')}{|x-x'|} \, dx' \]

Boundary value problems: often concerned with region of space with boundaries on which we know something about \( E \). Then \( \phi_{\text{hom}} \) is chosen to ensure these "boundary conditions" are satisfied.

As in

Conducting boundary: \( \phi = \text{constant} \)

Surface charge density: \( \Delta E \cdot \hat{n} = \sigma \)

\[ \text{Gauss:} \quad \int E \cdot dA = \frac{\rho}{\epsilon_0} \]

\[ \Rightarrow \quad (E \cdot \hat{n}) \, dA = \frac{\rho}{\epsilon_0} \]

\[ \Rightarrow \quad \sigma = \frac{\partial}{\partial t} \int (E \cdot \hat{n}) \, dA \]
Regardless of the presence of charges, the central problem in electrostatics is how to solve

$$\nabla^2 \phi = 0 \quad \text{(Laplace Equation)}$$

subject to boundary conditions:

- (i) $\phi$ specified (Dirichlet)
- or
- (ii) $\frac{\partial \phi}{\partial n}$ specified (Neumann)
- or
- (iii) mixed

Uniqueness of solution of Poisson with boundaries: [Gar8 16]

If $\phi_1, \phi_2$ are two solutions $\nabla^2 \phi = 0$ with $\phi_1, \phi_2$ specified on $S = \partial V$, then $\psi = \phi_2 - \phi_1$ satisfies $\nabla^2 \psi = 0$ and $\psi = 0$ or $\frac{\partial \psi}{\partial n} = 0$ on $\partial V$.

By Gauss's Theorem

$$\int_V \nabla \cdot (\psi \nabla \phi) \, dv = \int_{\partial V} (\psi \nabla \phi) \cdot \hat{n} \, ds = \int_{\partial V} \psi \frac{\partial \phi}{\partial n} \, ds$$

The RHS vanishes by assumption. The LHS

$$0 = \int_V \nabla \cdot (\nabla \psi \nabla \phi + \psi \nabla \psi) \, dv = \int_V \nabla \psi \cdot \nabla \phi \, dv$$

$\Rightarrow \psi = \text{Constant} \quad \Rightarrow |\phi_2 - \phi_1| = \text{Constant}$

(Note, for 2 functions $\psi, \chi$)

$$\int_V (\psi_1 \nabla^2 \psi_2 + \nabla \psi_1 \cdot \nabla \psi_2) \, dv = \int_{\partial V} \psi_1 \frac{\partial \chi}{\partial n} \, ds$$

is "Green's 1st identity"

$$\int_V (\psi_2 \nabla^2 \psi_1 - \psi_1 \nabla \psi_2) \, dv = \int_{\partial V} (\psi_2 \frac{\partial \chi}{\partial n} - \psi_1 \frac{\partial \chi}{\partial n}) \, ds$$

is "Green's 2nd identity"

(re or Green's Theorem)
Solving Laplace (PDE): separation of variables

Cartesian: \( \phi(x, y, z) = X(x) Y(y) Z(z) \)

\[
\frac{1}{\phi} \nabla^2 \phi = \frac{1}{x} X''(x) + \frac{1}{y} Y''(y) + \frac{1}{z} Z''(z) = 0
\]

The three functions of different arguments can add up to zero only if each one is a constant:

\[
\frac{1}{x} X'' = \alpha^2, \quad \frac{1}{y} Y'' = \beta^2, \quad \frac{1}{z} Z'' = \gamma^2
\]

with \( \alpha^2 + \beta^2 + \gamma^2 = 0 \), \( X \propto e^{\alpha x}, \; Y \propto e^{\beta y}, \; Z \propto e^{\gamma z} \)

The boundary conditions (b.c.'s) limit values of \((\alpha, \beta, \gamma)\). The solution is a linear combination.

Example: \( \phi_0(x, y, z) = \phi_0(x) \phi_0(y) \phi_0(z) \)

with \( \phi_0(0, y) = \phi_0(a, y) = 0 = \phi_0(x, 0) = \phi_0(x, b) \)

So take \( \alpha < 0, \beta < 0 \) above for oscillatory functions. Changing \( x \rightarrow -x, \; y \rightarrow -y \) above

\( x^2 + y^2 = \rho^2 \)

h.c. at \( z = 0, \; x = a, \; y = b \) \( = (A e^{i \alpha x} + B e^{-i \beta y}) \lim_{\rho \rightarrow 0} \rho \)

h.c. at \( z = 0, \; x = a, \; y = b \) \( = (A e^{i \alpha x} + B e^{-i \beta y}) \lim_{\rho \rightarrow 0} \rho \)

\( \Phi(x, y, z) \propto \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin(m \pi x / a) \sin(n \pi y / b) \left[ A_{mn} \sinh(y_{mn} z) + B_{mn} \cosh(y_{mn} z) \right] \)

where \( y_{mn} = \sqrt{(n b / a)^2 + (m a / b)^2} \). The b.c. at \( z = 0 \) gives

\( \Phi_0(x, y) \propto \sum_{m=1}^{\infty} A_{mn} \sin(m \pi x / a) \sin(n \pi y / b) \)

With \( \int_0^a \int_0^b \sin(m \pi x / a) \sin(n \pi y / b) \; dx \; dy = 0 \) for \( m \neq 0, \; n \neq 0 \)

\( A_{mn} = \frac{1}{ab} \int_0^a \int_0^b \sin(m \pi x / a) \sin(n \pi y / b) \; dx \; dy \)

Finally as \( \epsilon \rightarrow \infty \) we do not want \( \Phi \propto e^{\alpha x} \rightarrow \infty \) choose \( B_{mn} = -A_{mn} \)

CLEAR THAT THE SOLUTION IS MOST GENERAL, BUT IMPLEMENTING BOUNDARY CONDITIONS COMPLICATED UNLESS RECTANGULAR SYMMETRY IN PROBLEM → CONSIDER ALSO Spherical Coordinates
Cylindrical: $\Phi(\rho, \phi, z) = R(\rho) \tilde{\Phi}(\phi) Z(z)$  
(use $\phi$ = angle, $\Phi$ = potential)

\[ \frac{1}{\rho} \frac{d}{d\rho} \left( \rho \frac{dR}{d\rho} \right) + \frac{1}{\rho^2} \tilde{\Phi}'' + \frac{1}{\rho^2} Z'' = 0 \]

\[ \Rightarrow Z'' = \beta^2 Z, \quad \rho^2 \left[ \frac{1}{\rho^4} \frac{d}{d\phi} \left( \rho^4 \frac{d\tilde{\Phi}}{d\phi} \right) + \alpha^2 \right] = -\beta^2 \tilde{\Phi} = -\beta^2 \]

\[ \Rightarrow \tilde{\Phi}'' = -\beta^2 \tilde{\Phi} = \left[ \frac{\partial^2}{\partial \phi^2} + \frac{d}{d\phi} + \alpha^2 - \frac{\beta^2}{\rho^2} \right] \tilde{\Phi} = 0 \]

\( \Phi \) periodic except for "canonical" configurations

\[ \Phi_{\beta} \left( \frac{\rho, \phi, z}{\rho, \phi, z + 2\pi n + m \pi} \right) \]

\[ \Rightarrow \beta = m, \quad m \in \mathbb{Z} \quad \Phi = e^{i m \phi} . \]

\( z = e^{i n \phi} \)

Radial equation is Bessel's (see 203A, caution with signs)

\[ R(\rho) = J_n(\alpha \rho) . \quad \text{We'll review later.} \quad \text{See last quarter notes.} \]

Expansion in terms $J_n(\xi_{mn}) = 0$, say, if $\Phi(\rho = a, \phi, z) = 0$

\[ \Phi(\rho, \phi, z) = \sum_{m, n} c_{mn} J_n(\xi_{mn}) e^{i m \phi} e^{-\xi_{mn} \rho} \]

\( d_{\beta, \gamma} = \sum_{m,n} c_{mn} J_n(\xi_{mn}) e^{i m \phi} e^{-\xi_{mn} \rho} \)
Spherical coordinates $\mathbf{r} (r, \theta, \phi)$

Appropriate for problems with spherical symmetry: spherical boundaries.

Recall, a scalar has $\phi' (r') = \phi(r)$ with $r' = r, \quad r'^2 = 1$

With $r = 1 + \varepsilon, \quad \varepsilon \rightarrow 0$, infinitesimal $\quad r'^2 = 1 \Rightarrow \quad \varepsilon = -\varepsilon$

With $\phi' (r') = \phi (r'^2)$, we have $\delta \phi = \phi (r'^2) - \phi (r) = \phi (r - 0^2) - \phi (r) = -\varepsilon_i x_i \partial_i \phi$

$\Rightarrow$ the infinitesimal rotation is generated by $-\varepsilon_i x_i \partial_i = \varepsilon_{ij} x_j \partial_i$

Now an antisymmetric $3 \times 3$ matrix has $3 \times 2 = 3$ independent components, so we can parameterize $\varepsilon_{ij}$ as $\varepsilon_{ij} = \varepsilon^a \varepsilon_{aij}$, $\varepsilon^a, a = 1, 2, 3$ are parameters (infinitesimal)

(and $\varepsilon_{aij}$ is completely antisymmetric tensor with $\varepsilon_{123} = +1$).

$\Rightarrow \delta \phi = \varepsilon^a \varepsilon_{aij} x_j \partial_i = \varepsilon \cdot (\mathbf{x} \times \nabla)$

This should ring a bell! In QM $\mathbf{L} = \mathbf{r} \mathbf{p}$ $= -i \hbar \mathbf{r} \times \mathbf{p}$

Setting $\hbar = 1$ (because we are not doing QM) $\Rightarrow \mathbf{L} = -i \mathbf{r} \times \mathbf{p}$

We can use our knowledge from QM here:

$\mathbf{L} \times \mathbf{L} = i \varepsilon_{ijk} \mathbf{L} \mathbf{L}_k$

$\mathbf{L} \times (\mathbf{L} \times \mathbf{L}) = \mathbf{L} \cdot (\mathbf{L} \times \mathbf{L})

\begin{align*}
\text{Prof.} & \quad [\mathbf{L}_i, \mathbf{L}_j] = [L_i, L_j] \\
& = \alpha L_i \varepsilon_{ijk} \varepsilon_{kmn} \left( \delta_{mj} x_k x_n - \delta_{mk} x_j x_n \right) \\
& = \alpha \left( \delta_{kn} \varepsilon_{ijk} - \delta_{kn} \varepsilon_{ijk} \right) \\
& = \alpha \delta_{kn} \varepsilon_{ijk} x_k x_n \\
& = -\alpha \varepsilon_{ijk} L_k \\
& = i \varepsilon_{ijk} \mathbf{L}_k \\
\end{align*}$

$\mathbf{L}^2 = \mathbf{L} \cdot \mathbf{L}$, $\mathbf{L} = \mathbf{L}

\begin{align*}
[\mathbf{L}_i, \mathbf{L}_j] & = 0 \quad \Rightarrow \left[ L^+_i, L^+_j \right] = 0 \\
\mathbf{L}^+ = \frac{1}{\sqrt{2}} (L_1 \pm i L_2) & \Rightarrow \left[ L^+_i, L^+_j \right] = \frac{1}{\sqrt{2}} \left( -i \left[ L_1, L_1 \right] + i \left[ L_1, L_1 \right] \right) = L^+_i

\left[ L^+_1, L^+_3 \right] = \frac{1}{\alpha} \left( \left[ L_1, L_3 \right] \pm L_3 \left[ L_1, L_3 \right] \right) = \frac{1}{\alpha} \left( -i L_2 + L_1 \right) = \frac{1}{\alpha} \mathbf{L}^+ \end{align*}$
We can simultaneously diagonalize $L^2$ and one $L_i^2$, say $L_3$:

The eigenvectors are $Y_{lm}$, "spherical harmonics."

Before we find them, let's connect this to $\nabla^2 \phi = 0$

Note that $L_3 = (\mathbf{r} \times \nabla) \cdot (\mathbf{r} \times \nabla) = -\epsilon_{ijk} x_i \partial_{x_j} \epsilon_{mpq} x_j \partial_{x_p} \phi$

\[ = -\left( \delta_{ij} \delta_{pq} - \delta_{ip} \delta_{jq} \right) x_i x_j \partial_{x_p} \phi \]
\[ = -x_i \partial_{x_i} x_j \partial_{x_j} + x_j x_i \partial_{x_j} \phi \]
\[ = -\nabla \cdot \mathbf{r} + 3\mathbf{r} \cdot \nabla \phi \]
\[ = \left( \nabla \cdot \mathbf{r} \right)^2 + (\mathbf{r} \cdot \nabla)^2 - r^2 \nabla^2 \]

Now $\mathbf{r} \cdot \nabla = \frac{\partial}{\partial r}$ so
\[ \frac{1}{r} L_3 = \frac{1}{r} \frac{\partial}{\partial r} \left( r^2 \phi \right) + \frac{1}{r} \frac{\partial}{\partial r} - \nabla^2 \]
\[ = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \nabla^2 \]

\[ = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \nabla^2 \]
\[ \Rightarrow L_3 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \frac{\partial}{\partial \theta} = Y_{lm}' \quad \text{are functions of } \theta \times \phi, \quad Y_{lm}(\theta, \phi) \]

As we will see, $L_3 Y_{lm} = \ell(\ell+1) Y_{lm}$; $\ell = 0, 1, 2, \ldots$

So $\phi = \frac{1}{r} Y_{lm}(\theta, \phi)$ so $\nabla^2 \phi = 0 \Rightarrow \frac{1}{r} \frac{\partial}{\partial r} \left( r^2 \phi \right) = 0 \Rightarrow r^2 \frac{\partial^2 \phi}{\partial r^2} + \ell(\ell+1) \phi = 0$

This is homogeneous in $r \Rightarrow R = r^\alpha$ gives $\alpha(\alpha-1) = \ell(\ell+1) \Rightarrow \alpha = \ell + 1, \quad \alpha = -\ell$

\[ \phi(r, \theta, \phi) = \sum_{l,m} C_{lm} r^{\ell+1} Y_{lm}(\theta, \phi) + d_{lm} r^{-\ell-1} Y_{lm} \]

By the standard argument, the $Y_{lm}$'s form an orthonormal set. They are normalized,

\[ \int d\Omega Y_{lm}^*(\theta, \phi) Y_{mn}(\theta, \phi) = \delta_{lm} \delta_{mn} \]

and form a complete basis (in the space of normalizable functions on the unit sphere), so

\[ C_{lm} + d_{lm} r^{-\ell-1} = \int d\Omega Y_{lm}^*(\theta, \phi) \phi(r, \theta, \phi) \]

Either determine $C_{lm}$ at two radii to solve for both $C_{lm}$ and $d_{lm}$, or, very commonly, use some additional condition, e.g., regularity at origin ($r=0 \Rightarrow d_{lm} = 0$) or at $r=\infty$ ($\Rightarrow C_{lm} = 0$ for $l>0$).
Finding eigenvalues: \[ \mathcal{L}^2 \psi = \lambda \psi, \quad \mathcal{L}_3 \psi = \lambda' \psi. \]

To streamline notation, use \(\mathcal{L}_3 \ket{\lambda', \lambda'} = \lambda' \ket{\lambda', \lambda'}\). We really should write \(\ket{\lambda, \lambda'}\) for \(\mathcal{L}^2 \ket{\lambda, \lambda'} = \lambda \ket{\lambda, \lambda'}\) and \(\mathcal{L}_3 \ket{\lambda', \lambda'}\), but for now we concentrate on \(\lambda'\) for fixed \(\lambda\), so omit \(\lambda\) in \(\ket{\lambda, \lambda'}\).

We also need an inner product \(\langle \psi | \chi \rangle = \int d^3x \, \psi^* \chi\). Note that \(\mathcal{L}^2\) is hermitian w.r.t. this inner product. The hermitian conjugate \(\langle \chi | \mathcal{L}^2 | \psi \rangle = \langle \psi | \mathcal{L}^2 | \chi \rangle^* \)

\[ \langle \chi | \mathcal{L}^+ | \psi \rangle = \langle \chi | \mathcal{L} | \psi \rangle \]

Note also \(\mathcal{L}^+_\pm = \mathcal{L}^{-\pm}\).

We will show that \(\ket{\lambda'}\)'s come in discrete sets with \(\lambda' = -l, -l+1, \ldots, l-1, l\) for some integer \(l\).

First note: \(\mathcal{L}_3 \mathcal{L}^+_\pm \ket{\lambda'} = (\mathcal{L}_3 \mathcal{L}^+_\pm + \mathcal{L}^-_3 \mathcal{L}_3) \ket{\lambda'} = (\lambda' \pm \lambda) \mathcal{L}^+_\pm \ket{\lambda'} \)

\(\Rightarrow \mathcal{L}^+_\pm \ket{\lambda'}\) is an eigenvector of \(\mathcal{L}_3\) with eigenvalue \(\lambda' \pm \lambda\).

So we have a chain \(\ldots, \mathcal{L}^2 \ket{\lambda'}, \mathcal{L} \ket{\lambda'}, \ket{\lambda'}, \mathcal{L}^+_2 \ket{\lambda'}, \mathcal{L}^+_1 \ket{\lambda'}, \ldots\)

This will terminate if \(\mathcal{L}^+_l \ket{\lambda'} = 0\) for some \(\lambda' \neq l\). Assume \(\lambda' = l\).

Introduce proportionality constants into \(\mathcal{L}^+_l \ket{\lambda'} = c_l' \ket{\lambda'}\) (use \(y\) because the prime in \(x\) is firing).

and assume \(\ket{\chi}\) is normalized: \(\langle \chi | \chi \rangle = 1\) (any \(\chi\))

\[ \mathcal{L}^+_l \ket{\chi} = c_l \ket{\chi + 1}, \quad \text{and} \quad \mathcal{L}^-_l \ket{\chi} = d_l \ket{\chi - 1} \]

These are not independent: \(|| \mathcal{L}^+_l \ket{\chi} ||^2 = c_l^2 \langle \chi | \chi + 1 \rangle = \langle \chi | \mathcal{L}^+_l \ket{\chi} = c_l \langle \chi | \chi \rangle = c_l \)

\[ \Rightarrow c_l = d_{l+1} \]
Now we use \([L_+ L_-] |\gamma\rangle = L_3 |\gamma\rangle = \gamma |\gamma\rangle\):

\[
L_+ D_k - L_- P_{k+1} = \gamma
\]

\[
\Rightarrow C_{2k+1}^l - C_{2k+1}^{l-1} = \gamma
\]

and \(L_+ |l\rangle = 0\) is \(C_l = 0\). So we have

\[
C_{2k+1}^l = 0 
\Rightarrow C_{2k+1}^l = \frac{l}{2} (l + 1)
\]

\[
C_{2k}^l = 0 
\Rightarrow C_{2k}^l = \frac{l}{2} (l - 1)
\]

\[
\vdots
\]

\[
C_{2k+1}^l = \frac{l}{2} (l + 1) (2l + 1)
\]

This should not be negative; if we take \(2l = \text{integer}\) then for \(k = 2l\)

\[
C_{2k}^l = D_k = 0 \Rightarrow l - l = 0
\]

\[
\Rightarrow \text{The set of functions is } |l-l>, |l-l+1>, ..., |l>, U > = \text{all then } |m\rangle
\]

\[
|m| = -l, ..., l
\]

Let's get back to \(L^2 = L^2_1 + L^2_2 + L^2_3\).

Note \(k = 1\)

\[
L_1 L_+ + L_+ L_1 = \frac{1}{2} \left( (l_1 l_2) (l_1 l_2) - (l_1 l_2) (l_1 l_2) \right) = L^2_1 + L^2_2
\]

\[
\Rightarrow L^2 = L_+ L - L_+ L + L^2
\]

\[
L_+ L - L_+ L = C_{l=0}^l |m\rangle = C_{l=1}^l |m\rangle
\]

\[
L_+ L + L_+ L = C_{l=1}^l |m\rangle
\]

\[
\Rightarrow L^2 |m\rangle = \left( C_{l=1}^l + C_{l=1}^l \right) |m\rangle
\]

\[
\Rightarrow L^2 |m\rangle = \gamma (l_1l_2) |m\rangle
\]

\[
C_{l=0}^l + C_{l=1}^l = \frac{1}{2} (l_1 l_2) (l_1 l_2) = \frac{1}{2} (l_1 l_2) (l_1 l_2) \Rightarrow L^2 |m\rangle = \gamma (l_1l_2) |m\rangle
\]

So all our functions have the same \(L^2\) eigenvalue and are fully \(l\)-labelled

\[
\text{Labelled } |l, m\rangle, \text{ with } L^2 |l, m\rangle = \gamma (l_1l_2) |l, m\rangle, L_3 |l, m\rangle = m |l, m\rangle
\]

\[
l = \frac{1}{2} \mathbb{Z}, \quad m = l_1, l_2, ..., l_3, l
\]
Find eigenfunctions.

\[ L_+ = \frac{1}{\ell} (L_x, \ell L_\ell) = \frac{(\ell)}{\ell} \left[ (y^2 - 2 \theta) \partial_x + \frac{\ell}{\ell} (x \theta - \ell \theta) \right] \]

\[ = \frac{1}{\ell} \left[ (y \theta + x \theta) \partial_x + \frac{\ell}{\ell} (x \theta - \ell \theta) \right] \]

Note that \((\partial_x + \ell \beta) (x + iy) = 0\), \((\partial_x + \ell \beta) (x - iy) = 0\)

\[ \frac{1}{\ell} (\partial_x + \ell \beta) \left( x + iy \right) = 0 \quad \text{and} \quad \frac{1}{\ell} (\partial_x + \ell \beta) \left( x - iy \right) = 0 \]

So \( \ell \left( x + iy \right)^n \left( x - iy \right)^m = \ell \left( \frac{y + i x}{\ell} \right)^n \left( \frac{y - i x}{\ell} \right)^m = \ell \left( \frac{y + i x}{\ell} \right)^n \left( \frac{y - i x}{\ell} \right)^m = 0 \)

and \( \ell \left( \frac{y + i x}{\ell} \right)^n \left( \frac{y - i x}{\ell} \right)^m = \ell \left( \frac{y + i x}{\ell} \right)^n \left( \frac{y - i x}{\ell} \right)^m = 0 \)

\[ \ell \left( \frac{y + i x}{\ell} \right)^n \left( \frac{y - i x}{\ell} \right)^m = \ell \left( \frac{y + i x}{\ell} \right)^n \left( \frac{y - i x}{\ell} \right)^m = 0 \]

\[ A_{l \ell} \quad L_+ = \frac{\partial}{\partial x} (x \beta - y \ell) \]

To streamline notation, let \( x_\ell = \frac{x + iy}{\ell}, \quad y_\ell = \frac{y + i x}{\ell}, \quad \beta_\ell = \frac{\partial}{\partial x} \)

\[ \left\{ 0, \quad \beta_\ell x_\ell = 1, \quad \beta_\ell x_\ell = 0. \right\} \]

Then \( x = \frac{1}{\ell} (x_\ell + x), \quad y = \frac{1}{\ell} (y_\ell + y), \quad \beta = \frac{1}{\ell} (\beta_\ell + \beta), \quad \partial = \frac{1}{\ell} (\partial_\ell + \partial) \)

So \( L_3 = \frac{\ell}{\ell} \left( \left( x_\ell + y_\ell \right) \partial_x + \frac{\ell}{\ell} (x_\ell + y_\ell) \left( \partial_x + \partial_\ell \right) \right) = x \partial_x - y \partial_\ell \)

and in this notation \( L_+ = \frac{\ell}{\ell} \left( x \partial_x - y \partial_\ell \right) \)

This suggests \( |l, l\rangle = \ell^l |x_\ell^l \rangle \) with \( \ell \) a normalization constant.

Clearly \( L_+ |l, l\rangle = 0 \), \( L_3 |l, l\rangle = \ell |l, l\rangle \)

\[ L^\ell |l, l\rangle = \left( L_+ + L_+ \right) |l, l\rangle = \ell \ell |l, l\rangle \]

\[ = \ell \ell \left( -y_\ell + \ell \partial_\ell \right) + \ell^2 \ell |l, l\rangle \]

\[ = \ell \ell \left( -y_\ell + \ell \partial_\ell \right) + \ell^2 \ell |l, l\rangle \]

\[ = \ell \ell \left( -y_\ell + \ell \partial_\ell \right) + \ell^2 \ell |l, l\rangle \]

\[ = \ell \ell \left( -y_\ell + \ell \partial_\ell \right) + \ell^2 \ell |l, l\rangle \]

Omega! We have \( |l, l-k\rangle = \ell^k |l, l\rangle \)

For example \( |l, l-1\rangle = \ell^{l-1} |l, l\rangle \)

\[ = \ell^{l-1} \left( \left( x_\ell \partial_\ell - y \ell \partial_x \right) x_\ell - 2 \ell \partial_\ell \right) \]

\[ = 2 \ell^{l-1} x_\ell - \ell^{l-1} \]

\[ = 2 \ell^{l-1} x_\ell - \ell^{l-1} \]

\[ = 2 \ell^{l-1} x_\ell - \ell^{l-1} \]

\[ = 2 \ell^{l-1} x_\ell - \ell^{l-1} \]
In terms of $\theta, \phi$, $x^2 = \frac{1}{\sin \theta} \sin^2 \theta \cos^2 \phi \sin \phi = \frac{1}{\sin \theta} \sin^2 \theta \cos^2 \phi \sin \phi$

$\tilde{r} = r \cos \theta$

Since $|l, m\rangle$ defined above is proportional to $r^l$ for all functions, we can tabulate out and replace $|l, m\rangle \rightarrow \frac{1}{r^l} |l, m\rangle$, a function of $\theta$ and $\phi$ only.

This gives $Y_{lm}$, up to normalization,

$$Y_{lm}(\theta, \phi) = \frac{1}{\mathcal{N}_m} L^m \sin \theta \cos \phi$$

Normalization:

$$\int_0^{2\pi} \int_0^\pi |Y_{lm}|^2 \sin \theta \, d\theta \, d\phi = \delta_{ll'} \delta_{mm'}$$

Example: $l = 1$. From the above $Y_{01}$, $Y_{00}$, $Y_{11}$, $Y_{10}$

So $Y_{11} = N_1 \sin \theta \cos \phi$, $Y_{10} = N_0 \cos \phi$

$$\int_0^{2\pi} |Y_{11}|^2 \sin \theta \, d\phi = \int_0^\pi N_1^2 \sin^2 \theta \cos^2 \phi \, d\theta = \frac{2}{3} N_1^2 \pi$$

$$\int_0^{2\pi} |Y_{10}|^2 \sin \theta \, d\phi = \int_0^\pi N_0^2 \cos^2 \phi \, d\phi = \frac{1}{2} N_0^2 \pi$$

The phase is by convention: $Y_{00} = \frac{r}{\sqrt{\pi}} \cos \theta$ and

$$Y_{11} = \frac{1}{\sqrt{8\pi}} \sin \theta \cos \phi$$

Note that the eigenvalues with $2l + 1$ odd integer $m$, $l = 0, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$

are periodic in $\phi$ with period $2\pi$, not $\pi$: $e^{im\phi} \rightarrow e^{im\phi}$. Hence they do not play a role in solving Laplace's equation, but they do play a role in other physics (spinors).

Note: derivation of eigenvalues only depends on commutation relations and normalizable vectors. Applies equally to $\Sigma$ invariances. Then $\mathcal{U}$ are $l \times l$ on space of $(l, m)$ vectors.
This presentation emphasizes the connection to the rotation group and angular momentum. There are many other ways to introduce $Y_{lm}$'s and many additional developments. Here we list some facts.

- $Y_{lm}(-\hat{r}) = (-1)^m Y_{lm}(\hat{r})$
- $Y^*_{lm}(\hat{r}) = (-1)^m Y_{-lm}(\hat{r})$

- Completeness: $\sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y^*_{lm}(\theta, \phi) Y_{lm}(\theta', \phi') = \delta(\cos \theta - \cos \theta') \delta(\phi - \phi')$

- Addition Theorem: Let $\theta_{12} = \hat{r}_1 \cdot \hat{r}_2$ (and recall $Y_{l0}$ is $\phi$ independent).
  $$Y_{l0}(\theta_{12}, \phi) = \sqrt{\frac{2l+1}{4\pi}} \sum_{m=-l}^{l} Y^*_{lm}(\theta, \phi) Y_{lm}(\theta_{12}, \phi)$$

- Relation to Legendre polynomials: $Y_{l0}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} \ P_l(\cos \theta)$

  (And, more generally, to Associated Legendre function)

  $$Y_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} \frac{(l-m)!}{(l+m)!} P^m_l(\cos \theta) e^{im\phi}$$

  Generating function for $P_l(x)$:
  $$\frac{1}{\sqrt{1 + 2tx - t^2}} = \sum_{l=0}^{\infty} t^l P_l(x) \quad x \in [-1, 1]$$

- Generating function for $P_l +$ addition theorem $\Rightarrow$

  $$\frac{1}{|\hat{r}_1 - \hat{r}_2|^2} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{\delta^2}{(2\pi)^2} Y^*_{lm}(\hat{r}_1) Y_{lm}(\hat{r}_2)$$

  where $r_1 = \min(|\hat{r}_1|, |\hat{r}_2|)$ and $r_2 = \max(|\hat{r}_1|, |\hat{r}_2|)$. I have switched notation from $\hat{r}$ to $r$ to stay closer to textbook (Garg).
Example:

Conducting Sphere of radius $a$ with upper/lower hemispheres at potential $+V/-V$. 

It has azimuthal symmetry, so only $m=0$ contributes to expansion of $\phi$:

$$\phi(r, \theta) = \sum_{l=0}^{\infty} \frac{(2l+1)}{4\pi} \sum_{m=-l}^{l} \frac{1}{2} \left[ Y_{lm}(\theta, \phi) + \text{adj} Y_{lm}(\theta, \phi) \right] Y_{lm}^*$$

For $\phi$ inside sphere, $d_{l0} = 0$ (regularity at origin).

Inverting:

$$c_{l0} = \int \phi(r, \theta) Y_{l0}^* \, d\Omega$$

or, evaluating at $r = a$:

$$c_{l0} = \frac{1}{a^2} \frac{2\pi}{4\pi} \int \left( \frac{a^2 + 1}{4\pi} \right) \frac{P_l(\cos \theta)}{r} \, d\Omega \int_{-\pi}^{+\pi} \phi(\theta) \, d\phi$$

$$= \frac{V}{a^2} \frac{2\pi}{4\pi} \int \left( \frac{a^2 + 1}{4\pi} \right) \frac{P_l(\cos \theta)}{r} \, d\Omega \int_{-\pi}^{+\pi} \phi(\theta) \, d\phi$$

This vanishes for $l = \text{even}$, and we need to do the integral:

$$\int_{0}^{\pi} d\theta \frac{P_l(\cos \theta)}{r} \text{ for odd } l$$
From the generating function:

\[ \int_{0}^{1} dx \sum_{l=0}^{\infty} t^l \phi_l(x) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \int_{0}^{1} dx \phi_k(x) \]

which equals

\[ \int_{0}^{1} \frac{1}{\sqrt{1+t^2-x^2}} \frac{1}{1-t^2} \left( 1 + t \sqrt{1+t^2-x^2} \right) \, dx = \frac{1}{\left( 1+t^2 \right) \left( 1-t^2 \right)} \]

Use the Taylor expansion

\[ (1+x)^{s} = \sum_{k=0}^{\infty} \frac{s(s-1)\cdots(s-k+1) x^k}{k!}, \quad \text{with } s = \frac{1}{2}, \quad x = t^2 \]

\[ \frac{1}{\sqrt{1+t^2}} - \frac{1}{t} = \sum_{k=1}^{\infty} \frac{1}{k!} \frac{1}{2} \frac{1}{3} \cdots \frac{1}{k+1} t^{k-1} \]

And so

\[ \int_{0}^{1} \phi_{2k-1}(x) \, dx = \frac{(2k-1)!}{2^k k!} \frac{1}{(2k-3)!!} \]

So we have

\[ \phi(r, \theta) = \sqrt{\sum_{k=1}^{\infty} \sqrt{\prod_{j=1}^{k-1} \frac{1}{2^j j!}} \frac{1}{2^k k!} \frac{(-1)^{k+1}}{(2k-3)!!} Y_{2k-1}(\theta, \phi) \]

\[ = \sqrt{\sum_{k=1}^{\infty} (\frac{r}{a})^{2k-1} \frac{1}{2^k k!} \frac{(-1)^{k+1}}{(2k-3)!!} r_{2k-1}(\cos \theta) \]

\[ = \sqrt{\left[ \frac{2}{2a} r_1(\cos \theta) - \frac{2}{8(a)} r_3(\cos \theta) + \frac{1}{16} \frac{2}{64} (a) \cdot \frac{3}{6} r_5(\cos \theta) \cdots \right] } \]
Multiple expansion

Localized charge distribution $p$:

$$\phi (r) = \sum \frac{q_n}{|r - r_0|} = \int d^3p \frac{p(r')}{|r - r'|}$$

Physical idea of multipole expansion: at $r \gg R$ we should have $\phi \sim \frac{q}{r}$, where $q = \int d^3p \rho (r')$. Cones are by an expansion in:

$$\frac{q}{r} \sim \frac{q_0}{r} + \frac{q_1}{r^3} + \frac{q_2}{r^5} + \ldots$$

Expand in powers or $|r'/r| < 1$

We can do this in one swoop by using Taylor expansion of $1/r'$. But let's get some intuition of what this is by expanding by hand first. We will use a Taylor series for $f$ about $0$:

$$f(x) = f(0) + x \frac{\partial f}{\partial x} (0) + \frac{x^2}{2!} \frac{\partial^2 f}{\partial x^2} (0) + \ldots$$

$$\frac{1}{|r' - r|} = \frac{1}{r} + \frac{x' \frac{\partial f}{\partial x}}{r} \bigg|_{x' = 0} + \frac{1}{2!} \left. \frac{x^2 \frac{\partial^2 f}{\partial x^2}}{r} \right|_{x' = 0} + \ldots$$

$$= \frac{1}{r} + \frac{x'}{r} f_x (r) + \frac{1}{2!} \frac{x^2}{r} \frac{\partial f (r)}{\partial x} + \ldots$$

Stick this into $\int d^3p \frac{p(r')}{|r - r'|}$
More precisely:

Define \( q = \int \rho(r) \, r \, d^3 r \) monopole = charge

\[ \mathbf{d} = \int \rho(r) \mathbf{r} \, d^3 r \] dipole moment

\( D_{ij} = \int \rho(r) (3 x_i x_j - r^2 \delta_{ij}) \) quadrupole moment

The extra term \( r^{-2} \) in the definition of \( D_{ij} \) is included so that \( D_{ij} r_{ij} = 0 \) (i.e., it's traceless). This can be done freely because the coefficient \( f_{2ij}(r) \) is traceless as well, as we will see. The "3" is just an arbitrary normalization in the definition of \( D_{ij} \).

Note that under rotations,

\( q \rightarrow \text{scalar} \quad (l=0) \quad \mathbf{d} \rightarrow \text{vector} \quad (l=1) \quad D_{ij} \rightarrow l=2 \)

i.e., 2-index symmetric tensors transform in 1-to-1 correspondence with \( l, m \). Both have 5 (independent) components.

This is why we subtract the trace in the definition of \( D_{ij} \): a \( 3 \times 3 \) matrix \( M_{ij} \) in general has 9 components.

Trace: \( M_{ii} \rightarrow l=0 \) components

Antisymmetric: \( M_{ij} - M_{ji} \rightarrow l=1 \)

Symmetric-tensors \( M_{ij} + M_{ji} - \frac{2}{3} \delta_{ij} M_{kk} \rightarrow l=2 \)

Covariant: \( \partial_{ij} f(r^{-1}) = - \partial_{ji} f(r^{-1}) \)

\( f_{2ij} \) and \( \partial_{ij} f_{2ij} = - \delta_{ij} f_{2} + \frac{3}{r^2} x_i x_j \)

\( f_{2} = \frac{X_i}{r^3} \) f_{2ij} = \frac{3 x_i x_j - \delta_{ij} r^2}{r^5} \) note \( \delta_{ij} f_{2} = 0 \) as advertised.
So we have

\[ \phi(r) = \frac{q}{r} + \frac{\vec{d} \cdot \vec{r}}{r^3} + \frac{1}{2} \frac{x_i x_j - \frac{1}{3} \delta_{ij} r^2}{r^5} D_{ij} + \ldots \]

where \( \vec{d} = \frac{\vec{r}}{r} \) has \( n^2 = 1 \).

Clearly \( |d_i| \leq q \) \( |D_{ij}| \leq q^2 r^2 \) as expected.

Systematize:

\[ \phi(r) = \int \frac{d^3 p(r')}{(r-r')} = \int d^3 p(r') \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{\sqrt{4\pi}}{2l+1} \frac{r^l}{r^l} Y_{lm}^*(r') Y_{lm}(r) \]

\[ = \sum_{l=0}^{\infty} \frac{1}{2l+1} \sum_{m=-l}^{l} q_{2lm} Y_{lm}(r) \]

where \( q_{2lm} = \sqrt{\frac{4\pi}{2l+1}} \int d^3 p(r') r^l Y_{lm}^*(r') \) “2nd-order moment”

Note: \( \phi(r') = \phi(r) \Rightarrow \sum_{m=-l}^{l} q_{2lm} Y_{lm}(r') = \sum_{m=-l}^{l} q_{2lm} Y_{lm}(r) \)

\( \Rightarrow q_{2lm} = (-1)^m q_{2-lm} \) which is verified by its definition in terms of the integral over the real charge density \( p(r') \).

Relating to the \( l = 1 \):

\[ q_{11} = \frac{1}{2 \sqrt{3}} \int d^3 p(r') x'_x y'_y = \frac{1}{2 \sqrt{3}} \left( x_1 y'_x + y_1 x'_y \right) \]

\( \Rightarrow q_{11} = \frac{1}{2 \sqrt{3}} \int d^3 p(r') x' y'_x \)

Similarly \( q_{10} = d_2 \)
Exercise: Show 19.25 in Garg

\[ q_{20} = D_{20} \quad q_{21} = \pm \frac{1}{\sqrt{2}} (D_{x2} \mp i D_{y2}) \]

\[ q_{22} = \pm \frac{i}{\sqrt{10}} (D_{xx} - D_{yy} \mp 2i D_{xy}) \]

Assignment: Read about Earsenhaw's Theorem in Garg.

In a charge free region \(<\Phi>\) over a sphere equals \(\Phi_{\text{center}}\).
Charge distributions in external fields

A charge distribution characterized by $q$, $\vec{a}$, $\vec{d}$, etc... what gives its field away, in an external $\vec{E}$ field experiences forces and torques. What are they?

Take now $\phi(\vec{r}) = \text{potential due to external sources}$

The energy in this configuration is (with $q_a = \int d^3r \rho(\vec{r})$)

$$U = \sum_a q_a \phi(\vec{r}_a) = \int d^3r \rho(\vec{r}) \phi(\vec{r})$$

Now choose coordinate system with origin within $\rho$ (hect, choose the same one as was used to define multipoles):

expanding $\phi(\vec{r})$

$$\phi(\vec{r}) = \phi(\vec{0}) + \sum x_i \vec{\partial} \phi(\vec{0}) + \frac{1}{2} \sum x_i x_j \vec{\partial}_i \vec{\partial}_j \phi(\vec{0}) + \ldots$$

so

$$U = \int d^3r \rho(\vec{r}) \left[ \phi(\vec{0}) + \sum x_i \vec{\partial} \phi(\vec{0}) + \frac{1}{2} \sum x_i x_j \vec{\partial}_i \vec{\partial}_j \phi(\vec{0}) + \ldots \right]$$

[using $\nabla^2 \phi|_0 = 0$ since the external field is due to remote charges].
Examine result: contributions to $V$:

(I) Lowest: $q \phi(0)$

If we were to move the distribution to a new location, $\bar{r}$, we would have instead $q \phi(\bar{r})$. The force on this is

$$\bar{F} = -\nabla \bar{V}(\bar{r}) = q (-\nabla \phi) = q \bar{E}(\bar{r})$$

No surprise!

(ii) 1st correction: $d_i \phi \delta E_i = -\bar{d} \cdot \bar{E}$

As above, a rigid translation $\rightarrow -\bar{d} \cdot \bar{E}(\bar{r})$

Force $\bar{F} = -\nabla (\bar{d} \cdot \bar{E}) = \bar{d} \cdot \nabla \bar{E}$

or $\bar{F}_j = d_j \partial_j \bar{E} = d_j (\partial_j \bar{E}_i - \bar{E}_j) + (\bar{d} \cdot \nabla) \bar{E}_j$

But in static situation $\nabla \times \bar{E} = 0 \Rightarrow \bar{F} = (\bar{d} \cdot \nabla) \bar{E}$

Even if $\partial_j \bar{E}_j = 0$ ($\bar{E}$ = uniform) there is a torque

$$\bar{N} = \int \nabla \times (d^2 \rho(r) \bar{E}(r)) = \int (\bar{d} \cdot \nabla \rho(r)) \times (\bar{E}(0) + x \cdot \partial \bar{E}(0) + ...)$$

Lowest term

$$\bar{N} = \bar{d} \times \bar{E}(0)$$

(iii) 2nd cor: $\frac{1}{2} D_{ij} \partial_j \phi \delta E_i = -\frac{1}{2} D_{ij} \partial_j \partial_i \phi(0)$

$$\bar{F}_k = -\frac{1}{2} D_{ij} \partial_j \partial_k \bar{E}_i$$

and $N_i = \delta_{ijk} \left( \left( \bar{d} \cdot \rho(r) x_j x_k \right) \partial_i \bar{E}_k = \frac{1}{2} \delta_{ijk} D_{jm} \partial_m \bar{E}_k$ where we used $\nabla \times \bar{E} = 0$ to include the $\delta_{ijm}$ term at no price.
Charge on charge:

\[ f_1 \rightarrow \vec{R} \rightarrow f_2 \]

Energy of configuration: use \( f_1 \) as source of external field and \( f_2 \) in presence of this

\[ \phi = \frac{q_1}{r} + \frac{\vec{d} \cdot \vec{E}}{r^3} + \frac{1}{2} D_{ij} \delta_{ij} \frac{E_i E_j}{r^5} + \cdots \quad (1) \]

and

\[ U = \frac{q_2 \phi(0)}{4\pi} - \vec{d} \cdot \vec{E}(0) - \frac{1}{8\pi} D_{ij} \delta_{ij} E^2(0) + \cdots \quad (2) \]

Here "0" is in the \( f_2 \) distribution, so take that as \( \vec{R} \) from "center" of \( f_1 \); set \( \vec{r} = \vec{R} \) in (1). Then stick (1) into (2). Expanding \( 1/\vec{R} \) we have

\[ U = U(0) + U(1) + U(2) + \cdots \]

with

\[ U(0) = q_2 \left[ \frac{q_1}{r^2} + \cdots \right] = \frac{q_1 q_2}{r} \]

the potential energy of two point charges
Next
\[ U^{(i)} = q_i \left[ \frac{\vec{d}_i \cdot \vec{R}}{R^3} \right] + \left[ -d_2 \cdot \vec{E} \right] \]

where \[ \vec{E} = -\nabla \left( \frac{q_1}{r} \right) \bigg|_{r=R} \]

\[ = \frac{q_1}{R^3} \]

\[ \Rightarrow \quad U^{(i)} = q_i \frac{d_i \cdot \vec{R}}{R^5} - q_1 \frac{d_2 \cdot \vec{R}}{R^5} \]

(Looks asymmetric, but isn't: exchange \( q_1 \leftrightarrow q_2 \)
also exchanges \( \vec{R} \leftrightarrow -\vec{R} \).

At order \( 1/R^3 \), there are 4 \( D_{ij} \) terms and \( d_1, d_2 \) terms.
The latter are
\[ U^{(2)} = -\vec{d}_2 \cdot \vec{E} \]

with \[ \vec{E} = -\nabla \left( \frac{d_i}{r} \right) \bigg|_{r=R} = \frac{3 R_i R_j - \delta_{ij} R^2}{R^5} \]

\[ \Rightarrow \quad U^{(2)} = \frac{d_i d_j R^2}{R^5} - 3 (\vec{R} \cdot \vec{d}_1)(\vec{R} \cdot \vec{d}_2) \]

\[ \min \text{ for } \vec{d}_i \parallel \vec{d}_j, \max \text{ for } \vec{d}_i / R, \vec{d}_j / R^2 \] or vice versa.