6-24 After rearrangement, the Schrödinger equation is 
\[
\frac{d^2 \psi}{dx^2} = \frac{2m}{\hbar^2} \left( U(x) - E \right) \psi(x)
\]
with 
\[
U(x) = \frac{1}{2} m \omega^2 x^2
\]
for the quantum oscillator. Differentiating \( \psi(x) = C x e^{-\alpha x^2} \) gives 
\[
\frac{d\psi}{dx} = -2\alpha x \psi(x) + C e^{-\alpha x^2}
\]
and 
\[
\frac{d^2 \psi}{dx^2} = -2\alpha x \frac{d\psi}{dx} - 2\alpha \psi(x) - (2\alpha x)C e^{-\alpha x^2} = (2\alpha x)^2 \psi(x) - 6\alpha \psi(x).
\]
Therefore, for \( \psi(x) \) to be a solution requires 
\[
(2\alpha x)^2 - 6\alpha = \frac{2m}{\hbar^2} \left( U(x) - E \right) \psi(x) \Rightarrow \psi(x) = \frac{\psi(x)}{\frac{2m}{\hbar^2} \left( U(x) - E \right)}.
\]
Equating coefficients of like terms gives 
\[
2\alpha = \frac{m\omega^2}{\hbar^2} \quad \text{and} \quad 6\alpha = \frac{2mE}{\hbar^2}.
\]
Thus, \( \alpha = \frac{m\omega}{2\hbar} \) and 
\[
E = \frac{3\alpha h^2}{m} = \frac{3}{2} \hbar \omega.
\]
The normalization integral is 
\[
\int_{-\infty}^{\infty} \psi(x)^2 dx = 2C^2 \int_{-\infty}^{\infty} x^2 e^{-2\alpha x^2} dx
\]
where the second step follows from the symmetry of the integrand about \( x = 0 \). Identifying \( a \) with \( 2\alpha \) in the integral of Problem 6-32 gives 
\[
1 = 2C^2 \left( \frac{1}{8\alpha} \right) \left( \frac{\pi}{2a} \right)^4 \quad \text{or} \quad C = \left( \frac{32\alpha^3}{\pi} \right)^{1/4}.
\]
6-25 At its limits of vibration \( x = \pm A \) the classical oscillator has all its energy in potential form: 
\[
E = \frac{1}{2} m \omega^2 A^2 \quad \text{or} \quad A = \left( \frac{2E}{m\omega^2} \right)^{1/2}.
\]
If the energy is quantized as \( E_n = \left( n + \frac{1}{2} \right) \hbar \omega \), then the corresponding amplitudes are 
\[
A_n = \left( \frac{2n+1}{m\omega \hbar} \right)^{1/2}.
\]
6-32 The probability density for this case is 
\[
\psi_0(x)^2 = C_0^2 e^{-ax^2} \quad \text{with} \quad C_0 = \left( \frac{a}{\pi} \right)^{1/4}
\]
and \( a = \frac{m\omega}{2\hbar} \).

For the calculation of the average position \( \langle x \rangle = \int_{-\infty}^{\infty} x |\psi_0(x)|^2 dx \) we note that the integrand is an odd function, so that the integral over the negative half-axis \( x < 0 \) exactly cancels that over the positive half-axis \( x > 0 \), leaving \( \langle x \rangle = 0 \). For the calculation of \( \langle x^2 \rangle \), however, the integrand \( x^2 |\psi_0|^2 \) is symmetric, and the two half-axes contribute equally, giving 
\[
\langle x^2 \rangle = 2C_0^2 \int_0^{\infty} x^2 e^{-ax^2} dx = 2C_0^2 \left( \frac{1}{4a} \right) \left( \frac{\pi}{a} \right)^4.
\]
Substituting for \( C_0 \) and \( a \) gives 
\[
\langle x^2 \rangle = \frac{1}{2a} = \frac{\hbar}{2m\omega} \quad \text{and} \quad \Delta x = \left( \langle x^2 \rangle - \langle x \rangle^2 \right)^{1/2} = \left( \frac{\hbar}{2m\omega} \right)^{1/2}.
\]
6-33 (a) Since there is no preference for motion in the leftward sense vs. the rightward sense, a particle would spend equal time moving left as moving right, suggesting \( \langle p_x \rangle = 0 \).

(b) To find \( \langle p_x^2 \rangle \) we express the average energy as the sum of its kinetic and potential energy contributions: \( \langle E \rangle = \langle \frac{p_x^2}{2m} \rangle + \langle U \rangle = \langle \frac{p_x^2}{2m} \rangle + \langle U \rangle \). But energy is sharp in the oscillator ground state, so that \( \langle E \rangle = E_0 = \frac{1}{2} \hbar \nu \). Furthermore, remembering that \( U(x) = \frac{1}{2} m \omega^2 x^2 \) for the quantum oscillator, and using \( \langle x^2 \rangle = \frac{\hbar}{2m\omega} \),

\[
\langle p_x^2 \rangle = 2m(E_0 - \langle U \rangle) = 2m \left( \frac{\hbar \nu}{4} \right) = \frac{m \hbar \nu}{2}.
\]

(c) \( \Delta p_x = \left( \langle p_x^2 \rangle - \langle p_x \rangle^2 \right)^{1/2} = \left( \frac{m \hbar \nu}{2} \right)^{1/2} \)

6-34 From Problems 6-32 and 6-33, we have \( \Delta x = \left( \frac{\hbar}{2m\omega} \right)^{1/2} \) and \( \Delta p_x = \left( \frac{m \hbar \nu}{2} \right)^{1/2} \). Thus,

\[
\Delta x \Delta p_x = \left( \frac{\hbar}{2m\omega} \right)^{1/2} \left( \frac{m \hbar \nu}{2} \right)^{1/2} = \frac{\hbar}{2} \text{ for the oscillator ground state. This is the minimum uncertainty product permitted by the uncertainty principle, and is realized only for the ground state of the quantum oscillator.}
\]

6-35 Applying the momentum operator \( [p_x] = \left( \frac{\hbar}{i} \right) \frac{d}{dx} \) to each of the candidate functions yields

(a) \( [p_x] \{ A \sin(kx) \} = \left( \frac{\hbar}{i} \right) k \{ A \cos(kx) \} \)

(b) \( [p_x] \{ A \sin(kx) - A \cos(kx) \} = \left( \frac{\hbar}{i} \right) k \{ A \cos(kx) + A \sin(kx) \} \)

(c) \( [p_x] \{ A \cos(kx) + iA \sin(kx) \} = \left( \frac{\hbar}{i} \right) k \{ -A \sin(kx) + iA \cos(kx) \} \)

(d) \( [p_x] \{ e^{i(kx-a)} \} = \left( \frac{\hbar}{i} \right) k \{ e^{i(kx-a)} \} \)

In case (c), the result is a multiple of the original function, since

\[
-A \sin(kx) + iA \cos(kx) = i \left[ A \cos(kx) + iA \sin(kx) \right].
\]
The multiple is \( \left( \frac{h}{i} \right)^i = \hbar k \) and is the eigenvalue. Likewise for (d), the operation \( \left[ p_x \right] \) returns the original function with the multiplier \( \hbar k \). Thus, (c) and (d) are eigenfunctions of \( \left[ p_x \right] \) with eigenvalue \( \hbar k \), whereas (a) and (b) are not eigenfunctions of this operator.

7-1

(a) The reflection coefficient is the ratio of the reflected intensity to the incident wave intensity, or \( R = \frac{\int |1/2 |(1-i)|^2}{\int |1/2 |(1+i)|^2} \). But

\[ |1-i|^2 = (1-i)(1-i)^* = (1-i)(1+i) = |1+i|^2 = 2, \]

so that \( R = 1 \) in this case.

(b) To the left of the step the particle is free. The solutions to Schrödinger’s equation are \( e^{ix} \) with wavenumber \( k = \left( \frac{2mE}{\hbar^2} \right)^{1/2} \). To the right of the step \( U(x) = U \) and the equation is

\[ \frac{d^2 \psi}{dx^2} = \frac{2m}{\hbar^2} (U - E) \psi(x). \]

With \( \psi(x) = e^{-ix} \), we find \( \frac{d^2 \psi}{dx^2} = k^2 \psi(x) \), so that

\[ k = \left[ \frac{2m(U-E)}{\hbar^2} \right]^{1/2}. \]

Substituting \( k = \left( \frac{2mE}{\hbar^2} \right)^{1/2} \) shows that

\[ \left[ \frac{E}{(U-E)} \right]^{1/2} = 1 \]

or \( \frac{E}{U} = \frac{1}{2} \).

(c) For 10 MeV protons, \( E = 10 \) MeV and \( m = \frac{938.28 \text{ MeV}}{c^2} \). Using

\[ h = 197.3 \text{ MeV} \frac{\text{fm}}{c} \left( 1 \text{ fm} = 10^{-15} \text{ m} \right), \]

we find

\[ \delta = \frac{1}{k} = \frac{h}{(2mE)^{1/2}} = \frac{197.3 \text{ MeV} \frac{\text{fm}}{c}}{\sqrt{2 \left( \frac{938.28 \text{ MeV}}{c^2} \right)(10 \text{ MeV})}} = 1.44 \text{ fm}. \]

7-2

(a) To the left of the step the particle is free with kinetic energy \( E \) and corresponding wavenumber \( k_1 = \left( \frac{2mE}{\hbar^2} \right)^{1/2} \):

\[ \psi(x) = Ae^{ik_1x} + Be^{-ik_1x} \quad x \leq 0 \]

To the right of the step the kinetic energy is reduced to \( E - U \) and the wavenumber is now \( k_2 = \left[ \frac{2m(E-U)}{\hbar^2} \right]^{1/2} \):

\[ \psi(x) = Ce^{ik_2x} + D e^{-ik_2x} \quad x \geq 0 \]

with \( D = 0 \) for waves incident on the step from the left. At \( x = 0 \) both \( \psi \) and \( \frac{d\psi}{dx} \) must be continuous:

\[ \psi(0) = A + B = C \]

\[ \frac{d\psi}{dx} \bigg|_{x=0} = ik_1(A - B) = ik_2C. \]
(b) Eliminating $C$ gives $A + B = \frac{k_1}{k_2} (A - B)$. Thus,

$$ R = \left| \frac{B}{A} \right|^2 = \frac{(k_1/k_2 - 1)^2}{(k_1/k_2 + 1)^2} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} $$

$$ T = 1 - R = \frac{4k_1k_2}{(k_1 + k_2)^2} $$

(c) As $E \to U$, $k_2 \to 0$, and $R \to 1$, $T \to 0$ (no transmission), in agreement with the result for any energy $E < U$. For $E \to \infty$, $k_1 \to k_2$ and $R \to 0$, $T \to 1$ (perfect transmission) suggesting correctly that very energetic particles do not see the step and so are unaffected by it.

7-3 With $E = 25$ MeV and $U = 20$ MeV, the ratio of wavenumber is

$$ \frac{k_1}{k_2} = \left( \frac{E}{E-U} \right)^{\frac{I_P}{I_{E}}} = \left( \frac{25}{25-20} \right)^{\frac{I_P}{I_{E}}} = \sqrt{5} = 2.236 \right.$$ Then from Problem 7-2 $R = \frac{(\sqrt{5} - 1)^2}{(\sqrt{5} + 1)^2} = 0.146$ and $T = 1 - R = 0.854$. Thus, 14.6% of the incoming particles would be reflected and 85.4% would be transmitted. For electrons with the same energy, the transparency and reflectivity of the step are unchanged.

7-4 The reflection coefficient for this case is given in Problem 7-2 as

$$ R = \left| \frac{B}{A} \right|^2 = \frac{(k_1/k_2 - 1)^2}{(k_1/k_2 + 1)^2} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} \right.$$ The wavenumbers are those for electrons with kinetic energies $E = 54.0$ eV and $E - U = 54.0$ eV + 10.0 eV = 64.0 eV:

$$ \frac{k_1}{k_2} = \left( \frac{E}{E-U} \right)^{\frac{I_P}{I_{E}}} = \left( \frac{54 \text{ eV}}{64 \text{ eV}} \right)^{\frac{I_P}{I_{E}}} = 0.918 \right.$$ Then, $R = \frac{(0.918 6 - 1)^2}{(0.918 6 + 1)^2} = 1.80 \times 10^{-3}$ is the fraction of the incident beam that is reflected at the boundary.

7-5 (a) The transmission probability according to Equation 7.9 is

$$ \frac{1}{T(E)} = 1 + \left[ \frac{U^2}{4E(U-E)} \right] \sinh^2 \alpha L \text{ with } \alpha = \frac{2m(U - E)^{\frac{1}{2}I_P}}{h} \right.$$ For $E \ll U$, we find

$$ (\alpha L)^2 = \frac{2mU^2}{h^2} \gg 1 \text{ by hypothesis. Thus, we may write } \sinh \alpha L \approx \frac{1}{2} e^{\alpha L} \right.$$ Also

$$ U - E \approx U \right.$$ giving

$$ \frac{1}{T(E)} = 1 + \left( \frac{U}{16E} \right)e^{2\alpha L} = \left( \frac{U}{16E} \right)e^{2\alpha L} \text{ and a probability for transmission } P = T(E) = \left( \frac{16E}{U} \right)e^{-2\alpha L} \right.$$ (b) Numerical Estimates: $(\hbar = 1.055 \times 10^{-34} \text{ Js})$
1) For \( m = 9.11 \times 10^{-31} \) kg, \( U - E = 1.60 \times 10^{-21} \) J, \( L = 10^{-10} \) m; 
\[ \alpha = \frac{2m(U-E)}{\hbar} = 5.12 \times 10^8 \text{ m}^{-1} \text{ and } e^{-2\alpha L} = 0.90 \]

2) For \( m = 9.11 \times 10^{-31} \) kg, \( U - E = 1.60 \times 10^{-19} \) J, \( L = 10^{-10} \) m; 
\[ \alpha = 5.12 \times 10^9 \text{ m}^{-1} \text{ and } e^{-2\alpha L} = 0.36 \]

3) For \( m = 6.7 \times 10^{-27} \) kg, \( U - E = 1.60 \times 10^{-13} \) J, \( L = 10^{-15} \) m; 
\[ \alpha = 4.4 \times 10^{14} \text{ m}^{-1} \text{ and } e^{-2\alpha L} = 0.41 \]

4) For \( m = 8 \) kg, \( U - E = 1 \) J, \( L = 0.02 \) m; \( \alpha = 3.8 \times 10^{34} \text{ m}^{-1} \text{ and } e^{-2\alpha L} = e^{-1.5 \times 10^{33}} = 0 \)

Since the alpha particle has the combined mass of 2 protons and 2 neutrons, or about 3.755.8 MeV/\( c^2 \), the first approximation to the decay length \( \delta \) is

\[ \delta = \frac{\hbar}{(2ml)^{1/2}} = \frac{197.3 \text{ MeV fm}/c}{\left[ \frac{2(3.755.8 \text{ MeV}/c^2)(30 \text{ MeV})}{(2)} \right]^{1/2}} = 0.415 \text{ fm} . \]

This gives an effective width for the (infinite) well of \( R + \delta = 9.415 \text{ fm} \), and a ground state energy \( E_1 = \frac{-h^2 (197.3 \text{ MeV fm}/c)^2}{2(3.755.8 \text{ MeV}/c^2)(9.415 \text{ fm})} = 0.577 \text{ MeV} \). From this \( E \) we calculate \( U - E = 29.42 \text{ MeV} \) and a new decay length

\[ \delta = \frac{197.3 \text{ MeV fm}/c}{\left[ \frac{2(3.755.8 \text{ MeV}/c^2)(29.42 \text{ MeV})}{(2)} \right]^{1/2}} = 0.4197 \text{ fm} . \]

This, in turn, increases the effective well width to 9.4197 fm and lowers the ground state energy to \( E_1 = 0.576 \text{ MeV} \). Since our estimate for \( E \) has changed by only 0.001 MeV, we may be content with this value. With a kinetic energy of \( E_1 \), the alpha particle in the ground state has speed \( v_1 = \left( \frac{2E_1}{m} \right)^{1/2} = \left[ \frac{2(0.576 \text{ MeV})}{(3.755.8 \text{ MeV}/c^2)} \right]^{1/2} = 0.0175 \text{ c} \). In order to be ejected with a kinetic energy of 4.05 MeV, the alpha particle must have been preformed in an excited state of the nuclear well, not the ground state.

The collision frequency \( f \) is the reciprocal of the transit time for the alpha particle crossing the nucleus, or \( f = \frac{v}{2R} \), where \( v \) is the speed of the alpha. Now \( v \) is found from the kinetic energy which, inside the nucleus, is not the total energy \( E \) but the difference \( E - U \) between the total energy and the potential energy representing the bottom of the nuclear well. At the nuclear radius \( R = 9 \) fm, the Coulomb energy is

\[ \frac{kZe(2e)}{R} = 2Z \left( \frac{k^2 e^2}{\alpha_0} \right) \left( \frac{\alpha_0}{R} \right) = 2(88)(27.2 \text{ eV}) \left( \frac{5.29 \times 10^4 \text{ fm}}{9 \text{ fm}} \right) = 28.14 \text{ MeV} . \]

From this we conclude that \( U = -1.86 \text{ MeV} \) to give a nuclear barrier of 30 MeV overall. Thus an alpha with \( E = 4.05 \text{ MeV} \) has kinetic energy \( 4.05 + 1.86 = 5.91 \text{ MeV} \) inside the nucleus. Since the alpha particle has the combined mass of 2 protons and 2 neutrons, or about 3.755.8 MeV/\( c^2 \), this kinetic energy represents a speed
\[ v = \left( \frac{2E_k}{m} \right)^{1/2} = \left[ \frac{2(5.91)}{3755.8 \text{ MeV}/c^2} \right]^{1/2} = 0.056 \, c. \]

Thus, we find for the collision frequency \( f = \frac{v}{2R} = \frac{0.056c}{2(9 \text{ fm})} = 9.35 \times 10^{20} \text{ Hz}. \)