Physics 210B

Module I - Kinetic Theory

I. Mechanics Foundations

Boltzmann Eq. / Kinetic Theory Founded on Hamiltonian Mechanics

→ Review of Foundations in Classical Mechanics → Develop Kinetic Theory!

Statistical Dynamics is simply the marriage of mechanics and statistical treatment of a large number of particles (\(N \gg 1\)).

→ Key Concepts:

- Hamiltonian System, Liouville Theory
- Integrability
- Resonances and Chaos
- Ergodicity and Mixing
- Kolmogorov Dynamics
- Entropy

N.B.: Not self-contained
- See standard refs - list.
- Hamiltonian Systems

How Hamiltonian?

\[ L = L(\bar{z}, \dot{\bar{z}}, t) \]
\[ \bar{z} = \bar{z}(\phi, \dot{\phi}, \partial \phi / \partial x) \quad \phi \equiv \text{field} \]

\[ p = \partial L / \partial \dot{\bar{z}} \]

\[ H = H(p, \bar{z}) = p \dot{\bar{z}} - L \]

\[ \dot{\bar{z}} = -\frac{\partial H}{\partial p} \quad \bar{z} = \frac{\partial H}{\partial \phi} \]

- \( p, \bar{z} \) equal footing
- usual phase space parametrization
  \[ \varphi = \mathcal{F}(p, \bar{z}, t) \equiv \text{phase space density} \]
  \[ \rightarrow \text{distribution function} \]
No attractors $\iff$ No source, sinks, asymptotically stable cycles.

\begin{itemize}
  \item Yes
  \item No
\end{itemize}
\[ \int d^3 \rho \ E(\rho, \xi, t) = \eta(\xi, t), \text{ etc.} \]

**Liouville's Thm:**

"Phase volume conserved"

"Phase space density conserved along particle orbits."

\[ \frac{df}{dt} + \frac{\partial}{\partial \xi} \left( \xi \frac{df}{d\xi} \right) + \frac{\partial}{\partial \mathbf{p}} \left( \mathbf{p} \frac{df}{d\mathbf{p}} \right) = 0 \]

\[ \frac{d \xi}{dt} + \frac{d \mathbf{p}}{dt} = 0 = \mathbf{V}_p = 0 \]

Phase space flow incompressible for Hamiltonian system

\[ \frac{df}{dt} + \mathbf{V}_p \cdot \nabla_p f = 0 = \frac{df}{dt} \]

\[ \mathbf{V}_p = (\xi, \mathbf{p}) \]

\[ \Rightarrow \nabla \cdot \mathbf{V}_p = 0 \]

\begin{center}
\text{no attractors in Hamiltonian Mechanics.}
\end{center}
Related: Poincaré Recurrence Theorem

Define: phase space flow
\( g^t : \text{transformation at time } t \)
\( p(0), z(0) \rightarrow p(t), z(t) \) along trajectories

\[\begin{align*}
P & \xrightarrow{g^t} P' \\
\text{domain} & \xrightarrow{g^t} \text{downstream domain}
\end{align*}\]

First: Another look at Liouville ---
Now Hamiltonian equations constitute autonomous system.

\[
\begin{align*}
\dot{x} &= f(x) \\
\left( \begin{array}{c} \dot{x} \\ \dot{p} \end{array} \right) &= \left( \begin{array}{c} \partial f / \partial x \\ -\partial H / \partial x \end{array} \right)
\end{align*}
\]

defines \( f \)}
then for small change:

\[ g^t(x) = x + f(x) t + o(t^2) \]

so phase volume \( V_t \) at \( t \):

\[ V_t(t) = \int_{D_0} \left| \frac{\partial x'}{\partial x} \right| \]

\[ \text{Jacobian of transform} \]

\[ = \int_{D_0} \det \left| \frac{\partial g^t(x)}{\partial x} \right| \]

\[ \frac{\partial g^t(x)}{\partial x} = \frac{I}{t} + \frac{\partial F}{\partial x} t + o(t^2) \]

so small identity:

\[ \det \left( \frac{I}{t} + \frac{\partial F}{\partial x} t \right) \approx 1 + t \cdot tr A \]

\[ \therefore V_t(t) = \int d^o x \sum \left[ 1 + t + t \cdot tr \left[ \frac{\partial F}{\partial x} \right] \right] + o(t^2) \]
but \[ \frac{\partial f}{\partial t} = \mathbf{D} \cdot \mathbf{F} \]

and \[ \frac{\partial \mathbf{p}}{\partial t} = \mathbf{f} \quad \mathbf{D} \cdot \mathbf{p} = 0 \quad \text{so} \quad \mathbf{D} \cdot \mathbf{F} = 0. \]

\[ \mathbf{V}_b \left( \mathbf{r} + \mathbf{t} \right) = \mathbf{V}_b \left( \mathbf{r} \right) \quad \text{Phase volume constancy} \]

Recurrent

\[ \rightarrow \text{Fundamental to ergodic theory} \]

Loosely, "what goes around comes around arbitrarily closely", for \underline{Hamiltonian system}

\[ \mathbf{U} \times \text{system universe bounded} \]

\[ g^f \text{ Hamiltonian = volume preserving} \]
For any $x$ in $U$, we can define $B(x, \epsilon)$. A ball (neighborhood) in phase space around point $x \mapsto (\theta, \varphi)$ of radius $\epsilon$ then $\exists x' \in B(x)$ such that $J^n(x') \in B(x)$. There is a point in the $\epsilon$-ball of $x$ such that $n$-iterations of evolution operator yield a point in $\epsilon$-ball. Point/orbit recurs, arbitrarily closely.

N.B. Why is this of concern in statistical mechanics?
Proof:

Consider $g^n(B)$,

if each $g^n$ disjoint,

$$\lim_{n \to \infty} \bigcup g^n = \emptyset,$$ but $U$ bounded

$\Rightarrow$ contradiction!

So, must have:

$$g^n(B) \cap g^l(B) \neq \emptyset \quad \{\text{intersection of arbitrary closed not empty}\}$$

$$g^{m+n}(B) \cap B \neq \emptyset$$

So, $\exists$ some $x'$ arbitrarily close to $x$.

Q.E.D. / A.A.O.F.C.S.
Implications

- Box with particles
- Particles escape thru hole
- Eventually (3) will re-enter
  Time.

- Torus (why?)

\[
\begin{align*}
\mathbf{v}_1 &= x_1 \\
\mathbf{v}_2 &= x_2 \\
\phi &= x_1 \\
\phi &= x_2
\end{align*}
\]

Consider \( x_1, x_2 \) (translation)

then: if \( g^t(\mathbf{v}_1, \mathbf{v}_2) = (\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_2 + \mathbf{v}_2) \)
then \( \alpha_1/\alpha_2 \) irrational \( \Rightarrow \) winding
  'fills' the torus
  i.e. comes arbitrarily close to i.c.,
  but stays on torus.

\[ \Rightarrow \text{classic example of ergodicity} \]

N.B. linear on time.

\[ \Rightarrow \text{integrability and torus destruction} \]

Integrability \( \Rightarrow \) can canonically transform

\[ B \mathcal{I} \rightarrow \mathcal{I}, \Theta \]

\[ \frac{d\mathcal{I}}{dt} = 0, \quad \frac{d\Theta}{dt} = \omega (\mathcal{I}) \]

i.e. \{ all variables cyclic \}

\[ \mathcal{I} \text{ constant } \Rightarrow \text{IOMs} \]
motion defines torus
\((n \text{-torus})\)

\[
\frac{d\theta}{dt} = \omega_1 (T_1) + \\
\frac{d\phi}{dt} = \omega_2 (T_2) +
\]

scanning \(T_1, T_2 \Rightarrow\) define nested tori
\(\sim\sim\sim\)

i.e. box and particle: 2D

\[
\begin{cases}
\omega_1 = \frac{\pi^2 T_1}{ma^2} \\
\omega_2 = \frac{\pi^2 T_2}{mb^2} \\
E = T_1 \omega_1 + T_2 \omega_2
\end{cases}
\]

- motion on each toroidal surface ergodically, unless \(\omega_1/\omega_2\) rational

- set of surfaces \(\Rightarrow\) volume of phase space

- motion in conditionally periodic

\(\Rightarrow\) Poincare Recurrence guarantees \(\text{almost return}\) to i.e.
Begin two questions:
- what if unable to integrate?
  - integrate approximately

\[ H = H_0(I) + \epsilon H_1 \]

unperturbed integrable symmetry breaking perturbation

Integrate to some order or \( \in \),

c.e. transform \( J, \Theta \rightarrow I, \phi \)

\[ \frac{d}{dt} I = 0 \]
\[ \phi = \omega(I) \]

\( \Rightarrow \) How fragile are surfaces? Can nested structure be maintained with \( o(\epsilon) \) deformation?

Answer \( \Psi \) use canonical perturbation theory - i.e. PT which maintains Hamiltonian structure.
Perturbation expansion to canonical transformation.

1 DOF

\[ \phi = \Theta + \varepsilon \frac{\partial S}{\partial I} \]

\[ I = I - \varepsilon \frac{\partial \Omega}{\partial \omega} \]

\[ \omega = \omega_0 (I) + \varepsilon \frac{1}{2} \omega_1 (I) \]

\[ s_1 = -\sum H_{n_1} (I) \frac{e^{i n_\omega}}{c_n \omega_0 (I)} \]

\[ \kappa_1 = \langle H_1 \rangle \]

Beyond: \( (2 + \text{DOF}) \)

\[ \nabla \cdot \omega_0 (I) = 0 \]

\[ n_1 \omega_{0,1} + n_2 \omega_{0,2} = 0 \]

\[ n_1 \alpha_1 + n_2 \alpha_2 = 0 \]
Resonance!

- occurs at rational surfaces, i.e.

\[ \frac{x_1}{x_2} = \frac{-n_2}{n_1} \]

(sign irrelevant)

- winding ratio is rational number

\[ \frac{n_2}{n_1} = \text{pitch} = \frac{w_1}{w_2} \]

- trajectories close \( \Rightarrow \) don't ergodically cover the surface.

\( \Rightarrow \) identify where:

- perturbative integration fails
- surfaces most fragile

**Welcome to the small denominator problem!**

- crucial to Hamiltonian Chaos.

See Ott, Chapt. 07

Examples:
E.g. - Wave - particle

\( V = \omega / k \)

\( \frac{\partial S}{\partial t} = H - H' \)

field lines / foliations

\( \mathcal{E} = \frac{m}{n} \), \( \mathcal{E} = \mathcal{E}(r) \)

How forward? \( \dot{I} \)

If \( I = 1 \) (isolated) resonance secular perturbation theory (no avg. over fast variable)

Can transform into form:

\[
\langle H(\hat{I}_{1}, \phi_{1}) \rangle = \frac{1}{2} (\hat{I}_{1} - \hat{I}_{0}) \cdot \hat{S} \cdot \frac{\partial^{2} H_{0}}{\partial \hat{I}_{1}^{2}} - F \cdot \cos \phi_{1}
\]

\[
= \frac{\partial^{2} H_{0}}{\partial \hat{S}_{1}^{2}} \cdot \hat{I}_{0} \cdot \hat{I}_{0} - F \cdot \cos \phi_{1}
\]

\[ F = -2 e H_{\parallel} \]
~ pendulum

\[ \frac{\partial^2 \theta}{\partial t^2} = \frac{d \omega}{dt} + \theta \]

\( \text{differential} \)

\( \text{rotational} \)

\( \text{Shear} \)

\( \text{accidental resonance} \)

\( \text{Motion located in/about phase space island} \)

\( \text{separatrix} \)

\[ \text{divides phase space into bounded/trapped and circulating orbits} \]

\( \text{island} \rightarrow \text{island chain} \)

\( \text{foliater or distorted resonant surface} \)
- Width of separate x width

\[(A\Gamma)_{\text{max}} = 2CF/\gamma \]

\[\approx 2\left(\frac{-2\xi \beta \sigma}{\gamma} \right) \left(\frac{\Delta \gamma}{\sigma^2} \right)^{1/2}\]

i.e. if particle and wave:

\[h = (P + m_\omega/k)^2/2m + e^\lambda \cos \theta\]

\[AP = (e^\lambda m)^{1/2}\]

\[AV = (e^\lambda /m)^{1/2} \rightarrow \text{energy width}\]

\[\xi\]

\[= \text{Ramanant surfaces foliated/filamented}\]

\[- - - \Rightarrow \]

\[= \cdot \cdot \cdot \]

\[= \cdot \cdot \cdot \]
- not destroyed

- motion remains on surface, though surface ruffled.