a) Renormalization, cont'd

b) Long Time Tails and Mode-Mode Coupling Theory

a) Renormalization

- What is \( \epsilon^2 \)? - Reduction of thinning of \( d \)-o.-f. (c.f. chain)

- An approach \( \rightarrow \) Mori-Zwanzig Theory
  (in detail: Linear Chain)

\[(Q_t + L) P = 0\]

\[L = L_a + L_b + L_i\]

Slow Fast Couplings

\[\text{exhibit} \quad b \quad \text{exist} \quad \text{reg.(5)}\]

Kernel\: defined\:

\[(Q_t + L) P = 0 \rightarrow \left[ Q_t + L_a + \int ds \left[ \text{kernel}(s) \right] \right] P = \text{noise}\]
Quotes re: Renormalization

"In general, ordering the multitudes is just like ordering the few, or that it required a division into units."

- Sun Zi
"Art of War" Chapt. 5
(translated by M. Nylen)

"The shell game that we play — is called renormalization. But no matter how clever the word, it is what I would call a dirty process!"

- R.P. Feynman
"QED: The Strange Theory of Light and Matter"

"I must say that I am very dissatisfied with the situation, because this so-called 'good theory' does involve neglecting infinities which appear in its equations in some arbitrary way."

- P.A.M. Dirac
"Directions in Physics"
'"I disagreed with Dirac and argued the point with him ... Taking account of the difference between the bare charge and the mass of an electron and their measured values is not merely a trick that is invented to get rid of infinities; it is something that we would have to do even if everything was finite. There is nothing arbitrary or ad-hoc about the procedure; it is simply a matter of correctly identifying what we are actually measuring in the laboratory ..."

- S. Weinberg

"Dreams of a Final Theory"

"In the renormalization group method you take a structure you don't understand and convert it to another structure you don't understand, you keep doing it till you finally understand."

- Michael Berry
N.B.: Generically:

\[ \sim \begin{array}{c}
\text{coupling opr.} \\
\text{coupling opr.}
\end{array} \]

\[ \sim \begin{array}{c}
\text{reduction time of} \\
\text{test}
\end{array} \]

\[ \sim \begin{array}{c}
\text{phase space diffusing}
\end{array} \]

\[ D \rightarrow \int \left< \chi_1(t) \chi_2(t) \right> dt \]

\[ \Rightarrow \text{key element: coarse graining of fast d-o-Rs} \]

Chain \[ \rightarrow \text{assume fast d-o-Rs} \]

\[ \begin{array}{c}
\text{equivalent to known} \\
\text{known \( \rho_{2}(b) \)}
\end{array} \]

\[ \Rightarrow \text{How is this like/different from other incarnations of renormalization?} \]

\[ \begin{array}{c}
\text{Chain (2x2)}
\end{array} \]

\[ \begin{array}{c}
\text{projects} \\
\text{system onto}
\end{array} \]

\[ \begin{array}{c}
\text{system onto}
\end{array} \]

\[ \begin{array}{c}
\text{system onto}
\end{array} \]
Block spin - (T \to T_0, \text{ be diverse})

but no invariance argument.

\( Q \) Chain w. Self-Energy vs. Vacuum

\( \Delta \) Chain

\((\not{Q} + L) P = 0\)

\( \rightarrow (\not{Q} + L_i) P + \int \frac{1}{T} P = 0 \)

interactions with fast d-0-fs

\( \sim \) QED

\( \frac{1}{p - M_0} \rightarrow \frac{1}{p - m_0 + \Sigma} \)

Self energy interaction with vacuum pol. cloud
- Viscosity (today)

\[-\nu w + \nu_t h^2 \Rightarrow -\nu w + (\nu_0 + \nu_t) h^2\]

interaction with turbulent motion

\[\sum_{i} \frac{\omega_i l_i^2}{\omega^2 + (\omega l_i)^2}\]

\[\text{Fundamentally, all involve:}\]

- relevant, irrelevant split
- some aspect of coarse graining and equilibration of irrelevants.

- Pawn for model reduction.
Ex. \[ \text{Response Path for Noisy Burgers' Equation} \]

\[ \partial_t V + \nu \partial_x V - \partial_x \partial_x V = \xi \]

Burgers (also L/PZ)

Seek \( \partial V_k / \partial t \to \text{response} \)

Burgulence \( \leftrightarrow \) Burgulence (Jeffrey)

\[ 1D \quad p = 0 \quad \text{Navier-Stokes} \]

- **shocks**
  \[ \frac{dV}{dt} = -V^2 + \ldots \]

not captured in closure

- useful test.
  \( \Rightarrow \) Asymmetric Pdf \( V \)
  c.e. \( \nabla \rightarrow \text{shocks ramps} \)
  \( \rightarrow \) not captured in low order closure.
Point

\[ \frac{\partial \mathbf{v}}{\partial x} \mathbf{v} / \nu \frac{\partial^2 \mathbf{v}}{\partial x^2} \mathbf{v} \sim Re \]

do response dominated by nonlinearity

-Idea: NL couplings \(\rightarrow\) (turbulent) mixing.

so seek \( \mathbf{v} \). \( \frac{\partial^2 \mathbf{v}}{\partial x^2} \mathbf{v} \)

\[ \text{turbulent viscosity} \]

key physics: space - time scales

d.e. \( Re < 1 \) (Weak noise)

\[ 2 \kappa \mathbf{v} + \nu \mathbf{k}^2 \mathbf{v} + i \hbar \sum_n \mathbf{k} \mathbf{v}_n \mathbf{v}_n \mathbf{k} \mathbf{v} \mathbf{k} \mathbf{v} = \mathbf{F}_k \]

\[ (-i \omega + \nu \mathbf{k}^2) \mathbf{v}_n = \mathbf{F}_n \]

\( R_{\mathbf{v}} = -\partial \mathbf{v}_n / \partial \mathbf{v}_n = (i \omega + \nu \mathbf{k}^2)^{-1} \)

decay by viscosity
New for Re >> 1 = need faster (strong noise) mixing rate.

→ need extract effective time scale from nonlinearity

→ physics is nonlinear scrambling time (analogous Δω)

so seek:

\[ \alpha V_n + \nu k^3 V_n + C_w V_n = \Phi_n \]

seek response of test mode phase coherent with

\[ \Phi_n = (a e^{i \phi}) n \]

test of turbulent spectrum

\[ C_w \sim |\bar{V}|^2 \] (no phase content)

To calculate \( C_w \),

\[ \frac{(-\omega + \nu k^3)}{\omega} \sum_{\omega + \nu} \frac{V_n}{V_n} = \Phi_n \]
\[ \begin{align*} \frac{V_{u+\omega}}{e^{\omega_2}} \rightarrow V_{\omega_2} \rightarrow V_{\omega_2} \rightarrow V_{\omega_2} \text{ driven by direct best interaction of} \ V_{\omega_2}, V_{\omega_2} \\
\end{align*} \]

\[ \begin{align*} (-\omega + \nu k)^2 V_{\omega_2} + \frac{\nu^2}{\omega} \sum_{n \neq 0} \frac{V_{-\omega_2} V_{n+\omega_2}}{\omega - \omega} &= \frac{\partial}{\partial \omega} \\
\end{align*} \]

\[ \begin{align*} \left( \sum_{n \neq 0} \frac{V_{-\omega_2} V_{n+\omega_2}}{\omega - \omega} \right) \frac{\partial}{\partial \omega} &= C_\omega V_{\omega_2} \\
\end{align*} \]

\[ \begin{align*} \text{when calculated} \\
\end{align*} \]

\[ \frac{\partial V_{\omega_2}}{\partial \omega} = \frac{1}{-\omega + \nu^2 + C_\omega} \]

\[ \left( \begin{array}{c} \text{defined renormalized} \\
\text{viscosity} \\
\text{limits} \\
\text{reflects} \\
\text{response} \\
\text{scrambling} \\
\end{array} \right) \]

Now, to calculate:

\[ \left[ -\frac{c (\nu + \omega_2)}{c (\nu + \omega_2)^2 + C_{\omega_2}} \right] \frac{\partial}{\partial \omega} \]

\[ \begin{align*} &= -\frac{c (\nu + \omega_2)}{2} \left( \frac{V_{\omega_2} V_{\omega_2} + V_{\omega_2} V_{\omega_2}}{\omega_1 \omega_1 \omega_2 \omega_2} \right) \\
&= -\frac{c (\nu + \omega_2)}{2} V_{\omega_2} V_{\omega_2} \\
\end{align*} \]
N.B. Decomposition:

\[ NLT = C_{\text{nl}} V^{(\omega + d)}_{k+\omega} \frac{e^{i(k\cdot x)}}{\sqrt{2\pi}} + i(k\cdot k') (V_{k'} V_{k}) X \]

- all interactions other than best of those selected are absorbed into $C$.

* test field hypothesis : removal of 2 modes won't change $C$.

Now, define: \( NLT \) interaction

\[ L_{k+\omega}^{(\omega + d)} = -i(k\cdot k') + V(k\cdot k')^2 + C_{\text{nl}} \]

\( L = \) renormalized / dressed propagator

\[ V_{k+\omega}^{(\omega + d)} = L_{k+\omega}^{(\omega + d)} (-i(k\cdot k')) V_{k'} V_{k} \]
So, self-consistently:

\[
\epsilon_{k_0} \nu_{k_0} = \sqrt{\sum_{k_1} \frac{\nu_{k_1}}{\omega_{k_1}} L_{k_0 k_1} \frac{1}{\omega_{k_0} + \omega_{k_1}}} \nu_{k_1} \nu_{k_0} \]

\[
= \kappa^2 \sum_{k_1} \frac{1}{\omega_{k_1}} L_{k_0 k_1} \frac{1}{\omega_{k_0} + \omega_{k_1}} \left(1 + \frac{\nu_{k_1}^2}{\omega_{k_1}}\right) \nu_{k_0}
\]

\[
\sigma_{k_0} = \frac{1}{E\nu_{k_0} + \nu_{k_0}^2 + C_{\nu_{k_0}}}
\]

\[
C_{\nu_{k_0}} = \frac{\nu_{k_0}^2}{\omega_{k_0}} \kappa^2 = \kappa^2 \sum_{k_1} \frac{1}{\omega_{k_1}} L_{k_0 k_1} \frac{1}{\omega_{k_0} + \omega_{k_1}} \left(1 + \frac{\nu_{k_1}^2}{\omega_{k_1}}\right)
\]

"turbulent viscosity"

(n.b. \(h_{\nu} \omega\) dependent)

\(D\) defines renormalized propagator

\[\text{Sym.}\]

\[\text{Note recursive definition}\]
About $V_{fga}$

- at long wavelength (low frequency)
  \[ k < k^1 \]

- Markovian limit (Fokker-Planck)

\[ v^T = \sum_i \frac{1}{\nu_i} \frac{l^2}{w_i} = \sum_i \frac{V_{ik} l^2}{\nu_i (\nu_s + (k^1 \nu_s)^2)} \]

- effective transport \rightarrow sets NL/turbulent time scale

\[ v^T \sim \langle \nu^2 \rangle \tau_c \sim \nu_s l_c \]

\[ l_c \sim \nu \tau_c \]

- \( k^2 v^T \) is emergent time scale

\[ \tau_c \geq \tau_{NL} \]

\[ \tau_{NL} \sim \text{jet time scale} \]

- Contrast \( D = \frac{\langle \nu^2 \rangle \tau_c}{\nu_s} \) in B.M.

\[ \langle \nu^2 \rangle \tau_c = \theta T \] by E-N-T, or QLT.
- irreversible from turbulent scrambling.

To estimate

\[ \tau \sim \frac{\nu}{\nu_{\text{rms}}} \]

\[ \nu \sim \frac{\nu_{\text{rms}}}{\text{rms}} \]

\[ \sqrt{\frac{1}{\nu}} \sim \frac{1}{\tau} \]

\[ \omega \rightarrow \omega ' \quad \omega < \omega ' \]

no memory \( \tau \sim \) slow E-P Eqn.

\[ \tau \sim \frac{1}{\omega} \]

interaction behaves as memory-less quickly as in walk for \( \omega < \omega ' \), dwell

\[ \Rightarrow \text{Markovian} \quad \text{Slosh/Shift) } \nu < \nu ' \text{ dwl} \]

If not, feel time history or sloshing

\[ \Rightarrow \text{Non-Markovian} \]
Approximate Resp. is exact for what system?

Oscillators with Random Couplings Ensemble

[n.b. not random phase!]

see Krasichov '61.
Long Time Tails and Mode-Mode Coupling

- Long Time Tails (Alder & Wainwright 1968 → j et al 82.)

- Molecular dynamics (i.e. particles) simulation of fluid. Few configurations pulse it

- "tag a particle"

- Self-diffusion correlation

- Expect, for velocity correlation

\[
\langle \mathbf{v}(t) \mathbf{v}(t') \rangle = \langle \mathbf{v}(0) \rangle^2 e^{-t/T_{\text{ac}}}
\]

\[0 = \int_0^\infty \langle \mathbf{v}(t) \mathbf{v}(0) \rangle \, dt \rightarrow \langle \mathbf{v}(0) \rangle^2 T_{\text{ac}}
\]

but

Surprise!
Actually a long time tail
(power law)

8D (hard spheres)
\[ \langle \mathbf{v}(t) \rangle \sim t^{-3/2} \]
dimension.

2D (discs)
\[ \langle \mathbf{v}(t) \rangle \sim t^{-1} \]
\[ D \rightarrow \infty \]
Why?

How treat, theoretically?

Heuristics (see Appendix Resibois)

What happens

1) deliver impulse to tagged particle
(c) Shortly, target and impulsed particle shares its momentum with neighbors. Quasi-particle moves.

\[ \dot{V}(t) = \frac{\dot{V}(0)}{V(0)} \rightarrow \text{Euclid particle} \]

Velocity drops as plies of neighbors group/expands density of neighbors.

(d) What sets \( V_n \)?

- Response of system / fluid
- Candidates (modes)
  - Acoustic compression (sound)
    - Short time \( \omega \approx \text{v} \) \( \text{Shear} \)
  - Vortex pair (rigid in 3D)
    - Expands
      - \( \omega \approx \text{v} \) \( \text{Shear} \)

\[ V_n \approx (\text{Re}_s)^{1/2} \]

where \( \text{Re}_s \sim (rt)^{1/2} \)
\[ V_n \sim R_n \sim (ct)^{d/2} \]

Quasi-particle volume grows in time

\[ c_n \sim c_A \sim \frac{c_A}{(ct)^{d/2}} \]

\[ \langle V_A(V_A) \rangle \sim \frac{c_A^2}{(ct)^{d/2}} \]

Long time tail

Technically:

\[ \langle V_A(V_A) \rangle \sim \frac{c_A^2}{[c_A + D(t)]^{d/2}} \]

Self-wondering

Also:

Key here:

- Long time collective dynamics of system (fluid)
- Particle + collective coupling, has flavor of renormalization

\[ \omega = -ivz^2 \]
→ Now, how approach systematically ?

→ Mode - Mode Coupling Theory
  c.f. Kadanooff + Swift 168 *
  Zwanzig (book)
  Romeu + Resibois (review)
  Regener (after Kirkwood) (book)

→ built on Hilbert space picture

→ complete orthonormal set \( \{ \phi_j(x) \} \)
  of position \( X \) of system in
  phase space

\[
\langle \phi_j | \phi_n \rangle = \int dx \, \phi_j(x) \phi_n^*(x) \delta_{j,n} = \delta_{j,n}
\]

→ can recast Liouville Eqn as matrix equation

\[
D(t) = \frac{1}{\hbar} \begin{bmatrix} \langle V(t) \end{bmatrix} \rightarrow \text{exploits state vector approach}
\]

\[
= \frac{1}{\hbar} \sum_{j,k} \langle \phi_j | \phi_k \rangle \langle e^{tL} \phi_j | \phi_k \rangle \langle \phi_k | V \rangle
\]
What is \( e^{Ht} \psi_j \)? \( \rightarrow \) evolution of state vector.

Faster if \( \psi_j \) constructed from slow variables.

Slow \( \rightarrow \) conserved concentration

\[
\frac{d}{dt} C = - \frac{D \cdot J}{4} = D \frac{d^2 C}{\omega^2} \quad \omega \rightarrow 0
\]

\( \sim D^2 \) \( \rightarrow \) decay very slowly at large scale

\[
\frac{d}{dt} \nu = - \frac{\nu \cdot \Pi}{\omega} \quad \omega \rightarrow 0
\]

Fast: \( \frac{d}{dt} \nu = - \gamma \times \nu \) \( \rightarrow \) not conserved

\( \leftrightarrow \) Examples of slow variables:

\[
C(z, t) = \delta(\rho_0(z) - \nu)
\]

\( \rightarrow \) tagged

\[
C(z, t) = \sum \frac{C_2(t)e^{i\nu z}}{2} \quad \nu \rightarrow 0
\]

\[
C_2(LA) = e^{fL} C_2 = e^{-D^2 z^2} C_2
\]

\( \rightarrow \) different decay.
Likewise: \( \nu \rightarrow \) fluid velocity

mode

(long time \( \rightarrow \) incompressible)

\[ N(t) = \frac{1}{d} \sum \langle \nu | \psi_j \rangle e^{t \nu_j} | \psi_j \rangle \langle \psi_j | \nu \rangle \]

to calculate \( D \), need \( \sum \nu_j \)

\[ \langle \nu | \psi_j \rangle \neq 0 \]

\( \rightarrow \) i.e. seek projection

the particle velocity
onto a system

\( \rightarrow \) mode

For long time behavior, \( \) natural system

mode is a \( \) slow mode

\( \rightarrow \) But \( \langle \nu | \psi_2 \rangle = 0 \rightarrow \) projection

\( \) independent

\( \) depends on position

\( \) (tag is translational)

\( \) (invariant)
Consider the product of two slow variables \( \nu \in \mathbb{C} \), \( s/t \), product best is translationally covariant.

Product best of slow modes:

"mode-mode coupling"

So with normalization formulas, have mode coupled state vectors

\[
\begin{bmatrix}
\nu_j \\

\end{bmatrix} 
\quad \mathbf{U}_j = \mathbf{M}_j \mathbf{W}_j \mathbf{C}_j
\]

That is, state vector

\[
\langle \psi | \psi \rangle = \delta_{j,j'} \quad \text{etc.}
\]

\[
\langle \nu_j \mathbf{C}_j | \nu_j \mathbf{C}_j \rangle = \langle \nu_j \mathbf{W}_j \mathbf{C}_j | \nu_j \mathbf{W}_j \mathbf{C}_j \rangle = \frac{\mathbf{M}_j \mathbf{I}}{m}
\]
Then

\[ \langle \Psi(0) | \Psi(t) \rangle \equiv 0(t) \quad \text{correlation in the basis} \]

\[ 0(t) = \frac{d}{dt} \rho(x,t) + \int \frac{d}{dt} \sum_l \langle e^{ix|x|} \nu \Delta_l \bar{c}_l | \nu \Delta_l c_{-2} \rangle \]

where some slow modes \[ \rho(x,t) \]

\[ e^{+x|\nu|} \Delta_l \bar{c}_l = (e^{+x|\nu|} \nu \Delta_l) (e^{+x|\nu|} \nu \Delta_l) \]

\[ = (e^{+x|\nu|} \nu \Delta_l) e^{-x|\nu|^2 t} \nu \Delta_l \bar{c}_l \]

\[ v_{l-} = v_{l,\perp} + v_{l,\parallel} \cdot \quad v_{l,\parallel} = \frac{\nu}{Z^2} \cdot v_{l,\perp} \]

\[ 0 \cdot v = 0 \quad \quad \text{longitudinal} \quad \quad \text{transverse} \]

\[ v_{l,\perp} = \left( \frac{1}{2} - \frac{Z^2}{2} \right) \cdot v_{l,\perp} \]

\[ \Rightarrow v = v_{l,\perp} + 0 \quad \quad \text{\[ 0 \cdot v_{l,\perp} = 0 \]} \]

and \[ \nu_{l,\perp} \] decays due to shear viscosity only.
\[ \frac{\partial}{\partial t} v_{\pm 1} = -v_{\pm 1}^2 v_{\mp 1} + \text{noise} \]

\[ v_{\pm 1} = e^{\pm t} v_{\mp 1} = e^{-v_{\mp 1}^2 t} v_{\mp 1} \]

and \[ \overline{v} = 0 \]

\[ \langle e^{\pm t v_{\mp 1}^2} v_{\mp 1} \rangle = \frac{N^T}{\lambda} \left( \frac{I - 2 \xi \frac{I}{2} }{z^2} \right) e^{-v_{\mp 1}^2 t} \]

Finally:

\[ O(t) = D_0 e^{t} + \frac{d-1}{d} \frac{1}{\lambda} \sum_{\omega} e^{-\lambda \omega^2 t} \]

\[ = D_0 + \frac{d-1}{d} \frac{1}{\lambda} \sum_{\omega} e^{-\lambda \omega^2 t} \]

N.B.: In mode coupling time decay \( O(t) \)
set by the faster of the slow modes (i.e., viscous or diffusion).
\[ \sum_2 \to \left( \frac{1}{2\pi} \right) \int d^q z \]

\[ \phi = mN/L^q \]

and integrating over \( z \):

\[ \phi(t) = D_0(t) + \frac{d}{dt} \left\{ \frac{T}{8} \left[ \frac{1}{2} \left\{ \frac{1}{4\pi (0 + r)^5} \right\}^{1/2} \right] \right\} \]

long time tail

\[ \text{M.B.} \]

- long time tail results from slow diffusive decay of (slow) mode.

- \( \text{symptom of conservation} \)

- large scales \( \text{slowest of the slow modes} \)

Obviously result sensitive to d.o.f.'s mass structure.
\( d = 2 \), \( D_{xx} \to \infty \) as \( \langle v^2 \rangle \to \infty \)

Consistent with Stokes' Periplus → hydro friction or Stokes drag does not exist in 2D.

\[ v \sim 1/r^2 \text{ dipole} \]

\[ \phi \sim \ln r \]

\[ v \sim 1/r \]

Constant fall-off.

For stress \( \to \) M.C. \( V_x V_x \) basis (flux) \( \Rightarrow \eta \)

Stress correlation + thus \( \eta \) \( \Rightarrow \)

\( \sim 1/t^{1/2} \) tail.
György: Other approaches? Hard to see.

Nonlinear Langevin Eqs.

time scale separation in correlation C(t)

Fast

Slow

Is there another way?

See Pomeau & Rasbois. (many - )

also Bedeaux & Mazur et seq.

Renormalized continuity equation for tagged density given thermal velocity field.

Method of fluctuating hydrodynamics

c.i. derive total diffusivity in thermal flow field.

seek non-Markovian D. (long time)
\[ \rho \frac{\partial \bar{n}_i}{\partial t} = - \frac{\partial}{\partial x_j} \nabla \cdot \nabla \cdot \bar{n}_i \]

\[ \bar{n} = - \left( D_0 \nabla \bar{n}_i + \mathbf{v}(x,t) \cdot \nabla \bar{n}_i + \frac{\mathbf{f}_R}{\rho} \right) \]

\[ \nabla \cdot \mathbf{v}(x,t) \]

\[ \text{the point } \rightarrow \text{ Fluid convection adds to self-diffusion} \]

\[ \nabla \cdot \mathbf{v}(x,t) \]

\[ \text{take Fluid as ambient thermal} \]

\[ \mathbf{f}_R \text{ passive scalar (trapped) in thermal flow.} \]

\[ \text{drop } \mathbf{f}_R \text{ from continuity:} \]

\[ (\sigma \omega + D_0 \nabla^2) \bar{n}_i = \nabla \cdot \left( \frac{1}{\rho} \nabla \bar{n}_i \right) + \nabla \cdot \mathbf{f}_R(t=0) \]

\[ \text{where} \]

\[ \text{conventions! } \rightarrow \text{ cumbersome} \]
\[ \mathbf{A} \mathbf{u}_2 \mathbf{v} = \frac{1}{2 \omega} \int dz \int dw \left( \mathbf{v} \cdot \nabla \mathbf{u}_2 \right) \]

Advection operator arbitrary \( \mathbf{u}_2, \omega \)

\[ (\omega + 0 \mathbf{z}^2) \left( \tilde{\mathbf{u}}_{2\omega} + \tilde{\mathbf{u}}_{2} \right) \]

\[ = \mathbf{z} \cdot \mathbf{A} \left( \tilde{\mathbf{u}}_{2\omega} + \tilde{\mathbf{u}}_{2} \right) + \tilde{\mathbf{u}}_{2} \quad (t=0) \]

\[ \tilde{\mathbf{u}}_{2\omega} = \mathbf{z} \cdot \mathbf{A} \tilde{\mathbf{u}}_{2} \quad (t=0) \]

\[ \mathbf{u}_0 = (\omega + 0 \mathbf{z}^2)^{-1} \]

\[ \left( \mathbf{G}_0^{-1} - i \mathbf{z} \cdot \mathbf{A} \right) \tilde{\mathbf{u}}_{2\omega} = - \mathbf{G}_0 \tilde{\mathbf{u}}_{2\omega} \mathbf{G}_0^{-1} \tilde{\mathbf{u}}_{2\omega} \]

\[ \tilde{\mathbf{u}}_{2\omega} = (1 + i \mathbf{G}_0 \mathbf{A} \cdot \mathbf{G}_0)^{-1} \tilde{\mathbf{u}}_{2\omega} \]

\[ \tilde{\mathbf{u}}_{2\omega} = (\mathbf{A} + i \mathbf{G}_0 \mathbf{A}) (1 + i \mathbf{G}_0 \mathbf{A}) \tilde{\mathbf{u}}_{2\omega} \]

\[ \mathbf{J}_2 \mathbf{u} = (\mathbf{A} + i \mathbf{G}_0 \mathbf{A}) (1 + i \mathbf{G}_0 \mathbf{A}) \tilde{\mathbf{u}}_{2\omega} \]

and \[ \mathbf{J} = - \mathbf{D}_0 \mathbf{u} \]

\[ \mathbf{D}_0 \mathbf{u} = \mathbf{G}_0^{-1} \tilde{\mathbf{u}}_{2\omega} \]
\[
D = D_0 + \frac{1}{2} \left< \mathbf{g} \cdot \mathbf{AG}_0 \mathbf{g} \cdot \mathbf{A} \right> + \text{h.o.t.}
\]

\text{Oseen flow} \sim \frac{\mathbf{u} \cdot \mathbf{r}^2}{(\mathbf{u} \cdot \mathbf{r})^2}

\text{and:}

\[\mathbf{W}^2 = \left( 1 - \frac{3}{2} \frac{\mathbf{r} \cdot \mathbf{r}}{\mathbf{r}^2} \right) \mathbf{I} \mathbf{A} \left[ \frac{1}{(\mathbf{u} \cdot \mathbf{r})^2} \right]^p \]

\[\mathbf{D} = \mathbf{D} - \mathbf{D}_0\]

and expanding \( \mathbf{D} \) at low \( \mathbf{u} \):

\[\mathbf{D} \sim \frac{1}{\mathbf{u}^2} \sim t^{-3/2}\]

(No integrate over \( z \), leaving \( \mathbf{u} \) dependence.)

TBC

\[\text{Point is that long-time tail emerges from collective fluid effects entirely effective self-diffusion.}\]