L2.10 - H Theorem.

→ To H-Theorem !

→ Recall:
  - Hamiltonian system
  - Non-dissipative interactions
  - From Liouville (for $N - N_A \gg 1$) to Boltzmann:

    → key is diluteness ordering
    \[
    d < N^{-1/3} < \text{mfp} < L
    \]

    → Molecular Chaos Assumption
    Detailed Balance

    \[
    \rightarrow f_N \text{ eqn.} \rightarrow f_0 \text{ eqn.}
    \]

    → Boltzmann Equation.
\frac{df}{dt} + \nabla \cdot \mathbf{f} = C(f)

\text{Collision operator}

C(f) = \int d\mathbf{r}_2 \frac{\partial V}{\partial \mathbf{r}_2} \cdot \frac{\partial}{\partial \mathbf{r}_1} \left[ f_1(\mathbf{r}_1, \mathbf{r}_2) f_2(\mathbf{r}_2) \right]

\text{Nonlinear}

in reality, every particle (WMF)

is both a
test particle and
a field particle.
- What does it say/mean?
- What makes it remarkable?
- What does it rest upon? (Assumptions)
- How to prove?
Onward:

- Can also write Boltzmann Equation as change in occupation of state $\rightarrow \Phi$.

\[ \frac{df}{dt} = \text{rate of change of } f \text{ due to collisions (interactions)} \]

\[ \frac{df}{dt} = \text{rate scattering in} - \text{rate scattering out} \]

(anticipates Master Equation)

\[ c_{in} \]

\[ c_{out} \]

\[ \Phi' \]

\[ \Phi \]

\[ \Delta \]

\[ \delta \]

\[ \delta' \]

\[ \text{interaction} \]
\[
\text{in} = \int dp' \int dp_i \int d\gamma_i \ F(p') F(p_i') w(p_i', \gamma_i; p, \gamma)
\]
\[
\text{out} = \int dp_i \int dp_i' \int d\gamma_i \ F(p_i) F(p_i') w(p_i', \gamma_i; p_i, \gamma_i)
\]
as  \quad W = W^T \quad \text{(reversible)}

\[
\frac{df(p)}{dt} = \int dp_i \int dp_i' \int d\gamma_i \ w(p_i', \gamma_i; p, \gamma) \times \\
( f(p') F(p_i') - f(p) F(p_i) )
\]

\text{N.B.} \quad p + p_i = p_i' + p_i
\]
so detailed balance applies.

\[
-\quad f = \frac{\rho}{\rho_c} = c \exp \left[ - \frac{(E + p \cdot v)}{T} \right]
\]
N.B. - What about \( V(x) \)?

\[ \frac{ds}{dt} = 0 \]

... will show Maxwellian renders...

... conservation energy...
and:
- dilute: non-overlapping cylinders
- collisions as 'point events'

\[ 0 < \mathbf{\bar{r}} < \lambda_{\text{emp}} \]

Also:
\[ \int \left[ \nu_1 \to \sigma \left( \nu_1 - \nu_2 \right) \right] \]
\[ + \text{ integrates.} \]

\[ \omega d^3 \mathbf{v}_1 d^3 \mathbf{v}_j = \text{unit} \ dT \]

relates transition (probability) to familiar items like cross-section.

- \( \lambda_{\text{emp}} \), \( \lambda_{\text{emp}} = \frac{1}{n \pi} \)

Onward: \( \Rightarrow \) H-Theorem.

- a gas left alone will evolve to an equilibrium of maximal entropy.

- evolution accompanied by entropy production.
\[
\frac{dS}{dt} \geq 0
\]

- Evolution is to uniform Maxwellian

- Ideal gas:

\[
S' = \int dx \int dp \ f \ ln(c/e) \quad \text{(discussed later)}
\]

\[
\int dx \int dp \ f \ ln(f)
\]

\[
\frac{dS}{dt} = -\int dr \ \left[ \frac{df}{dt} \ ln f + \frac{f}{f} \ \frac{df}{dt} \right]
\]

\[
= -\int dr \ \left[ c(e) \ ln f + c(f) \right]
\]

\[
\int dr \ c(e) = 0 \quad \text{show!}
\]

\[
\frac{dS}{dt} = -\int dr \ c(e) \ ln f
\]
\[
\frac{ds}{dt} = - \int dx \int dp \int dp' \int dp'' \int dp''' \left( \frac{\partial F}{\partial p'} \frac{\partial F}{\partial p''} - \frac{\partial F}{\partial p} \frac{\partial F}{\partial p''} \right)
\]

Lemma

\[
\int (u(p) c(q)) dp = \frac{1}{2} \int d^4p \left( \psi' + \psi - \psi' - \psi' \right)\psi' \psi',
\]

explicitly:

\[
\int dp \ u(p) c(q) = \int \psi' \psi \left[ \int d^4p \ w(p, p', p, p') \right] \psi' \psi' d^4p
\]

\[
- \int \psi \psi \left[ \int d^4p \ w(p, p', p, p') \right] \psi' \psi' d^4p
\]

To show:

Now, on \( \mathfrak{2} \):

\( \rightarrow \) interchange \( p, p' \rightarrow p', p \)

\[\text{flip at point } 1\]

\[\text{use } W = W^T \] (micro-reversibility)
\[
\sum_{\text{ca}} \psi \ 
= \int d^4 \phi \left\{ (\text{c}(\phi) - \text{c}(\phi')) \mathcal{W} \left( \phi, \phi', \phi'^1, \phi'^2 \right) + \phi'^1 \phi'^2 \right\}
\]

Now consider:

\[
\text{and interchange}\quad \text{ca}
\]

\[
\text{c.e.}\ \phi, \phi'\ \text{with}\ \phi, \phi'
\]

\n
N.B.: up-down symmetry equivalent (no reason \(\Rightarrow\))

\[
\frac{1}{2} \int d^4 \phi \left\{ (\text{c}(\phi) - \text{c}(\phi')) \mathcal{W} \left( \phi, \phi', \phi'^1, \phi'^2 \right) + \phi'^1 \phi'^2 \right\}
\]

Proves Lemma 5
Now, \( \ell = \ln f \)

So, from Lemma:

\[
\frac{d\ell}{dt} = -\frac{1}{2} \int \frac{dx}{d^4p} \left( \ln f + \ln f_i - \ln f' \right) \\
- \ln f_i \right) \cdot \frac{f'}{f} \\
= \frac{1}{2} \int \frac{dx}{d^4p} \cdot \frac{f'}{f} \cdot \ln \left( \frac{f' f_i}{f f_i} \right)
\]

define \( x = \frac{f' f_i}{f f_i} \)

\[
\frac{d\ell}{dt} = \frac{1}{2} \int \frac{dx}{d^4p} \cdot \frac{f'}{f} \cdot x \cdot \ln x
\]

Since: \( \int \theta(x) \, dx = 0 \)

have \( \int w f_i \left( x - 1 \right) \, d^4p \, dx = 0 \)
As a case of writing zero in a complicated way!

so adding 0 to $\frac{dS}{dt}$ expression:

$$\frac{dS}{dt} = \frac{1}{2} \int a \frac{dx}{w} \ln x \left[ x \ln x - x + 1 \right]$$

The entropy production rate, $\dot{F}$

\[ F'(x) = x \ln x - x + 1 \]

\[ F'(1) = 1 + \ln 1 - 1 = 0 \]

\[ F'(\infty) = \infty \]

\[ \frac{dS}{dt} \geq 0 \]

$\rightarrow$ **H-Theorem**
\[
\frac{d\delta}{dt} = 0 \quad \text{for } x = 1
\]

\[f f_1 = f' f_1'
\]

\[\ln f + \ln f_1 = \ln f' + \ln f_1'
\]

\[\Rightarrow \ln f = c + f \cdot V - \alpha \epsilon
\]

\[\frac{d\delta}{dt} = 0 \Rightarrow \text{Maxwellian Distribution}
\]

Note:

1) KEYS: \( W = W' \Rightarrow \text{Detailed Balance} \)

\[f(f', Q) = f(Q') f(Q) \quad \text{Factorization (Molecular Chaos)}
\]

2) \[\frac{d\delta}{dt} = 0 \Leftrightarrow \text{CCR} = 0
\]

Collisions drive system to equilibrium.
3) $dx$ irrelevant

Entropy produced locally.

i.e. $F$ relaxes to local Maxwellian then to uniform Maxwellian (i.e. transport: $\frac{\partial}{\partial t}$)

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**Essence of H-Theorem:**

Macrosopic irreversibility from microscopically reversible dynamics

+ Molecular Chaos ("micro-chaos")
Some observations:

- If no a-priori concept/idea of equilibrium distribution, how denote it?

Recall: $\frac{dS}{dt} = 0$, for $x = 1$

$x = 1 \Rightarrow S' = S$

$\ln S' + \ln S = \ln S + \ln S$

as labels in collision arbitrary, i.e.

\[ \rho_i \rightarrow \rho_i' \Rightarrow \text{etc.} \]

So $\ln S + \ln S = \text{const.}$

What is conserved? (Dynamically): own/age conserved
- energy (kinetic energy / particle)
- momentum
- number

\[ \ln F = a + b \cdot \text{p} + c \cdot \frac{\text{p}^2}{2m} \]

\( c < 0 \) for normalizability

N.B.: Angular momenta not independent as collision event at 1 position.

\[ f = c' \left[ \frac{-p^2}{2mT} + \frac{p \cdot V}{T} \right] \]

\( c' = \Lambda \)

\( \Lambda, T, V \) can be \( \xi, T(\xi), V(\xi) \) can all be functions of \( x \) for input < 1
Thus, derived equilibrium distribution function from $\mathcal{H}$-Thm.

\[ \text{(2) How reconcile?} \]

- reversible Hamiltonian dynamics
  - $-dS/dt \geq 0$

Related: What happened to Poincaré Recurrence?

Partition:

- statistical description: $F(x, k, t)$

- coarse graining:

  (recall Lyapunov exponents)

\[ \text{MAP Partition} \rightarrow \text{sets resolution scale} \]

($S$ is integrated quantity)
Why Significant?

- Partition Hills small details in phase volume evolution

\[ \alpha \]

\[ t = 0 \]

\[ \to \]

\[ A \]

exact

With coarse graining (smearing) in entropy production timescale

local phase space density modified by coarse graining

\[ \text{coarse grained area.} \]

\[ \text{initial} \]

\[ P_0 A_0 = A \]

\[ F = \frac{P_0 A_0}{A_0} \]

\[ \bar{F} < P_0 \quad \text{as} \quad A_0 < A_\infty. \]
- prediction of close recurrences impossible as partition sets resolution limit.

N.B.: $\delta_{\text{mass}} = \frac{1}{\sqrt{N}}$

$\gamma_0^{-1} = \gamma_c \equiv \sqrt{w/\text{mass}}$

etc