Lecture 7 (Oct. 26)

We now turn to the subject of small oscillations. We assume that the kinetic energy is homogeneous of degree two in the generalized velocities: \( T = \frac{1}{2} T_{\sigma_1} (q_1, \ldots, q_n) \dot{q}_\sigma \dot{q}_\sigma \), and that the potential \( U(q_1, \ldots, q_n) \) is degree zero in the \( \{ \dot{q}_\sigma \} \). The equations of motion are then obtained as follows:

\[
L = T - U \Rightarrow \begin{cases} 
\rho_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma} = T_{\sigma_1} (q) \dot{q}_\sigma, \\
F_\sigma = \frac{\partial L}{\partial q_\sigma} = \frac{1}{2} \frac{\partial T_{\sigma_1}}{\partial q_\sigma} \dot{q}_\sigma \dot{\dot{q}}_\sigma - \frac{\partial U(q)}{\partial q_\sigma}
\end{cases}
\]

Thus, \( \dot{\rho}_\sigma = F_\sigma \) says

\[
T_{\sigma_1} \ddot{q}_\sigma + \left( \frac{\partial T_{\sigma_1}}{\partial q_\sigma} - \frac{1}{2} \frac{\partial T_{\sigma_1}}{\partial q_\sigma} \right) \dot{q}_\sigma \dot{\dot{q}}_\sigma = -\frac{\partial U}{\partial q_\sigma}
\]

This may be written as

\[
\dot{q}_\sigma + \Gamma^\lambda_{\mu\nu} \dot{q}_\mu \dot{q}_\nu = A_\lambda, \quad \text{with}
\]

\[
\Gamma^\lambda_{\mu\nu} = \frac{1}{2} T^{-1}_{\lambda_\sigma} \left( \frac{\partial T_{\mu\nu}}{\partial q_\sigma} + \frac{\partial T_{\nu\mu}}{\partial q_\sigma} - \frac{\partial T_{\nu\mu}}{\partial q_\sigma} \right) \text{ Christoffel symbols}
\]

\[
A_\lambda = -T^{-1}_{\lambda_\sigma} \frac{\partial U}{\partial q_\sigma}
\]
- Static equilibrium: \( \dot{\bar{q}}_\sigma = 0 \quad \forall \sigma \in \{1, \ldots, n\} \Rightarrow \) 
\( \frac{\partial U}{\partial \bar{q}_\sigma} = 0 \quad \forall \sigma \); \( n \) equations in \( n \) unknowns \( \{\bar{q}_1, \ldots, \bar{q}_n\} \)

Generically this has point-like solutions, \( \{\bar{q}_1, \ldots, \bar{q}_n\} \).

Let's write \( \bar{q}_\sigma = \bar{q}_\sigma + \eta_\sigma \) and expand the Lagrangian to quadratic order in the \( \eta_\sigma \) and \( \dot{\eta}_\sigma \):

\[
L = \frac{1}{2} T_{\sigma\tau} \eta_\sigma \eta_\tau - \frac{1}{2} V_{\sigma\tau} \eta_\sigma \eta_\tau + \ldots
\]

where

\[
T_{\sigma\tau} = T_{\sigma\tau}(\bar{q}) = \left. \frac{\partial^2 T}{\partial \bar{q}_\sigma \partial \bar{q}_\tau} \right|_{\bar{q}}
\]

\[
V_{\sigma\tau} = \left. \frac{\partial^2 U}{\partial \bar{q}_\sigma \partial \bar{q}_\tau} \right|_{\bar{q}}
\]

So to quadratic order, \( L = \frac{1}{2} \dot{\eta}^T T \dot{\eta} - \frac{1}{2} \eta^T V \eta \)

• Method of small oscillations

The idea here is to express the \( \eta_\sigma \) in terms of normal modes, \( \tilde{\xi}_i \), which diagonalize the equations of motion,

\[
T_{\sigma\tau} \tilde{\xi}_i = -V_{\sigma\tau} \tilde{\xi}_i
\]

This being a linear problem, we write \( \eta_\sigma = A_{\sigma i} \tilde{\xi}_i \) and demand

\[
A^T A = I
\]

\[
A^T V A = \text{diag}(w_1^2, \ldots, w_n^2)
\]

\( \text{n} \times \text{n} \text{ real matrix} \)
The vector form of the linearized EL eqns is

\[ T \ddot{\eta} = -V \dot{\eta} \]

so

\[ TA \ddot{x} = -VA \dot{x} \]

Multiplying on the left by \( A^T \), we then have

\[ \left( A^T A \right) \ddot{x} = -\left( A^T VA \right) \dot{x} \]

\[ \equiv I \quad \equiv \text{diag}(\omega_1^2, \ldots, \omega_n^2) \]

Thus we have \( n \) decoupled second order ODEs:

\[ \ddot{x}_i = -\omega_i^2 \dot{x}_i \]

with solutions

\[ x_i(t) = C_i \cos(\omega_i t) + D_i \sin(\omega_i t) \]

with \( 2n \) constants of integration \( \{C_i, D_i\} \) with \( i \in \{1, \ldots, n\} \).

Note \( \dot{x} = A \ddot{x} \) yields \( \ddot{x} = A^{-1} \dot{x} = A^T \ddot{x} \), thus

\[ \eta_0(t) = \sum_i A_{i0} \left[ C_i \cos(\omega_i t) + D_i \sin(\omega_i t) \right] \]

Multiplying on the left by \( A^T \), we obtain

\[ C_i \cos(\omega_i t) + D_i \sin(\omega_i t) = A_{i0}^T \ddot{\eta}_0, \quad \eta_0(t) \]

and thus

\[ C_i = A_{i0}^T \ddot{\eta}_0(0) \]

\[ D_i = \omega_i^{-1} A_{i0}^T \ddot{\eta}_0(0) \quad (\text{no sum on } i) \]
At this point, we have the complete solution to the problem for arbitrary initial conditions \( \{q_0(0), \dot{q}_0(0)\} \). The matrix \( A_{\sigma i} \) is called the modal matrix. If all the generalized coordinates have dimensions \([q_0] = L\),

\[
[T_{\sigma 1}] = M, \quad [V_{\sigma 1}] = \frac{E}{L^2} = \frac{M}{T^2}
\]

\[
[A_{\sigma i}] = M^{-\frac{1}{2}}, \quad [\dot{q}_i] = M^\frac{1}{2}L
\]

Why can we demand \( A^t TA = I \) and \( A^t VA = \text{diag}(\omega_1^2, ..., \omega_n^2) \)?

Proof by construction:

(i) Since \( T_{\sigma 1} \) is symmetric, there exists \( O_1 \in O(n) \) such that \( O_1^t T_{\sigma 1} O_1 = T_d \), where \( T_d \) is diagonal. Additionally, the entries of \( T_d \) are all positive because the kinetic energy is in general positive (only zero if \( \dot{q}_\sigma = 0 + \sigma \)).

(ii) \( T_d \) being positive definite, we may construct its square root \( T_d^{\frac{1}{2}} \) simply by taking the square root of each diagonal entry. Note then that

\[
T_d^{-\frac{1}{2}} O_1^t T_{\sigma 1} T_d^{-\frac{1}{2}} = T_d^{-\frac{1}{2}} T_d T_d^{-\frac{1}{2}} = I
\]

(iii) The matrix \( T_d^{-\frac{1}{2}} O_1^t V O_1 T_d^{-\frac{1}{2}} \) is symmetric, and hence diagonalized by some \( O_2 \in O(n) \). Thus,
we have two matrices $O_1$ and $O_2$ such that
\[
O_2^T T_d^{-1/2} O_1^T + O_1 T_d^{-1/2} O_2 = 1
\]
\[
O_2^T T_d^{-1/2} O_1^T V O_1 T_d^{-1/2} O_2 = \text{diag}(w_1^2, \ldots, w_n^2)
\]

Therefore the modal matrix is
\[
A = O_1 T_d^{-1/2} O_2
\]

(NB: $A$ not orthogonal!)

We can see that it is in general not possible to simultaneously diagonal three symmetric matrices. Two is the limit!

- How to find the modal matrix
  (i) Assume $\eta_\sigma(t) = \text{Re} \bar{\Psi}_\sigma e^{-i \omega t}$. Then from the EL eqn $T \ddot{\eta} = -V \dot{\eta}$ we have $(w^2 T - V)_{\sigma\sigma} \Psi_\sigma = 0$. 
  In order to have nontrivial solutions, we demand
  \[
  \det (w^2 T - V) = 0
  \]
  This yields an $n^{th}$ order polynomial equation in $w^2$.
  Its $n$ roots are the $n$ normal mode frequencies, $w_i^2$.
  (ii) Next, find the eigenvectors $\Psi^{(i)}_\sigma$ by demanding
  \[
  \sum_{\sigma} (w_i^2 T_{\sigma\sigma} - V_{\sigma\sigma}) \Psi^{(i)}_\sigma = 0
  \]
Since $\omega^2 T - V$ is defective, these equations are $(n-1)$ inhomogeneous linear equations for $\{\psi_1^{(i)}, \psi_2^{(i)}, \ldots, \psi_n^{(i)}\}$ yielding the ratios $\{\psi_2^{(i)}/\psi_1^{(i)}, \ldots, \psi_n^{(i)}/\psi_1^{(i)}\}$. It then follows (see §5.3.3) that $\psi_{(i)}^* \psi_{(j)} = 0$ if $i \neq j$.

In fact, this is only guaranteed if $\omega_i^2 \neq \omega_j^2$, but for degenerate eigenvalues $\omega_i^2 = \omega_j^2$, we may still choose the eigenvectors to be orthogonal (w.r.t. $T$) via the Gram–Schmidt process. Finally, we may choose to normalize each eigenvector, so that

$$<\psi_{(i)}^{(i)}|\psi_{(j)}^{(j)}> = \psi_{(i)}^{*} T_{(i)}^{(j)} \psi_{(j)}^{(j)} = \delta_{i,j}$$

(iii) The modal matrix is then given by $A_{(i)} = \psi_{(i)}^{*}$.

(iv) Since $\ddot{\eta} = A_{(i)} \ddot{x}$ and $A^T T A = 1$, $A^{*} = A^T$ and $\ddot{x} = A^T \ddot{\eta}$.

- Example: the double pendulum

  (For simplicity, choose $l_1 = l_2 = \ell, m_1 = m_2 = m$)

  \begin{align*}
  x_1 & = l \sin \theta_1, \quad y_1 = -l \cos \theta_1 \\
  x_2 & = l \sin \theta_1 + \ell \sin \theta_2, \quad y_2 = -l \cos \theta_1 - \ell \cos \theta_2 \\
  T & = \frac{1}{2} m (\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2) - \frac{1}{2} m \ell^2 (2 \dot{\theta}_1^2 + 2 \cos (\theta_1 - \theta_2) \dot{\theta}_1 \dot{\theta}_2 + \dot{\theta}_2^2) \\
  V & = -mg \ell (2 \cos \theta_1 + \cos \theta_2) \quad \text{equilibrium @ } \theta_1 = \theta_2 = 0
  \end{align*}

\[T = \begin{pmatrix} 2m\ell^2 & m\ell^2 \\ m\ell^2 & m\ell^2 \end{pmatrix}, \quad V = \begin{pmatrix} 2mg \ell & 0 \\ 0 & mgl \end{pmatrix}\]
Let $\omega_0^2 = g/l$. Then

$$\omega^2 T - V = ml^2 \begin{pmatrix} 2\omega^2 - 2\omega_0^2 & \omega^2 \\ \omega^2 & \omega^2 - \omega_0^2 \end{pmatrix}$$

And

$$\det(\omega^2 T - V) = (ml^2)^2 \left\{ 2(\omega^2 - \omega_0^2)^2 - \omega^4 \right\}.$$

Setting $\det(\omega^2 T - V) = 0$ then yields $\omega^2 = (2+\sqrt{2})\omega_0^2$.

Find

$$A = \begin{pmatrix} \psi_1^{(+)} & \psi_1^{(-)} \\ \psi_2^{(+)} & \psi_2^{(-)} \end{pmatrix} = \frac{1}{2\sqrt{ml^2}} \begin{pmatrix} \sqrt{2+\sqrt{2}} & \sqrt{2-\sqrt{2}} \\ -\sqrt{2}.\sqrt{2+\sqrt{2}} & \sqrt{2}.\sqrt{2-\sqrt{2}} \end{pmatrix} \sigma = 1$$

Note that $\vec{\psi}^{(+)} \propto \left( \frac{1}{\sqrt{2}} \right)$ and $\vec{\psi}^{(-)} \propto \left( \frac{1}{\sqrt{2}} \right)$

Normal mode shapes:

In the low frequency normal mode, the two masses oscillate in phase, while in the high frequency normal mode, they are $\pi$ out of phase.
**Zero modes**

Recall that to each continuous one-parameter family of coordinate transformations

\[ \varrho_0 \rightarrow \tilde{\varrho}_0(\varrho, S), \quad \tilde{\varrho}_0(\varrho, S = 0) = \varrho_0 \]

leaving \( L \) invariant corresponds a conserved "charge",

\[ \lambda = \sum_{\sigma} \frac{\partial L}{\partial \tilde{\varrho}_0} \frac{\partial \tilde{\varrho}_0}{\partial S} \bigg|_{S=0}, \quad \frac{d\lambda}{dt} = 0 \]

Let us label the various one-parameter invariances with a label \( k \). For small oscillations,

\[ \frac{\partial L}{\partial \tilde{\varrho}_0} = \frac{\partial L}{\partial \tilde{\eta}_0} = T_{\sigma \sigma}, \quad \frac{\partial \tilde{\varrho}_0}{\partial S} \bigg|_{S=0} \]

which says

\[ C_{k \sigma} = \sum_{\sigma} T_{\sigma \sigma} \frac{\partial \tilde{\varrho}_0}{\partial S_k} \bigg|_{S=0} \]

so that

\[ \ddot{\xi}_k = \sum_{\sigma} C_{k \sigma} \eta_0 \]

is a zero mode, satisfying \( \ddot{\xi}_k = 0 \). (As written it is unnormalized. Thus, in systems with continuous symmetries, associated with each such symmetry is a zero mode of the corresponding small oscillations problem.

**Example 1:** \( L = \frac{1}{2} M_1 \dot{x}_1^2 + \frac{1}{2} M_2 \dot{x}_2^2 - \frac{1}{2} k(x_2 - x_1 - a)^2 \)

\[ M_1 \rightarrow 2M, \quad M_2 \rightarrow 2M, \quad \frac{1}{2} \Rightarrow \frac{1}{2} \]

\( \Rightarrow X \) (cm) is a ZM

\[ x = \frac{1}{2}(x_1 + x_2), \quad x = x_2 - x_1 \]
Example 2

Consider the system to the right, for which

\[ T = \frac{1}{2} R^2 (m_1 \dot{\phi}_1^2 + m_2 \dot{\phi}_2^2 + m_3 \dot{\phi}_3^2) \]

and

\[ U = \frac{1}{2} k R^2 \left[ (\phi_2 - \phi_1 - \chi)^2 + (\phi_3 - \phi_2 - \chi)^2 + (2\pi + \phi_1 - \phi_3 - \chi)^2 \right] \]

where \( \phi_3 - 2\pi < \phi_1 < \phi_2 < \phi_3 < \phi_1 + 2\pi \), and where \( RX = a \) is the unstretched length of each spring.

The equilibrium configuration is

\[ \phi_1^0 = 5, \quad \phi_2^0 = 5 + \frac{2\pi}{3}, \quad \phi_3^0 = 5 + \frac{4\pi}{3} \]

where \( 3 \) is an arbitrary continuous parameter, corresponding to the continuous translational symmetry that is present. Find

\[
T = \begin{pmatrix}
m_1 R^2 & 0 & 0 \\
0 & m_2 R^2 & 0 \\
0 & 0 & m_3 R^2
\end{pmatrix}, \quad
V = \begin{pmatrix}
2k R^2 & -k R^2 & -k R^2 \\
-k R^2 & 2k R^2 & -k R^2 \\
-k R^2 & -k R^2 & 2k R^2
\end{pmatrix}
\]

and

\[
\omega^2 T - V = k R^2 \begin{pmatrix}
\frac{\omega_l^2}{\nu_l^2} - 2 & 1 & 1 \\
1 & \frac{\omega_l^2}{\nu_l^2} - 2 & 1 \\
1 & 1 & \frac{\omega_l^2}{\nu_l^2} - 2
\end{pmatrix}, \quad \nu_j^2 = \frac{k}{m_j}
\]
The characteristic polynomial is

\[
P(w^2) = \det(w^2T - V) = (kR^2)^2 \cdot \tilde{P}(w^2)
\]

\[
\tilde{P}(w^2) = \frac{w^6}{v_1^4 v_2^2 v_3^2} - 2\left(\frac{1}{v_1^4 v_2^4} + \frac{1}{v_2^6} + \frac{1}{v_3^2 v_1^2}\right) w^4
\]

\[+ 3\left(\frac{1}{v_1^2} + \frac{1}{v_2^2} + \frac{1}{v_3^2}\right) w^2
\]

This is cubic in \(w^2\), but since there is no \((w^2)^0\) term, \(w^2\) divides \(\tilde{P}(w^2)\), i.e., \(\tilde{P}(w^2) = w^2 \tilde{Q}(w^2)\), where \(\tilde{Q}(w^2)\) is a quadratic function of its argument. Thus the normal mode frequencies are

\[
\begin{align*}
\omega_1^2 &= 0 \\
\omega_2^2 &= \sqrt{\frac{v_1^4 + v_2^4 + v_3^4}{4} + (v_1^2 - v_2^2) + (v_2^2 - v_3^2) + (v_3^2 - v_4^2)}
\end{align*}
\]

To find the modal matrix, set \((w_j^2T - V)\psi(i) = 0\):

\[
\begin{pmatrix}
\frac{\omega_j^2}{v_1^2} - 2 & 1 & 1 \\
1 & \frac{\omega_j^2}{v_2^2} - 2 & 1 \\
1 & 1 & \frac{\omega_j^2}{v_3^2} - 2
\end{pmatrix}
\begin{pmatrix}
\psi(1) \\
\psi(2) \\
\psi(3)
\end{pmatrix}
= 0
\]

which yields \(\psi(1) = C_j \sqrt{3 - \frac{\omega_j^2}{v_1^2}},\) where

\[
C_j = \left[\sum_{\sigma=1}^{3} m_\sigma \left(3 - \frac{\omega_j^2}{v_\sigma^2}\right)^{-2}\right]^{-1/2}
\]

for normalization.
Note for the zero mode \((j=1)\) we have
\[ A_{\sigma 1} = \psi_{\sigma}^{(1)} = \frac{C_1}{3} = (m_1+m_2+m_3)^{-1/2} \quad \forall \sigma \in \{1,2,3\} \]
Thus,
\[ \delta_{\sigma 1} = A_{\sigma 1} T_{\sigma 0} \eta_0 \]
\[ = (m_1+m_2+m_3)^{-1/2} R^2 (m_1 \eta_1 + m_2 \eta_2 + m_3 \eta_3) \]
is the normalized zero mode. This is consistent with Noether's theorem, which says
\[ \Lambda = \sum_{\sigma=1}^{3} \frac{\partial L}{\partial \dot{\phi}_{\sigma}} \dot{\phi}_{\sigma} = R^2 (m_1 \dot{\phi}_1 + m_2 \dot{\phi}_2 + m_3 \dot{\phi}_3) \]
with \(\dot{\Lambda} = 0\). Note that \(\dot{\Lambda} = 0\) always, and not only in the limit of small deviations from static equilibrium.

**Chain of identical masses and springs**

\[ L = \frac{1}{2} m \sum_{\sigma} \dot{x}_{\sigma}^2 - \frac{1}{2} k \sum_{\sigma} (x_{\sigma+1} - x_{\sigma} - a)^2 + \tau \sum_{n} (x_{n+1} - x_{n}) \]
Clearly \( p_{\sigma} = \frac{\partial L}{\partial \dot{x}_{\sigma}} = m \dot{x}_{\sigma} \). If the chain is finite, with \(n\) running from 1 to \(N\), then
\[ F_1 = \frac{\partial L}{\partial x_1} = k (x_2 - x_1 - a) - \tau \]
\[ F_N = \frac{\partial L}{\partial x_N} = -k (x_N - x_{N-1} - a) + \tau \]
\[ F_\sigma = \frac{\partial L}{\partial x_\sigma} = k (x_{\sigma+1} + x_{\sigma-1} - 2x_j) \quad \sigma \in \{2, \ldots, N-1\} \]
The last equation says that $F_\sigma = 0 \not\equiv \sigma \in \{1, \ldots, N\}$ if

$$x_{\sigma + 1} - x_\sigma = b \quad \sigma \in \{1, \ldots, N-1\}$$

where $b$ is a constant. Plugging this into the first equations then yields $b = a + k^{-1} \zeta$.

If the chain is a periodic ring with $x_{N+1} = x_1 + C$, then $b = C/N$ is the only solution. We’ll solve the problem in this case of periodic boundary conditions (PBCs). In the limit $N \to \infty$, the bulk behavior won’t differ between the two cases. Writing

$$x_\sigma = \sigma b + u_\sigma + s \quad \sigma \in \{1, \ldots, N\}$$

we have

$$L = \frac{1}{2} m \sum_{\sigma = 1}^{N} \dot{u}_\sigma^2 - \frac{1}{2} k \sum_{\sigma = 1}^{N} (u_{\sigma + 1} - u_\sigma)^2 - k(b-a)C - \frac{1}{2} Nk(b-a)^2$$

The last two terms arise when $b \neq a$ due to the fact that the springs are all (equally) stretched in the static equilibrium configuration. These terms are both constants which we henceforth drop. The EL equations are then

$$\ddot{u}_\sigma = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{u}_\sigma} \right) = \frac{\partial L}{\partial u_\sigma} = k(u_{\sigma + 1} + u_{\sigma - 1} - 2u_\sigma)$$

with $u_{N+1} = u_1$. These $N$ coupled ODEs may easily be solved $u_{\sigma + 1} - u_\sigma - k(u_\sigma - u_{\sigma - 1})$
by transforming to Fourier space coordinates, viz.

\[ u_\sigma = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{2\pi i j \sigma / N} \hat{u}_j \quad \overset{\leftrightarrow}{\Rightarrow} \quad \hat{u}_j = \frac{1}{\sqrt{N}} \sum_{\sigma=1}^{N} e^{-2\pi i j \sigma / N} u_\sigma \]

Note that \( \hat{u}_j \) is complex, with

\[ \hat{u}_{N-j} = \frac{1}{\sqrt{N}} \sum_{\sigma=1}^{N} e^{2\pi i j \sigma / N} u_\sigma = \hat{u}_j^* \]

Let's count degrees of freedom. The set \( \{u_1, \ldots, u_N\} \) constitutes \( N \) real degrees of freedom. For \( N \) even, \( \hat{u}_N \) and \( \hat{u}_{N/2} \) are real, while \( \hat{u}_j \) for \( j \in \{1, \ldots, \frac{1}{2} N-1\} \) are complex and satisfy \( \Re \hat{u}_{N-j} = \Re \hat{u}_j \) and \( \Im \hat{u}_{N-j} = -\Im \hat{u}_j \). The number of real degrees of freedom is then

\[ \text{DoF} = 2 + 2 \times \left( \frac{1}{2} N - 1 \right) = N \quad \checkmark \]

If \( N \) is odd, then \( \hat{u}_N \) is again real, but there is no mode \( \hat{u}_j \) with \( j = \frac{1}{2} N \). We again have \( \hat{u}_{N-j} = \hat{u}_j^* \), this time for \( j \in \{1, \ldots, \frac{1}{2} (N-1)\} \). The number of real degrees of freedom is

\[ \text{DoF} = 1 + 2 \times \frac{1}{2} (N-1) = N \quad \checkmark \]

We now have

\[ M \frac{1}{\sqrt{N}} \sum_{\sigma=1}^{N} e^{-2\pi i j \sigma / N} \ddot{u}_\sigma = k \frac{1}{\sqrt{N}} \sum_{\sigma=1}^{N} e^{-2\pi i j \sigma / N} \left(u_{\sigma+1} + u_{\sigma-1} - 2u_\sigma\right) \]

\[ m \ddot{u}_j = -2k \left[1 - \cos\left(\frac{2\pi j}{N}\right)\right] \hat{u}_j \]
Thus we may write \( \ddot{u}_j = -w_j^2 u_j \) with
\[
w_j = 2\sqrt{\frac{k}{m}} | \sin(\frac{\pi j}{N}) |
\]
The solution for each normal mode is
\[
\dot{u}_j(t) = C_j e^{-iw_j t} e^{i\delta_j}
\]
where \( C_{N-j} = C_j \) and \( \delta_{N-j} = -\delta_j \) for all \( j \neq \{ \frac{N}{2}, N \} \), and \( \delta_{\frac{N}{2}} = \delta_{N} = 0 \). The \( \{C_j, \delta_j\} \) are all real constants.

The modal matrix is then
\[
A_{\sigma j} = \frac{1}{\sqrt{Nm}} e^{2\pi i j \sigma / N},
\]
where we have now included the \( m^{-1/2} \) factor. Note
\[
T_{\sigma \sigma'} = m \delta_{\sigma \sigma'}
\]
\[
V_{\sigma \sigma'} = 2k \delta_{\sigma \sigma'} - k \delta_{\sigma, \sigma+1} - k \delta_{\sigma, \sigma-1}.
\]

The Kronecker deltas are understood to be modulo \( N \), i.e.,
\[
\delta_{\sigma \sigma'} = \begin{cases} 1 & \text{if } \sigma' = \sigma \mod N \\ 0 & \text{otherwise} \end{cases}
\]
Thus, the matrix forms of \( T \) and \( V \) are
\[
T = \begin{pmatrix} m & 0 & \cdots & 0 \\ 0 & m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m \end{pmatrix}, \quad V = \begin{pmatrix} 2k & -k & 0 & \cdots & 0 & -k \\ -k & 2k & -k & 0 & \cdots & 0 \\ 0 & -k & 2k & -k & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -k & 2k & -k \\ -k & 0 & \cdots & -k & 2k & -k \end{pmatrix}
\]
Using the equation \( \frac{1}{N} \sum_{j=1}^{N} e^{2\pi i (j-j')/N} = \delta_{jj'} \) we can prove that \( A^T A = I \) and \( A^T V A = \text{diag}(\omega_1^2, \ldots, \omega_N^2) \).

Continuum limit: We take
\[
    u_\sigma(t) \to u(x=\sigma b, t)
\]
and
\[
    u_{\sigma+1} - u_\sigma = u(x+b) - u(x) = b \frac{\partial u}{\partial x} + \frac{1}{2} b^2 \frac{\partial^2 u}{\partial x^2} + \ldots
\]
Thus,
\[
    T = \frac{1}{2} m \sum_\sigma \dot{u}_\sigma^2 \to \frac{1}{2} m \int dx \left( \frac{\partial u}{\partial t} \right)^2
\]
\[
    V = \frac{1}{2} k \sum_\sigma (u_{\sigma+1} - u_\sigma)^2 \to \frac{1}{2} k \int dx \left( b \frac{\partial u}{\partial x} \right)^2 + \ldots
\]
and we may write
\[
    S = \int dt \int dx \mathcal{L}(\{u_\sigma\}, \{\dot{u}_\sigma\}, t) = \int dt \int dx \mathcal{L}(u, \partial_x u, \partial_t u, t)
\]
where
\[
    \mathcal{L}(u, \partial_x u, \partial_t u, t) = \frac{1}{2} \rho \left( \frac{\partial u}{\partial t} \right)^2 - \frac{1}{2} \tau \left( \frac{\partial u}{\partial x} \right)^2
\]
with \( \rho = m/b = \text{mass density} \) and \( \tau = k b = \text{"tension"} \) is the Lagrangian density. Suppose the Lagrangian is of the form
\[
    L = \sum_\sigma L_\sigma(u_\sigma, \dot{u}_\sigma, \frac{u_{\sigma+1} - u_\sigma}{b}, t)
\]
We have

\[ L = \sum \sigma L_\sigma (u_\sigma, \dot{u}_\sigma, \frac{u_{\sigma+1}-u_\sigma}{b}, t) \]

The EL eqns are then

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{u}_\sigma} \right) = \frac{\partial L}{\partial u_\sigma} + \frac{1}{b} \frac{\partial L_{\sigma-1}}{\partial u'_\sigma} - \frac{1}{b} \frac{\partial L_\sigma}{\partial u'_\sigma} \]

Now

\[ \frac{(\partial L_\sigma/\partial u'_\sigma)-(\partial L_{\sigma-1}/\partial u'_\sigma)}{b} = \frac{\partial}{\partial x} \frac{\partial L_\sigma}{\partial u'_\sigma} + \ldots \]

and writing

\[ L_\sigma (u_\sigma, \dot{u}_\sigma, \frac{u_{\sigma+1}-u_\sigma}{b}, t) = \frac{1}{b} \mathcal{L} (u_\sigma, \dot{u}_\sigma, \frac{u_{\sigma+1}-u_\sigma}{b}, \sigma b, t) \]

\[ = \frac{1}{b} \mathcal{L} (u, \partial_t u, \partial_x u, x, t) \]

we have

\[ S = \int dt \int d\sigma \mathcal{L} (u, \partial_t u, \partial_x u, x, t) \]

and the equations of motion

\[ \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{L}}{\partial \dot{u}} \right) + \frac{\partial}{\partial x} \left( \frac{\partial \mathcal{L}}{\partial \ddot{u}} \right) = \frac{\partial \mathcal{L}}{\partial \dot{u}} \]

More about this in chapter 9 of the lecture notes.