• Satellites and spacecraft

Recall: \[ T = \frac{2\pi}{\sqrt{GM_E}} (R_E + h)^{3/2} \quad (m_s \ll M_E) \]

LEO = “Low Earth Orbit” \( h \ll R_E = 6.37 \times 10^6 \text{ m} \)
So find \( T_{\text{LEO}} = 1.4 \text{ hr} \).

Problem: \( h_p = 200 \text{ km} \), \( h_a = 7200 \text{ km} \)
\[ a = \frac{1}{2} (R_E + h_p + R_E + h_a) = 10071 \text{ km} \]
\[ T_{\text{sat}} = \left( a/R_E \right)^{3/2} \cdot T_{\text{LEO}} \approx 2.65 \text{ hr} \]

• Read §§ 4.5 and 4.6

Lecture 6 (Oct. 21)

• A rigid body is a collection of point particles whose separations \( |\vec{r}_i - \vec{r}_j| \) are all fixed in magnitude. Six independent coordinates are required to specify completely the position and orientation of a rigid body. For example, the location of the first particle \((i)\) is specified by \( \vec{r}_i \), which is three coordinates. The second \((j)\) is then specified by a direction unit vector \( \hat{n}_{ij} \), which requires two additional coordinates (polar and azimuthal angle). Finally, a third particle, \( k \), is then fixed by its angle relative to the \( \hat{n}_{ij} \) axis. Thus, six generalized coordinates in all are required.
Usually, one specifies three CM coordinates $\vec{r}$, and three orientational coordinates (e.g. the Euler angles). The equations of motion are then

$$\ddot{\vec{P}} = \sum_i m_i \vec{r}_i \quad , \quad \vec{P} = \vec{F}_{\text{ext}} \quad (\text{external force})$$

$$\ddot{\vec{L}} = \sum_i m_i \vec{r}_i \times \vec{\dot{r}}_i \quad , \quad \vec{L} = \vec{N}_{\text{ext}} \quad (\text{external torque})$$

- Inertia tensor

Suppose a point within a rigid body is fixed. This eliminates the translational motion. If we measure distances relative to this fixed point, then in an inertial frame,

$$\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r} \quad ; \quad \vec{\omega} = \text{angular velocity}$$

The kinetic energy is then

$$T = \frac{1}{2} \sum_i m_i \left( \frac{d\vec{r}_i}{dt} \right)^2 = \frac{1}{2} \sum_i (\vec{\omega} \times \vec{r}_i) \cdot (\vec{\omega} \times \vec{r}_i)$$

$$= \frac{1}{2} \sum_i m_i \left[ \omega^2 \vec{r}_i^2 - (\vec{\omega} \cdot \vec{r}_i)^2 \right] = \frac{1}{2} I_{\alpha\beta} \omega_\alpha \omega_\beta$$

where $I_{\alpha\beta}$ is the inertia tensor,

$$I_{\alpha\beta} = \sum_i m_i \left[ \vec{r}_i^2 \delta_{\alpha\beta} - r_\alpha r_\beta \right] \quad \text{(discrete)}$$

$$= \int d^3r \rho(\vec{r}) \left[ \vec{r}^2 \delta_{\alpha\beta} - r_\alpha r_\beta \right] \quad \text{(continuous)}$$

Diagonal elements of $I_{\alpha\beta}$ are moments of inertia, while off-diagonal elements are products of inertia.
coordinate transformations
\{ \hat{e}_1, \hat{e}_2, \hat{e}_3 \} = \text{orthonormal basis}; \hat{e}_\alpha \cdot \hat{e}_\beta = \delta_{\alpha \beta}

Orthogonal basis transformation:
\hat{e}'_\alpha = R_{\alpha \mu} \hat{e}_\mu \quad ; \quad \hat{e}'_\alpha \cdot \hat{e}'_\beta = R_{\alpha \mu} R_{\beta \nu} \hat{e}_\mu \cdot \hat{e}_\nu = (R^T R)_{\alpha \beta} = \delta_{\alpha \beta}

Let \( \hat{A} = A^\mu \hat{e}_\mu \) be a vector with \( A^\alpha \) the components. Then
\[ \hat{A}' = A'^\mu \hat{e}'_\mu = A^\mu R_{\alpha \mu} \hat{e}'_\alpha \Rightarrow A'^\alpha = R_{\alpha \mu} A^\mu \]
coordinate transformation

How does the inertia tensor transform?
\[ I'^{\alpha \beta} = \int d^3 r' \rho(r') \left[ \hat{v}'^2 \delta^{\alpha \beta} - r'^{\alpha} r'^{\beta} \right] \]
\[ = \int d^3 r \rho(r) \left[ \hat{v}^2 \delta^{\alpha \beta} - R_{\alpha \mu} r^{\mu} R_{\beta \nu} r^{\nu} \right] \]
\[ = R_{\alpha \mu} I^{\mu \nu} R^T_{\nu \beta} \quad \text{, since } \rho'(r') = \rho(r) \]
i.e. \( \hat{v}' = R \hat{v} \) is the transformation rule for vectors, and \( I' = R I R^T \) the rule for tensors. For scalars, \( s' = s \). Note \( \hat{w} \) is a vector, as is \( \hat{L} \), but
\[ T = \frac{1}{2} w_\alpha I_{\alpha \beta} w_\beta \quad \text{is a scalar} \]

Note: \( T = \frac{1}{2} R^T_{\alpha \mu} w'_\mu I_{\alpha \beta} R^T_{\beta \nu} w'_\nu = \frac{1}{2} w'_\mu \left( R_{\mu \alpha} I_{\alpha \beta} R^T_{\beta \nu} \right) w'_\nu \)
\[ = \frac{1}{2} w'_\mu I'_{\mu \nu} w'_\nu = T' \quad (\hat{w} = R^T \hat{w}') \]
- The case of no fixed point

If there is no fixed point, choose CM as instantaneous origin for the body-fixed frame:

\[ \dot{\vec{R}} = \frac{1}{M} \sum_i m_i \dot{\vec{r}}_i = \frac{1}{M} \int d^3 \rho(\vec{r}) \dot{\vec{r}} \]

\[ M = \sum_i m_i = \int d^3 \rho(\vec{r}) = \text{total mass} \]

Then

\[ T = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} I_{\alpha \beta} \omega^\alpha \omega^\beta \]

\[ L_\alpha = \varepsilon_{\alpha \beta \gamma} M R^\beta \dot{R}^\gamma + I_{\alpha \beta} \omega^\beta \]

• Parallel axis theorem

Suppose we have \( I_{\alpha \beta} \) in a body-fixed frame.

Now shift the origin from \( \vec{0} \) to \( \vec{a} \). A mass at position \( \vec{r}_i \) is located at \( \vec{r}_i - \vec{a} \) as a result. Thus,

\[ I_{\alpha \beta}(\vec{a}) = \sum_i m_i \left[ (\vec{r}_i - 2\vec{a} \cdot \vec{r}_i + \vec{a}^2) \delta_{\alpha \beta} - (r_i^\alpha - d^\alpha)(r_i^\beta - d^\beta) \right] \]

If \( \vec{r}_i \) in the original frame is wrt the CM, then \( \sum_i m_i \vec{r}_i = 0 \), and we have

\[ I_{\alpha \beta}(\vec{a}) = I_{\alpha \beta}^{CM} + M \left( \sum_i \dot{\vec{r}}_i^2 \delta_{\alpha \beta} - \dot{d}^\alpha d^\beta \right) \]

Since we are only translating the origin, the coordinate axes remain parallel. Hence this result is known as the parallel axis theorem.
Example: Uniform cylinder of radius $a$, height $L$

With origin at CM,

$$I_{zz}^C = \int d^3 r \rho(r^2) (x^2 + y^2)$$

$$= 2\pi \rho L \int_0^a r^3 = \frac{\pi}{2} \rho L a^4$$

$$= \frac{1}{2} M a^2 \quad \text{since} \quad M = \pi a^2 L \rho$$

Displace origin to surface: $\vec{d} = a \hat{\rho}$

Distance $s$ ranges from 0 to $s_0$, with

$$a^2 = (s_0 \cos \alpha)^2 + (s_0 \sin \alpha - a)^2$$

$$= s_0^2 + a^2 - 2a s_0 \sin \alpha \Rightarrow s_0 = 2a \sin \alpha$$

Thus,

$$I_{zz}' = \rho L \int_0^\pi \int_0^{2a \sin \alpha} r^2 s^3 \, dr \, ds \, \sin^2 \alpha$$

$$= \frac{3}{2} Ma^2$$

Using parallel axis theorem: $\vec{d} = a \hat{\chi}$

$$I_{zz}' = I_{zz}^C + M (\vec{d}^2 \delta_{zz} - \vec{d}^2 \vec{d}^2)$$

$$= \frac{1}{2} Ma^2 + Ma^2 = \frac{3}{2} Ma^2 \quad \checkmark$$

No need for trigonometry or integration!

- Read § 8.3.1 (inertia tensor for right triangle)
Planar mass distributions:

If \( p(x,y,z) = \sigma(x,y) \delta(z) \), then \( I_{xz} = I_{yz} = 0 \)

Furthermore,

\[
I_{xx} = \int dx \int dy \, \sigma(x,y) y^2 \\
I_{yy} = \int dx \int dy \, \sigma(x,y) x^2 \\
I_{xy} = -\int dx \int dy \, \sigma(x,y) xy
\]

and \( I_{zz} = I_{xx} + I_{yy} \). Only 3 parameters.

Principal axes of inertia

In general, if you have a symmetric matrix and you diagonalize it, good things will happen. Recall that basis transformation \( e'_\alpha = R_{\alpha \mu} e_\mu \) entails the transformation rules for vectors and tensors,

\[
\vec{A}' = R \vec{A} , \quad I' = R I R^T
\]

i.e. \( A'^{\alpha}_\mu = R_{\alpha \mu} A^{\alpha}_\mu \), \( I'_{\alpha \beta} = R_{\alpha \mu} I_{\mu \nu} R_{\nu \beta} \).

Since \( I = I^T \) is symmetric, we can find a new orthonormal basis \( \{ e'_\mu \} \) with respect to which \( I' \) is diagonal. Dropping the primes, we have that in a diagonal basis,

\[
I = \text{diag}(I_1, I_2, I_3) , \quad \vec{L} = (I_1 w_1, I_2 w_2, I_3 w_3) \\
T = \frac{1}{2} w_\alpha I_{\alpha \mu} w_\beta = \frac{1}{2} (I_1 w_1^2 + I_2 w_2^2 + I_3 w_3^2)
\]
How to diagonalize $I_\alpha \beta$ (or any real symmetric matrix):

1) Find the diagonal elements of $I'$, which are the eigenvalues of $I$, by solving $P(\lambda) = \det(\lambda I - I) = 0$. If $I_\alpha \beta$ is of rank $n$, $P(\lambda)$ is a polynomial in $\lambda$ of order $n$.

2) For each eigenvalue $\lambda_a \ (a = 1, \ldots, n)$, solve the $n$ equations

$$\sum_{\nu=1}^{n} I_{\mu \nu} \psi_{\nu}^{a} = \lambda_a \psi_{\mu}^{a}$$

where $\psi_{\mu}^{a}$ is the $\mu^{th}$ component of the $a^{th}$ eigenvector $\hat{\psi}^{a}$. Since $(\lambda_a I - I)$ is degenerate, the above equations are linearly dependent, and we may solve for the $(n-1)$ ratios $\{\psi_{2}^{a}/\psi_{1}^{a}, \ldots, \psi_{n}^{a}/\psi_{1}^{a}\}$.

3) Since $I_\alpha \beta$ is real and symmetric, its eigenfunctions corresponding to distinct eigenvalues are necessarily orthogonal. Eigenvectors corresponding to degenerate eigenvalues may be chosen to be orthogonal via the Gram-Schmidt procedure. Finally, the eigenvectors are normalized, thus

$$\langle \hat{\psi}^{a} | \hat{\psi}^{b} \rangle = \sum_{\mu=1}^{n} \psi_{\mu}^{a} \psi_{\mu}^{b} = \delta^{ab}$$

4) The matrix elements of $R$ are then given by $R_{\alpha \mu} = \psi_{\mu}^{a}$, i.e. the $a^{th}$ row of $R$ is the eigenvector $\hat{\psi}^{a}$, which is the $a^{th}$ column of $R^T$. 
5) The eigenvectors are complete and orthonormal.

Completeness: \[ \sum \psi^a \psi^a = R_{\mu \rho} R_{\nu \rho} = (R^T R)_{\mu \nu} = \delta_{\mu \nu} \]

Orthogonality: \[ \sum \psi^a \psi^b = R_{\mu \rho} R_{\nu \rho} = (R R^T)_{ab} = \delta_{ab} \]

See § 8.4 Eqns. 8.32 – 8.38 for an example

- Euler's equations

We choose our coordinate axes such that \( I_{\alpha \beta} \) is diagonal. Such a choice \( \{ \hat{e}_\alpha \} \) are called principal axes of inertia. We further choose the origin to be located at the CM. Thus

\[
\mathbf{\dot{\omega}} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}, \quad I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}, \quad \mathbf{\ddot{L}} = I \mathbf{\dot{\omega}} = \begin{pmatrix} I_1 \omega_1 \\ I_2 \omega_2 \\ I_3 \omega_3 \end{pmatrix}
\]

The equations of motion are then

\[
\mathbf{\dot{N}}_{\text{ext}} = \left( \frac{d\mathbf{\ddot{L}}}{dt} \right)_{\text{inertial}} = \left( \frac{d\mathbf{\ddot{L}}}{dt} \right)_{\text{body}} + \mathbf{\dot{\omega}} \times \mathbf{\ddot{L}}
\]

in inertial frame

\[
= I \mathbf{\dot{\omega}} + \mathbf{\dot{\omega}} \times (I \mathbf{\dot{\omega}})
\]

Here we have used the important relation

\[
\left( \frac{d\mathbf{\hat{A}}}{dt} \right)_{\text{inertial}} = \left( \frac{d\mathbf{\hat{A}}}{dt} \right)_{\text{body}} + \mathbf{\dot{\omega}} \times \mathbf{\hat{A}},
\]

valid for any vector \( \mathbf{\hat{A}} \). Let's derive this important result.
- Interlude: accelerated coordinate systems (§ 7.1)

Consider an inertial frame with fixed coordinate axes \( \hat{e}_\mu \), and a rotating frame with axes \( \hat{e}'_\mu \), where \( \mu \in \{1, \ldots, d\} \). The two frames share a common origin which is fixed within the body.

Any vector \( \vec{A} \) may be written as
\[
\vec{A} = \sum_{\mu} A_\mu \hat{e}_\mu = \sum_{\mu} A'_\mu \hat{e}'_\mu
\]

Thus in the inertial frame
\[
(\frac{d\vec{A}}{dt})_{\text{inertial}} = \sum_{\mu} \frac{dA_\mu}{dt} \hat{e}_\mu
\]
\[
= \sum_{\mu} \frac{dA'_\mu}{dt} \hat{e}'_\mu + \sum_{\mu} A'_\mu \frac{d\hat{e}'_\mu}{dt}
\]

*This is \((d\vec{A}/dt)_{\text{body}})*

What is \(d\hat{e}'_\mu/dt\)? Since the basis \( \{\hat{e}'_\nu\} \) is complete, we may expand
\[
d\hat{e}'_\mu = \sum_{\nu} d\Omega_{\mu\nu} \hat{e}'_\nu \quad \Rightarrow \quad d\Omega_{\mu\nu} = d\hat{e}'_\mu \cdot \hat{e}'_\nu
\]

But \(d(\hat{e}'_\mu \cdot \hat{e}'_\nu) = \hat{e}'_\mu \cdot \hat{e}'_\nu + \hat{e}'_\mu \cdot \hat{e}'_\nu = d\Omega_{\mu\nu} + d\Omega_{\nu\mu} = 0\)

Thus, \(d\Omega_{\mu\nu}\) is a real, antisymmetric, infinitesimal \(d\times d\) matrix.
A $d \times d$ real antisymmetric matrix has $\frac{1}{2} d(d-1)$ independent entries. For $d=3$, we may write

$$d\Omega_{\mu\nu} = \sum_{\sigma} \varepsilon_{\mu\nu\sigma} d\Omega_{\sigma}$$

and we define $\omega_{\sigma} = d\Omega_{\sigma}/dt$. This yields

$$\frac{d\hat{e}_{\mu}}{dt} = \hat{\omega} \times \hat{e}_{\mu}$$

and we have

$$\left( \frac{d\hat{A}}{dt} \right)_{\text{inertial}} = \left( \frac{d\hat{A}}{dt} \right)_{\text{body}} + \hat{\omega} \times \hat{A}$$

is valid for any vector $\hat{A}$. We may then write

$$\frac{d}{dt} \bigg|_{\text{inertial}} = \frac{d}{dt} \bigg|_{\text{body}} + \hat{\omega} \times$$

so long as we apply this to vectors only. Applied to the vector $\hat{\omega}$ itself, this yields $\hat{\omega}_{\text{inertial}} = \hat{\omega}_{\text{body}}$. Applied twice,

$$\frac{d^2\hat{A}}{dt^2} \bigg|_{\text{inertial}} = \frac{d^2\hat{A}}{dt^2} \bigg|_{\text{body}} + \frac{d\hat{\omega}}{dt} \times \hat{A} + 2\hat{\omega} \times \frac{d\hat{A}}{dt} \bigg|_{\text{body}} + \hat{\omega} \times (\hat{\omega} \times \hat{A})$$

This formula contains the description of centrifugal and Coriolis forces, which you can read about in chapter 7 of the notes. But for now, back to rigid body dynamics...
Euler’s equations along body-fixed principal axes:

\[
\frac{d\mathbf{\omega}}{dt} = \left( \frac{d\mathbf{L}}{dt} \right)_{\text{inertial}} = \left( \frac{d\mathbf{L}}{dt} \right)_{\text{body}} + \mathbf{\hat{w}} \times \mathbf{L} = \mathbf{I} \mathbf{\ddot{\omega}} + \mathbf{\hat{w}} \times (\mathbf{I} \mathbf{\dot{\omega}}) = \mathbf{N}^\text{ext}
\]

Component by component,

\[
\begin{align*}
I_1 \mathbf{\ddot{\omega}}_1 &= (I_2 - I_3) \mathbf{w}_2 \mathbf{w}_3 + N_1^\text{ext} \\
I_2 \mathbf{\ddot{\omega}}_2 &= (I_3 - I_1) \mathbf{w}_3 \mathbf{w}_1 + N_2^\text{ext} \\
I_3 \mathbf{\ddot{\omega}}_3 &= (I_1 - I_2) \mathbf{w}_1 \mathbf{w}_2 + N_3^\text{ext}
\end{align*}
\]

These three equations are coupled and nonlinear. The components \(N^\text{ext}_\alpha\) must be evaluated along the body-fixed principal axes. The simplest case is when there is no net external torque, which is the case when a body moves in free space, but also in a uniform gravitational field:

\[
\mathbf{N}^\text{ext} = \sum \mathbf{i} \times (m_i \mathbf{g}) = \left( \sum m_i \mathbf{r}_i \right) \times \mathbf{g}
\]

In a body fixed frame with the origin at the CM, the term in parentheses vanishes, hence \(\mathbf{N}^\text{ext} = 0\), and

\[
\begin{align*}
\mathbf{\ddot{w}}_1 &= \left( \frac{I_2 - I_3}{I_1} \right) \mathbf{w}_2 \mathbf{w}_3 \\
\mathbf{\ddot{w}}_2 &= \left( \frac{I_3 - I_1}{I_2} \right) \mathbf{w}_3 \mathbf{w}_1 \\
\mathbf{\ddot{w}}_3 &= \left( \frac{I_1 - I_2}{I_3} \right) \mathbf{w}_1 \mathbf{w}_2
\end{align*}
\]
- Torque-free symmetric tops:

Suppose \( I_1 = I_2 \neq I_3 \). Then \( \dot{w}_3 = 0 \), hence \( \omega_3 = \text{const} \). The remaining two equations are

\[
\dot{w}_1 = \left( \frac{I_1 - I_3}{I_1} \right) w_3 w_2, \quad \dot{w}_2 = \left( \frac{I_3 - I_1}{I_1} \right) w_3 w_1
\]

hence \( \dot{w}_1 = -\Omega w_2 \), \( \dot{w}_2 = +\Omega w_1 \), with \( \Omega = \left( \frac{I_3 - I_1}{I_1} \right) w_3 \).

Thus,

\[
\omega_1(t) = \omega_1 \cos(\Omega t + \delta), \quad \omega_2(t) = \omega_1 \sin(\Omega t + \delta), \quad \omega_3(t) = \omega_3
\]

where \( \omega_1 \) and \( \delta \) are constants of integration.

Therefore, in the body-fixed frame, \( \ddot{\hat{w}}(t) \) precesses about \( \hat{e}_3 \) (\( \equiv \hat{e}_3^{\text{body}} \)) with frequency \( \Omega \) at an angle \( \lambda = \tan^{-1}(\omega_1/\omega_3) \). For the earth, this is called the Chandler wobble, and \( \lambda \approx 6 \times 10^{-2} \text{ rad} \), meaning that the north pole moves by about four meters during the wobble. Again for earth, \( (I_3 - I_1)/I_1 = \frac{1}{305} \), hence the precession period is predicted to be about 305 days. In fact, the period of the Chandler wobble is about 14 months, which is a substantial discrepancy, attributed to the mechanical properties of the earth (elasticity and fluidity): the earth isn't solid!
Asymmetric tops

In principal, we may invoke energy and angular momentum conservation,

\[
E = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} I_3 \omega_3^2
\]

\[
\mathbf{L}^2 = I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2
\]

and obtain \( \omega_1 \) and \( \omega_2 \) in terms of \( \omega_3 \). Then

\[
\dot{\omega}_3 = \left( \frac{I_1 - I_2}{I_3} \right) \omega_1 \omega_2
\]

becomes a nonlinear first order ODE. Using Lagranges method and extremizing the energy at fixed \( \mathbf{L}^2 \), we obtain the following:

<table>
<thead>
<tr>
<th>Conditions ( \omega_2 = \omega_3 = 0 )</th>
<th>energy ( E ) ( \frac{1}{2} I_1 \omega_1^2 = \frac{L^2}{2I_1} )</th>
<th>extremum classification ( I_i &lt; I_j &lt; I_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega_2 = \omega_3 = 0 )</td>
<td>MAX</td>
<td>SP</td>
</tr>
<tr>
<td>( \omega_1 = \omega_3 = 0 )</td>
<td>SP</td>
<td>MAX</td>
</tr>
<tr>
<td>( \omega_1 = \omega_2 = 0 )</td>
<td>MIN</td>
<td>MIN</td>
</tr>
</tbody>
</table>

We can then analyze the nonlinear ODE \( \dot{\omega}_3 = f(\omega_3) \). This is somewhat unpleasant.
We can however easily linearize the equations of motion about a known solution. For example, \( \omega_1 = \omega_2 = 0 \) and \( \omega_3 = \omega_0 \) is a solution of Euler's equations. Let us then write \( \dot{\omega} = \omega_0 \hat{e}_3 + \delta \omega \). Then

\[
\delta \omega_1 = \left( \frac{I_2 - I_3}{I_1} \right) \omega_0 \delta \omega_2 + O(\delta \omega_2 \delta \omega_3)
\]

\[
\delta \omega_2 = \left( \frac{I_3 - I_1}{I_2} \right) \omega_0 \delta \omega_1 + O(\delta \omega_1 \delta \omega_3)
\]

\[
\delta \omega_3 = 0 + O(\delta \omega_1 \delta \omega_2)
\]

Thus, we have \( \delta \omega_1 = -\Omega^2 \delta \omega_1 \) and \( \delta \omega_2 = -\Omega^2 \delta \omega_2 \) with

\[
\Omega^2 = \frac{(I_3 - I_1)(I_3 - I_2)}{I_1 I_2} \omega_0^2
\]

The solution is \( \delta \omega_1(t) = e^{i \Omega t} \cos(\Omega t + \eta) \), in which case

\[
\delta \omega_2(t) = \omega_0^{-1} \frac{I_1}{I_2 - I_3} \delta \omega_1 = \left( \frac{I_1 (I_3 - I_1)}{I_2 (I_3 - I_2)} \right)^{1/2} e^{i \Omega t} \sin(\Omega t + \delta)
\]

If \( \Omega \in \mathbb{R} \), \( \delta \omega_1(t) \) and \( \delta \omega_2(t) \) are harmonic functions with period \( 2\pi / \Omega \). This is the case when \( I_3 > I_{1,2} \) or \( I_3 < I_{1,2} \). But if \( I_3 \) is in the middle, i.e. \( I_1 < I_3 < I_2 \) or \( I_2 < I_3 < I_1 \), then \( \Omega^2 < 0 \), \( \Omega \in i \mathbb{R} \), and the behavior is exponential, i.e. \( \dot{\delta \omega}(t) = \omega_0 \hat{e}_3 \) is unstable.
- Read § 8.5.1 (example problem for Euler's equations)

- Euler's angles

The dimension of the orthogonal group $O(n)$ is

$$\dim O(n) = \frac{1}{2} n(n-1)$$

Thus in dimension $n=2$, a rotation is specified by a single parameter, i.e. the planar angle. In $n=3$ dimensions, we require three parameters in order to specify a general rotation, i.e. a general orientation of an object with respect to some fiducial orientation. These three parameters are often taken to be Euler's angles $\{\phi, \theta, \psi\}$.

- General rotation matrix $R(\phi, \theta, \psi) \in SO(3)$:

Start with an orthonormal triad $\{\hat{e}_n\}$. We first rotate by $\phi$ about the $\hat{e}_3$ axis:

$$\hat{e}_n' = R_{\mu\nu}(\phi, \hat{e}_3^0) \hat{e}_\nu^0 ; \quad R(\phi, \hat{e}_3^0) = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The next step is to rotate by $\theta$ about $\hat{e}_1$:

$$\hat{e}_n'' = R_{\mu\nu}(\theta, \hat{e}_1') \hat{e}_\nu^1 ; \quad R(\theta, \hat{e}_1') = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix}$$
Constructing a general rotation in \( \text{SO}(3) \) using Euler's angles \( \{ \phi, \theta, \psi \} \)

Finally, rotate by \( \psi \) about \( \hat{e}_3' \):

\[
\hat{e}_\mu' = \hat{e}_\mu'' = R_{\mu \nu}(\psi, \hat{e}_3') \hat{e}_\nu'' ; \quad R(\psi, \hat{e}_3') = \begin{pmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

Multiply the three matrices to get \( \hat{e}_\mu = R_{\mu \nu}(\phi, \theta, \psi) \hat{e}_\nu \) with

\[
R(\phi, \theta, \psi) = \begin{pmatrix}
\cos \psi \cos \phi - \sin \phi \sin \psi \cos \theta & \cos \psi \sin \phi + \sin \psi \cos \theta \cos \phi & \sin \psi \sin \theta \\
-\sin \phi & \cos \phi & 0 \\
\sin \theta \sin \phi & -\sin \theta \cos \phi & \cos \theta
\end{pmatrix}
\]

See the figure at the top of this page.
Next we relate the components of $\hat{\omega}$ to the derivatives $\{\dot{\phi}, \dot{\theta}, \dot{\psi}\}$. This is accomplished by writing

$$\hat{\omega} = \dot{\phi} \hat{e}_\phi + \dot{\theta} \hat{e}_\theta + \dot{\psi} \hat{e}_\psi$$

where (consult previous figure)

$$\begin{align*}
\hat{e}_\phi &= \sin \theta \sin \psi \hat{e}_1 + \sin \theta \cos \psi \hat{e}_2 + \cos \theta \hat{e}_3 = \hat{e}_o \\
\hat{e}_\theta &= \cos \psi \hat{e}_1 - \sin \psi \hat{e}_2 \text{ ("line of nodes")}
\end{align*}$$

We may now read off

$$\begin{align*}
\omega_1 &= \hat{\omega} \cdot \hat{e}_1 = \dot{\theta} \sin \theta \sin \psi + \dot{\theta} \cos \psi \\
\omega_2 &= \hat{\omega} \cdot \hat{e}_2 = \dot{\phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi \\
\omega_3 &= \hat{\omega} \cdot \hat{e}_3 = \dot{\phi} \cos \theta + \dot{\psi}
\end{align*}$$

Note that:

$$\begin{align*}
\dot{\phi} &\leftrightarrow \text{precession} \quad \dot{\theta} &\leftrightarrow \text{nutation} \quad \dot{\psi} &\leftrightarrow \text{axial rotation}
\end{align*}$$

In spinning tops, axial rotation is sufficiently fast that it appears to us as a blur. We can, however, discern precession and nutation. The rotational kinetic energy is then

$$T_{rot} = \frac{1}{2} I_1 (\dot{\theta} \sin \theta \sin \psi + \dot{\theta} \cos \psi)^2 + \frac{1}{2} I_2 (\dot{\phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi)^2 + \frac{1}{2} I_3 (\dot{\phi} \cos \theta + \dot{\psi})^2$$
The canonical momenta are then

\[ p\phi = \frac{\partial T}{\partial \dot{\phi}}, \quad p\theta = \frac{\partial T}{\partial \dot{\theta}}, \quad p\psi = \frac{\partial T}{\partial \dot{\psi}} \]

and the angular momentum vector is

\[ \mathbf{L} = p\phi \hat{\boldsymbol{e}}_\phi + p\theta \hat{\boldsymbol{e}}_\theta + p\psi \hat{\boldsymbol{e}}_\psi \]

Note that we don't need to specify the reference frame when writing \( \mathbf{L} \) — only for time-derivatives of vectors must we specify inertial or body-fixed frame.

- Torque-free symmetric top: \( \mathbf{N}^{\text{ext}} = 0 \)

Let \( I_1 = I_2 \). Then

\[ T = \frac{1}{2} I_1 (\dot{\phi}^2 + \sin^2 \theta \dot{\phi}^2) + \frac{1}{2} I_3 (\cos \theta \dot{\phi} + \dot{\psi})^2 \]

The potential is \( U = 0 \) so the Lagrangian is \( L = T \).

Since \( \phi \) and \( \psi \) are cyclic in \( L \), their momenta are conserved:

\[ p\phi = \frac{\partial L}{\partial \dot{\phi}} = I_1 \sin^2 \theta \dot{\phi} + I_3 \cos \theta (\cos \theta \dot{\phi} + \dot{\psi}) \]

\[ p\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3 (\cos \theta \dot{\phi} + \dot{\psi}) \]

Since \( p\psi = I_3 \omega_3 \), we have \( \omega_3 = \text{const.} \), as we have already derived from Euler's equations.
Let's solve for the motion. Note that \( \hat{L} \) is conserved in the inertial frame, i.e., \( (\hat{L})_{\text{inertial}} = 0 \). We choose \( \hat{e}_3^o = \hat{e}_\phi = \hat{L} \). From \( \hat{e}_\phi \cdot \hat{e}_\psi = \cos \Theta \), we have \( p_\psi = \hat{L} \cdot \hat{e}_\psi = L \cos \Theta \) and conservation of \( p_\psi \) thus entails \( \dot{\theta} = 0 \). From

\[
\dot{p}_\theta = I_1 \dot{\theta} = \frac{\partial L}{\partial \dot{\theta}} = (I_1 \cos \Theta \dot{\phi} - p_\psi) \sin \Theta \dot{\phi}
\]

and \( \dot{\theta} = 0 \), we conclude \( \dot{\phi} = p_\psi / I_1 \cos \Theta \). Now, from the equation for \( p_\psi \), we have

\[
\dot{\phi} = \frac{p_\psi}{I_3} - \cos \Theta \dot{\phi} = \left( \frac{1}{I_3} - \frac{1}{I_1} \right) p_\psi = \left( \frac{I_3 - I_1}{I_3} \right) \omega_3
\]

as we had derived from Euler's equations.

• Symmetric top with one point fixed:
  
  Now gravity exerts a torque. The Lagrangian is

\[
L = \frac{1}{2} I_1 (\dot{\phi}^2 + \sin^2 \Theta \dot{\phi}^2) + \frac{1}{2} I_3 (\cos \Theta \dot{\phi} + \dot{\psi})^2 - Mgl \cos \Theta
\]

where \( l \) is the distance from the fixed point to the CM. Let us now analyze the motion of this system.
The dreidel (Yid. סְדִיֶּל, Heb. דְרִיֶּל = spinner) is a symmetric top. Fourfold rotational symmetry is good enough to guarantee $I_1 = I_2$ and $I_{12} = 0$.

We have that $\phi$ and $\psi$ are still cyclic, so

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = I_1 \sin^2 \theta \dot{\phi} + I_3 \cos \theta (\cos \theta \dot{\phi} + \dot{\psi})$$

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3 (\cos \theta \dot{\phi} + \dot{\psi})$$

are again conserved. Thus,

$$\dot{\phi} = \frac{p_\phi - p_\psi \cos \theta}{I_1 \sin^2 \theta}, \quad \dot{\psi} = \frac{p_\psi}{I_3} - \frac{(p_\phi - p_\psi \cos \theta) \cos \theta}{I_1 \sin^2 \theta}$$

Energy $E = T + U$ is conserved:

$$E = \frac{1}{2} I_1 \dot{\phi}^2 + \frac{(p_\phi - p_\psi \cos \theta)^2}{2 I_1 \sin^2 \theta} + \frac{p_\psi^2}{2 I_3} + Mg l \cos \theta$$

effective potential $U_{\text{eff}}(\theta)$
Again:

\[ E = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{(P\phi - P\psi \cos \theta)^2}{2I_1 \sin^2 \theta} + \frac{P\psi^2}{2I_3} + Mg l \cos \theta \]

Straightforward analysis (see lecture notes, ch. 8, p. 18) reveals that \( U_{\text{eff}}(\theta) \) has a single minimum at \( \theta_0 \{0, \pi\} \), and that \( U_{\text{eff}}(\theta) \) diverges as \( \theta \to 0 \) and \( \theta \to \pi \).

Thus, the equation of motion,

\[ I_1 \ddot{\theta} = -U'_{\text{eff}}(\theta) \]

yields two turning points, which we label \( \theta_a \) and \( \theta_b \), satisfying \( E = U_{\text{eff}}(\theta_a, b) \). Now we have already derived the result

\[ \dot{\phi} = \frac{P\phi - P\psi \cos \theta}{I_1 \sin^2 \theta} \]

Thus we conclude that if \( P\psi \cos \theta_b < P\phi < P\psi \cos \theta_a \) then \( \dot{\phi} \) will change sign when \( \theta \) reaches \( \theta^* = \cos^{-1} \left( \frac{P\phi}{P\psi} \right) \).

This leads to two types of motion, as shown below.

Note that \( \hat{e}_3 = \sin \theta \sin \phi \hat{e}_1 - \sin \theta \cos \phi \hat{e}_2 + \cos \theta \hat{e}_3 \).

\( \phi \): precession
\( \theta \): nutation
\( \psi \): axial angle
$C_{4v}$

$\rho(x, y) = \rho(-y, x)$

$I_{\alpha\beta} = \int dx \int dy \rho(x, y) \left[ r^2 \delta_{\alpha\beta} - r x_{\alpha} y_{\beta} \right]$}

$(x, y) \rightarrow (-y, x) \rightarrow (-x, -y), (y, -x) \rightarrow (x, y)$

$\begin{array}{c|c|c}
\frac{\pi}{2} & \pi & \frac{3\pi}{2} \\
\frac{2\pi}{2} & & \\
\end{array}$

$I_{xx} = \int_{-a}^{a} \int_{-a}^{a} \rho(x, y) y^2 dx dy = \int_{-a}^{a} \int_{-a}^{a} \rho(y', -x') x'^2 dy dx$

$I_{xy} = \int_{-a}^{a} \int_{-a}^{a} \rho(x, y) xy dx dy = \int_{-a}^{a} \int_{-a}^{a} \rho(y', -x') y' x' dy dx$}

$I_{cube} = \frac{1}{6} Ma^2$ (1)

$I_{xx} < I_{yy}$ (2) isotropic

$I_{xy} = 0$ (3) cubic

$2 \text{- fold symmetry}$

$C_{2v}$

$21 \text{ components}$