Snell's law:

\[ \frac{\theta_1}{\sin \theta_1} = \frac{\theta_2}{\sin \theta_2} \]

\[ T(y) = \frac{1}{v_1} \sqrt{x_1^2 + (y-y_1)^2} + \frac{1}{v_2} \sqrt{x_2^2 + (y_2-y)^2} \]

\[ \frac{dT}{dy} = \frac{1}{v_1} \frac{y - y_1}{\sqrt{x_1^2 + (y-y_1)^2}} - \frac{1}{v_2} \frac{y_2 - y}{\sqrt{x_2^2 + (y_2-y)^2}} = 0 \]

Thus with \( v_j = c/n_j \) we have \( n_1 \sin \theta_1 = n_2 \sin \theta_2 \)

Now consider a sequence of slabs with differing \( v_j \).

We must have

\[ \frac{\sin \theta_j}{v_j} = \frac{\sin \theta_{j+1}}{v_{j+1}} \]

\[ \frac{\sin \theta(x)}{v(x)} = p = \text{constant} \]

We'll see that \( p \) corresponds to conserved momentum in mechanics. Note that

\[ \sin \theta(x) = \frac{y'(x)}{\sqrt{1 + [y'/x]^2}} = p v(x) \]

which yields

\[ y' = \frac{pv}{\sqrt{1 - p^2 v^2}} \Rightarrow y(x) = y(x_0) + \int_{x_0}^{x} ds \frac{pv(s)}{\sqrt{1 - p^2 v^2(s)}} \]
\[
\frac{d}{dx} \frac{y'}{\sqrt{1+(y')^2}} = \frac{y''}{\sqrt{1+(y')^2}} - \frac{y'y''}{\sqrt{1+(y')^2}^3} - \frac{v'y'}{\sqrt{1+(y')^2}}
\]

Thus,

\[
y'' - (\ln v)' [1+(y')^2] y' = 0
\]

Of course this may be integrated once to yield

\[
\frac{y'(x)}{\sqrt{1+ [y'(x)]^2}} = P\nu(x)
\]

Functional Calculus

- **Functions**: eat numbers, excrete numbers
  
  E.g. \( f: \mathbb{R} \rightarrow \mathbb{R} \), \( f(x) = -\frac{1}{2} x^2 + \frac{1}{4} x^4 \)
  
  Extremization: demand \( df=0 \) to lowest order in \( dx \)
  
  \[
f(x+d)=f(x)+f'(x)dx+\frac{1}{2}f''(x^*)(dx)^2+\ldots
\]

  Thus, \( df=0 \) in \( dx \rightarrow 0 \) limit says \( f'(x^*)=0 \), i.e. if \( f'(x^*)=0 \)
  
  Then \( x^* \) is an extremum. To second order,
  
  \[
  f''(x^*) > 0 \Rightarrow \text{minimum}, \quad f''(x^*) < 0 \Rightarrow \text{maximum}, \quad f''(x^*) = 0 \Rightarrow \text{inflection}
  \]
Multivariable functions: \( f : \mathbb{R}^n \rightarrow \mathbb{R} \)

\[
f(x^* + dx) = f(x^*) + \sum_{j=1}^{n} \left( \frac{\partial f}{\partial x_j} \right)_{x^*} dx_j + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \left( \frac{\partial^2 f}{\partial x_j \partial x_k} \right)_{x^*} dx_j dx_k + \ldots
\]

Extremum \( \Rightarrow \left. \frac{\partial f}{\partial x_j} \right|_{x^*} = 0 \quad \forall \ j = 1, \ldots, n
\)

Hessian matrix: \( H_{jk} = \left. \frac{\partial^2 f}{\partial x_j \partial x_k} \right|_{x^*} \) real, symmetric

eigenvalues of \( H \): \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \)
All \( \lambda_j > 0 \) \( \Rightarrow \) \( x^* \) local minimum
All \( \lambda_j < 0 \) \( \Rightarrow \) \( x^* \) local maximum
Some positive, some negative eigenvalues \( \Rightarrow \) \( x^* \) inflection pt

**Functionals**: functionals eat functions, excrete numbers

Typically, functionals are integrals, e.g.

\[
F[y(x)] = \int_{x_L}^{x_R} dx \left\{ \frac{1}{2} K \left( \frac{dy}{dx} \right)^2 + \frac{1}{2} a y^2 + \frac{1}{4} b y^4 \right\}
\]

Consider a class of functionals of the form

\[
F[y(x)] = \int_{x_L}^{x_R} dx \ L(y, y', x)
\]

where \( L(y, y', x) \) is a specified function of three variables, e.g.

\[
L = \frac{1}{2} K (y')^2 + \frac{1}{2} a y^2 + \frac{1}{4} b y^4
\]

Note this class may be extended to

\[
G[y(x)] = \int_{x_L}^{x_R} dx \ L(y, y', y'', x)
\]

Etc.
We now compute the functional variation by computing
\[ \delta F = F[y(x) + \delta y(x)] - F[y(x)] \]
\[ = \int_{x_L}^{x_R} d\xi \left\{ L\left(y' + \delta y', y + \delta y, \xi \right) - L(y', y, \xi) \right\} \]
\[ = \int_{x_L}^{x_R} d\xi \left\{ \frac{\partial L}{\partial y'} \delta y' + \frac{\partial L}{\partial y} \delta y + \ldots \right\} \quad \delta y' = \frac{d}{d\xi} \delta y \]
\[ = \int_{x_L}^{x_R} d\xi \left\{ \frac{d}{d\xi} \left( \frac{\partial L}{\partial y'} \delta y \right) + \left[ \frac{\partial L}{\partial y} - \frac{d}{d\xi} \left( \frac{\partial L}{\partial y'} \right) \right] \delta y \right\} \]
\[ = \left. \frac{\partial L}{\partial y'} \right|_{x_L}^{x_R} \delta y(x_R) - \left. \frac{\partial L}{\partial y'} \right|_{x_L}^{x_L} \delta y(x_L) + \int_{x_L}^{x_R} d\xi \left[ \frac{\partial L}{\partial y} - \frac{d}{d\xi} \left( \frac{\partial L}{\partial y'} \right) \right] \delta y \]

Suppose \( y(x) \) is fixed at the endpoints, in which case \( \delta y(x_L) = \delta y(x_R) = 0 \)

Then since \( \delta y(x) \) elsewhere on \([x_L, x_R]\) is arbitrary, we conclude that
\[ \frac{\delta F}{\delta y(x)} = \left[ \frac{\partial L}{\partial y} - \frac{d}{d\xi} \left( \frac{\partial L}{\partial y'} \right) \right] = 0 \quad \forall \ x \in [x_L, x_R] \]

Since \( L = L(y', y, x) \), the above equation is a second order ODE, known as the Euler–Lagrange equation. NB: If \( y(x_L, x_R) \) are not fixed, then we also require
\[ \left. \frac{\partial L}{\partial y'} \right|_{x_L, x_R} = 0 \text{ as well as } \frac{\partial L}{\partial y} - \frac{d}{d\xi} \left( \frac{\partial L}{\partial y'} \right) = 0 \]
in order that \( \delta F = 0 \).
Graphical representation:

The variation \( \delta y(x) \) resembles the following

\[
\delta F[y(x)] = F[y(x) + \delta y(x)] - F[y(x)]
\]

\[
\frac{\partial L}{\partial y'} \delta y' = \frac{\partial L}{\partial y} \frac{d}{dx} \delta y = \frac{d}{dx} \left[ \frac{\partial L}{\partial y'} \delta y \right] - \frac{d}{dx} \left( \frac{\partial L}{\partial y'} \right) \delta y
\]

\[
\frac{d}{dx} \frac{\partial L}{\partial y'} \delta y' = \frac{d}{dx} + y'' \frac{\partial}{\partial y'} + y' \frac{\partial}{\partial y}
\]
We now consider two important special cases:

1. \( \frac{\partial L}{\partial y} = 0 \), i.e. \( L(y, y', x) \) independent of \( y \)

   Then EL eqn says \( \frac{\partial}{\partial y} \left( \frac{\partial L}{\partial y'} \right) - \frac{d}{dx} \left( \frac{\partial L}{\partial y'} \right) = 0 \), which may be integrated once to yield \( \frac{\partial L}{\partial y'} = P \), where \( P = \) constant. This is then a first order ODE in \( y(x) \). Example: \( L = \frac{1}{v(y)} \sqrt{1+(y')^2} \). Then

   \[
   P = \frac{\partial L}{\partial y'} = \frac{y'}{v\sqrt{1+(y')^2}} = \frac{1}{v_0}
   \]

   \[
   \Rightarrow \quad \frac{dy}{dx} = \frac{v(x)}{\sqrt{v_0^2 - v^2(x)}} \quad \text{with} \quad v_0 = 1/P
   \]

2. \( \frac{\partial L}{\partial x} = 0 \), i.e. \( L(y, y', x) \) independent of \( x \)

   Define \( H = y' \frac{\partial L}{\partial y'} - L \). Then

   \[
   \frac{dH}{dx} = \frac{d}{dx} \left( y' \frac{\partial L}{\partial y'} - L \right)
   \]

   \[
   = y'' \frac{\partial L}{\partial y'} + y' \frac{d}{dx} \left( \frac{\partial L}{\partial y'} \right) - \frac{\partial L}{\partial y'} y'' - \frac{\partial L}{\partial y} y' - \frac{\partial L}{\partial x}
   \]

   \[
   = y' \left[ \frac{d}{dx} \left( \frac{\partial L}{\partial y'} \right) - \frac{\partial L}{\partial y} \right] = 0 \quad \text{if EL satisfied}
   \]

   Thus, \( \frac{\partial L}{\partial x} = 0 \Rightarrow \frac{dH}{dx} = 0 \Rightarrow H \) is constant

   \[
   y' \frac{\partial L}{\partial y'} - L = H \quad \text{again a first order ODE}
   \]
If \( L(y, y', x) = L_0(y, y', x) + \frac{d}{dx} \Delta(y, x) \), then
\[
F[y(x)] = \int_{x_L}^{x_R} L_0(y, y', x) + \Delta(y(x), x) - \Delta(y(x), x) \, dx
\]
If \( \delta y(x_1, x_2) = 0 \) (fixed endpoints), then the \( \Delta \) term makes no contribution to the EL eqns, which are then
\[
\frac{\partial L_0}{\partial y} - \frac{d}{dx} \left( \frac{\partial L_0}{\partial y'} \right) = 0
\]

- **Functional Taylor series**: 

\[
F[y + \delta y] = F[y] + \int_{x_L}^{x_R} dx_1 K_1(x_1) \delta y(x_1)
\]
\[+ \frac{1}{2!} \int_{x_L}^{x_R} dx_1 \int_{x_L}^{x_R} K_2(x_1, x_2) \delta y(x_1) \delta y(x_2)
\]
\[+ \frac{1}{3!} \int_{x_L}^{x_R} dx_1 \int_{x_L}^{x_R} \int_{x_L}^{x_R} K_3(x_1, x_2, x_3) \delta y(x_1) \delta y(x_2) \delta y(x_3)
\] + \( O(\delta y^4) \)

Thus,
\[
K_n(x_1, \ldots, x_n) = \frac{\delta^n F}{\delta y(x_1) \cdots \delta y(x_n)} = n^{th} \text{ functional derivative}
\]

- **Examples**: §3.3 in the lecture notes

- **More on functionals**: §3.4
Mechanics

Hamilton's principle: \[ \delta S = 0 \] where

\[ S[q(t)] = \int_{t_1}^{t_2} \frac{1}{2} L(q, \dot{q}, t) \, dt \]

with \( q = \{ q_1, \ldots, q_n \} \) = set of generalized coordinates

The function \( L(q, \dot{q}, t) \) is the Lagrangian, and is given by \( L = T - U \), where \( T \) = kinetic energy and \( U \) = potential energy. Typically \( T = T(q, \dot{q}) \) is a quadratic form in the generalized velocities \( \{ \dot{q}_i \} \), i.e. \( T(q, \dot{q}) = T_0(q) \dot{q}_i \dot{q}_i \). For example

\[ T = \frac{1}{2} m \dot{x}^2 = \frac{1}{2} m \left( \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) \quad \text{Cartesian} \quad (x, y, z) \]

\[ \frac{m}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix} \ddot{r} = \frac{1}{2} m \left( \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right) \quad \text{polar} \quad (r, \theta, \phi) \]

The potential energy \( U \) is most often a function of \( q \), but \( U = U(q, \dot{q}) \) applies, e.g., for charged particles in a magnetic field, where \( U \) = scalar potential

\[ U(x, \dot{x}) = q \phi(x) - \frac{q}{c} \vec{A}(x) \cdot \frac{d\vec{x}}{dt} \]

\( \vec{A} \) = charge \quad \text{vector potential}

Free particle \( \Rightarrow L = \frac{1}{2} m \dot{\vec{x}}^2 \) (§ 3.6.3)

NB: In general \( L = \frac{1}{2} T_0(q, t) \dot{q}_i \dot{q}_i \) - \( U(q, \dot{q}, t) \)