Lecture 11 (Nov. 9)

Start with the Lagrangian density

\[ \mathcal{L} = \frac{1}{2} \mu(x) y'^2 - \frac{1}{2} \tau(x) y'^2 - \frac{1}{2} v(x) y^2 \]

The last term corresponds to a harmonic potential attracting the string at each \( x \) value to \((x, y=0)\). In fact, if

\[ \mu(x) = \mu_0 + m \delta(x) \quad , \quad v(x) = KS(x) \]

then we recover the problem of a string with an attached point mass that is connected to the point \((0, 0)\) by a spring. The EL equations are found to be

\[
- \frac{\partial}{\partial x} \left[ \tau(x) \frac{\partial y}{\partial x} \right] + v(x) y = -\mu(x) \frac{\partial^2 y}{\partial t^2}
\]

This equation is time-translation invariant because the coefficients are autonomous (i.e. \( \tau(x), v(x), \) and \( \mu(x) \) do not depend on time \( t \)). This means that the partial differential operator (PDO)

\[
\hat{Q} = \mu(x) \frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial x} \tau(x) \frac{\partial}{\partial x} + v(x)
\]
for which \( \hat{Q} y(x,t) = 0 \), commutes with the PDO \( \partial / \partial t : [\hat{Q}, \partial / \partial t] = 0 \). This means that the solutions to \( \hat{Q} y(x,t) = 0 \) may be written as

\[
y(x,t) = \psi(x) e^{-i\omega t}
\]

Furthermore, since \( y^*(x,t) \) is a solution, then we may write

\[
y(x,t) = \psi(x) \cos(\omega t + \phi)
\]

We are left with the equation

\[
\hat{K} \psi(x) = \mu(x) \omega^2 \psi(x)
\]

where

\[
\hat{K} = -\frac{d}{dx} \tau(x) \frac{d}{dx} + \nu(x)
\]

is an ordinary differential operator (ODO). The equation

\[
\hat{K} \psi(x) = -\frac{d}{dx} \left[ \tau(x) \frac{d\psi(x)}{dx} \right] + \nu(x) \psi(x) = \mu(x) \omega^2 \psi(x)
\]

is known as the Sturm–Liouville equation.

The simplest example is when \( \tau(x) = \tau \) and \( \mu(x) = \mu \) are constants, and \( \nu(x) = 0 \). Then \( \hat{K} = -\tau \frac{d^2}{dx^2} \).
and the solutions to the SL eqn are of the form
\[ \psi(x) = A e^{ikx} \]
where \( k^2 = \mu w^2/T = \omega^2/c^2 \) with \( c=(T/\mu)^{1/2} \) = wave speed.
I.e. \( \psi(x) = A e^{\pm iw x/c} \), so \( y(x) = f(ct-x) + g(ct+x) \).

**Boundary conditions** - We consider four classes:

1. **Fixed endpoints**: \( \psi(x)=0 \) for \( x = x_L, R \)
2. **Natural**: \( \tau(x) \psi'(x) = 0 \) for \( x = x_L, R \)
3. **Periodic**: \( \psi(x+L) = \psi(x) \) where \( L = x_R - x_L \)
   [Also require \( \tau(x) = \tau(x+L) \).]
4. **Mixed homogeneous**: \( \alpha \psi(x) + \beta \psi'(x) = 0 \) for \( x = x_L, R \)
   [Same \( \alpha, \beta \) at both endpoints.]

**Eigenfunction properties**:
The SL equation is an eigenvalue equation:
\[ -\frac{d}{dx} \left( \tau(x) \psi'_n(x) \right) + \nu(x) \psi_n(x) = \omega_n^2 \mu(x) \psi_n(x) \]  \hspace{1cm} (A)
for a given choice of BCs. Suppose we have a second soln,
\[ -\frac{d}{dx} \left( \tau(x) \psi'_m(x) \right) + \nu(x) \psi_m(x) = \omega_m^2 \mu(x) \psi_m(x) \]  \hspace{1cm} (B)
Multiply (B) by \( \psi^*_n(x) \) and (A*) by \( \psi_m(x) \) and subtract:
\[
\psi_n^* \frac{d}{dx} \left[ I \psi_m \right] - \psi_m \frac{d}{dx} \left[ I \psi_n^* \right] = (\omega_n^* - \omega_m^2) \mu \psi_m \psi_n^*
\]
\[
= \frac{d}{dx} \left[ I \psi_n^* \psi_m - I \psi_m \psi_n^* \right]
\]

Now integrate from \( x_L \) to \( x_R \):
\[
(\omega_n^* - \omega_m^2) \int_{x_L}^{x_R} dx \mu(x) \psi_n^*(x) \psi_m(x) = \tau(x) \left[ \psi_n^*(x) \psi_m(x) - \psi_m(x) \psi_n^*(x) \right]_{x_L}^{x_R}
\]
\[
= 0
\]
because the term in square brackets vanishes for any of the four boundary conditions. Thus,
\[
(\omega_n^* - \omega_m^2) \langle \psi_n | \psi_m \rangle = 0
\]
where the inner product is
\[
\langle \psi | \phi \rangle = \int_{x_L}^{x_R} dx \mu(x) \psi^*(x) \phi(x)
\]
Since \( \langle \psi_n | \psi_n \rangle > 0 \), we have that \( \omega_n^2 \in \mathbb{IR} \). (Note this does not preclude \( \omega^2 < 0 \) in which case \( \omega_n \in i\mathbb{IR} \).)

When \( \omega_m^2 = \omega_n^2 \), we have \( \langle \psi_n | \psi_m \rangle = 0 \). For degenerate eigenvalues, we may invoke the Gram-Schmidt method, which orthogonolizes the eigenfunctions within a degenerate subspace. Since the SLE is linear, we may then demand orthonormality:
\[
\langle \psi_n | \psi_m \rangle = \delta_{mn}
\]
Furthermore when the functions $\mu(x)$, $\tau(x)$, $\nu(x)$ are all real, and when, in the case of mixed homogeneous BCs, $\alpha/\beta \in \mathbb{R}$, we may choose $\psi_n(x) \in \mathbb{R} \neq n$. Another aspect of the eigenspectrum, which is more difficult to prove (so we won't) is completeness:

$$\mu(x) \sum_{n=0}^{\infty} \psi_n^*(x) \psi_n(x') = \delta(x-x')$$

Note that we have labeled the eigenvalues and eigenfunctions with a discrete integer index $n \in \{0, 1, \ldots, \infty\}$, and we may demand $\omega_0^2 < \omega_1^2 < \omega_2^2 < \ldots$. Any square integrable, or $L^2$, function $f(x)$, for which $\langle f | f \rangle < \infty$, can be expanded in the eigenfunctions, viz.

$$f(x) = \sum_{n=0}^{\infty} f_n \psi_n(x)$$

$$f_n = \langle \psi_n | f \rangle = \int_{x_l}^{x_R} dx \mu(x) \psi_n^*(x) f(x)$$

NB: What is true is that $\| f - \sum_{n=0}^{\infty} f_n \psi_n \| = 0$, where $\| h \| = \langle h | h \rangle$ is the norm of $h$. Note that this does not guarantee that $\sum_{n=0}^{\infty} f_n \psi_n(x)$ converges to $f(x)$ pointwise for all $x \in [x_l, x_R]$. Rather, the convergence holds "almost everywhere", which is to say for all $x \in [x_l, x_R]$ except on a set of measure zero.
Variational method

Define the functional \[ \omega^2[\psi(x)] = \frac{N[\psi(x)]}{D[\psi(x)]} \]

with

\[ N[\psi(x)] = \frac{1}{2} \int_{x_L}^{x_R} \{ \tau(x) \psi'(x)^2 + v(x) \psi(x)^2 \} \]

\[ D[\psi(x)] = \frac{1}{2} \int_{x_L}^{x_R} d x \mu(x) \psi(x)^2 \]

Then the variation of \( \omega^2[\psi] \) is

\[ \delta \omega^2 = \frac{\delta N}{D} - \frac{N \delta D}{D^2} \]

Thus, if we demand \( \delta \omega^2 = 0 \), we have

\[ \delta N = \frac{N}{D} \delta D = \omega^2 \delta D \]

and since

\[ \frac{\delta N}{\delta \psi(x)} = - \frac{d}{d x} \left[ \tau(x) \psi'(x) \right] + v(x) \psi(x) \]

\[ \frac{\delta D}{\delta \psi(x)} = \mu(x) \psi(x) \]

we see that \( \delta \omega^2 = 0 \) yields the SLE,

\[ \frac{\delta N}{\delta \psi(x)} = - \frac{d}{d x} \left[ \tau(x) \psi'(x) \right] + v(x) \psi(x) = \omega^2 \mu(x) \psi(x) = \omega^2 \frac{\delta D}{\delta \psi(x)} \]

Note also that the variation of \( \delta N \) contains
\[ N[\psi(x)] = \frac{1}{2} \int_{x_L}^{x_R} \left\{ \tau(x) \psi'(x)^2 + \nu(x) \psi(x)^2 \right\} dx \equiv \int_{x_L}^{x_R} L_N(\psi, \psi', x) \]

\[ D[\psi(x)] = \frac{1}{2} \int_{x_L}^{x_R} \mu(x) \psi(x)^2 \equiv \int_{x_L}^{x_R} L_D(\psi, \psi', x) \]

\[ L_N(\psi, \psi', x) = \frac{1}{2} \tau(x) \psi'^2 + \frac{1}{2} \nu(x) \psi^2 \]

\[ L_D(\psi, \psi', x) = \frac{1}{2} \mu(x) \psi^2 \]

\[ \frac{\delta N}{\delta \psi(x)} = \frac{\partial L_N}{\partial \psi} - \frac{d}{dx} \frac{\partial L_N}{\partial \psi'} = \nu(x) \psi - \frac{d}{dx} \left[ \tau(x) \psi' \right] \]

\[ \frac{\delta D}{\delta \psi(x)} = \frac{\partial L_D}{\partial \psi} - \frac{d}{dx} \frac{\partial L_D}{\partial \psi'} = \mu(x) \psi \]

**Fourier analysis**: \( \psi_n(x) \rightarrow \psi_k(x) = e^{ikx} \)

\[ f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \hat{f}(k) e^{ikx} \]

\[ \hat{f}(k) = \int_{-\infty}^{\infty} dx f(x) e^{-ikx} = \langle \psi_k | f \rangle \]

\[ \langle k | k' \rangle = \int_{-\infty}^{\infty} dx e^{i(k-k')x} = 2\pi \delta(k-k') \quad \text{replaces } \delta_{kk'} \]

**Completeness**: \( \delta(x-x') = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-x')} \)
a boundary term \( \tau(x) \psi'(x) \delta \psi(x) \bigg|_{x_L}^{x_R} \), which vanishes for any of our first three classes of boundary conditions, i.e. fixed endpoints \( (\delta \psi(x_L,R) = 0) \), natural \( (\tau(x_L,R) \psi'(x_L,R) = 0) \), or periodic \( (f(x) = f(x + L) \) for \( f(x) = \psi(x) \) and \( f(x) = \tau(x) \)).

In order to accommodate the fourth class of BC, i.e. mixed homogeneous, with \( \alpha \psi(x) + \beta \psi'(x) = 0 \) for \( x = x_L, R \), if we redefine \( \omega^2 = \tilde{N}/D \), where

\[
\tilde{N}(\psi(x)) = N(\psi(x)) + \frac{\alpha}{2\beta} \left\{ \tau(x_R) \psi(x_R)^2 - \tau(x_L) \psi(x_L)^2 \right\}
\]

In fact, for all for classes of BC we can take

\[
\omega^2 [\psi(x)] = \frac{N(\psi(x))}{D(\psi(x))} = \frac{\frac{1}{2} \int_{x_L}^{x_R} d\psi(x) \left[ \psi(x) \left( -\frac{d}{dx} \frac{\tau(x)}{\psi(x)} + \nu(x) \right) \right] \psi(x)}{\frac{1}{2} \int_{x_L}^{x_R} \mu(x) \psi^2(x)}
\]

Thus, expanding \( \psi(x) = \sum_{n=0}^{\infty} C_n \psi_n(x) \), we have

\[
\omega^2 [\psi(x)] = \omega^2 (C_0, \ldots, C_\infty) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{C_n^2}{C_n^2}
\]

Then

\[
\frac{\partial \omega^2}{\partial C_j} = (\omega_j - \omega^2)C_j = 0 \quad \text{for all } j \in \{0, 1, \ldots, \infty\}
\]

Solutions:

\[
C_j = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}
\]

with \( \omega^2 = \omega_k^2 \) (\( k^{th} \) s.o.l.)
Example: string with mass point in center

\[ \mu(x) = \mu + m \delta(x - \frac{L}{2}) \; ; \; \tau(x) = \tau \; ; \; \nu(x) = 0 \]

Here \( x_L = 0 \) and \( x_R = L \). Then

\[ \omega^2[\psi] = \frac{\frac{1}{2} \tau \int_0^L \psi'(x)^2 \, dx}{\frac{1}{2} \mu \int_0^L \psi^2(x) \, dx + \frac{1}{2} m \psi^2(L/2)} \]

Now consider a trial function

\[ \psi(x) = \begin{cases} Ax^\alpha & \text{for } x \in [0, \frac{L}{2}] \\ A(L-x)^\alpha & \text{for } x \in [\frac{L}{2}, L] \end{cases} \]

Here we have a single variational parameter, \( \alpha \).

\[ \int_0^L \psi'(x)^2 \, dx = 2A^2 \int_0^{L/2} x^\alpha x^{2\alpha-2} \, dx = A^2 \cdot \frac{2\alpha^2}{2\alpha-1} \left( \frac{L}{2} \right)^{2\alpha-1} \]

\[ \int_0^L \psi^2(x) \, dx = 2A^2 \int_0^{L/2} x^{2\alpha} \, dx = A^2 \cdot \frac{2}{2\alpha+1} \left( \frac{L}{2} \right)^{2\alpha+1} \]

\[ \psi^2(L/2) = A^2 \left( \frac{L}{2} \right)^{2\alpha} \]

\[ \omega^2[\psi] = \frac{\tau \left( \frac{\alpha^2}{2\alpha-1} \right) \left( \frac{L}{2} \right)^{2\alpha-1}}{\mu \left( \frac{1}{2\alpha+1} \right) \left( \frac{L}{2} \right)^{2\alpha+1} + \frac{1}{2} m \left( \frac{L}{2} \right)^{2\alpha}} \]

\[ C = \left( \frac{\tau}{\mu} \right)^{1/2} \]

\[ M = \mu L \]
Best variational estimate ⇒ set \( \frac{d\omega^2(\alpha)}{d\alpha} = 0 \):

\[
\frac{d\omega^2}{d\alpha} = 0 \Rightarrow 4\alpha^2 - 2\alpha - 1 + (\alpha - 1)(2\alpha + 1)^2 \frac{m}{M} = 0
\]

This is a cubic equation. For \( m/M \to 0 \), we have

\[
4\alpha^2 - 2\alpha - 1 = 0 \Rightarrow \alpha = \frac{1}{4}(1 + \sqrt{5}) = 0.809.
\]

Find then \( \omega = 11.09 \frac{c}{L} \Rightarrow \omega = 3.330 \frac{c}{L} \). The exact result we know is \( \Psi_0(x) = (2/L)^{1/2} \sin \left( \pi x / L \right) \) with \( \omega_0 = \pi c / L \), and our variational frequency is about 6.00% higher. For \( m/M \to \infty \), the string's inertia is negligible.

Then \( \Psi(x) \) describes an isosceles triangle, and

\[
\text{mij} = -2\pi \cdot \left( \frac{y}{\frac{1}{2}L} \right) \Rightarrow \omega = 2 \sqrt{\frac{c}{mL}} = \frac{2}{L} \sqrt{\frac{c}{\mu}} \cdot \frac{\mu L}{m} = \frac{2c}{L} \sqrt{\frac{\mu}{m}}
\]

The variational solution yields \( \alpha = 1 \) and \( \omega^2 = \omega_0^2 \) exactly.

Note \( \alpha = 1 \) corresponds to a triangular shape.

Our example involved just one variational parameter. We could have more, e.g.

\[
\Psi(x) = Ax^\alpha + Bx^\beta \quad (0 \leq x \leq \frac{L}{2})
\]

\[
\Psi(L-x) = \Psi(x)
\]

Variation parameters: 3 \((\alpha, \beta, B/A)\)

Or: \( A = C \cos \gamma, \ B = C \sin \gamma \Rightarrow (\alpha, \beta, \gamma) \)
Another basis: \( \psi_n(x) = \left( \frac{L}{2} \right)^{1/2} \sin \left( \frac{n \pi x}{L} \right) \)

\[
\begin{align*}
\int_0^L \psi_m(x) \psi_n(x) \, dx &= \delta_{mn} \\
\int_0^L \psi'_m(x) \psi'_n(x) \, dx &= -\int_0^L \psi'_m(x) \psi''_n(x) \, dx = \left( \frac{n \pi}{L} \right)^2 \delta_{mn}
\end{align*}
\]

So take \( \psi(x) = \sum_{n=1}^\infty C_n \psi_n(x) \)

\[
\psi'' = -\left( \frac{n \pi}{L} \right)^2 \psi_n
\]

\[
\omega^2[\psi] = \frac{\frac{1}{2} \int_0^L \psi'^2(x) \, dx}{\frac{1}{2} \mu \int_0^L \psi^2(x) \, dx + \frac{1}{2} m \psi^2(\frac{1}{2} L)}
\]

\[
= \frac{\frac{1}{2} \mu \sum_{n} \left( \frac{n \pi}{L} \right)^2 C_n^2}{\frac{1}{2} \mu \sum_{j} C_j^2 + \frac{1}{L} \left( \sum_{j} C_j \sin \left( \frac{j \pi x}{2} \right) \right)^2} \left( -1 \right)^k \delta_{j,2k-1} \left[ \sum_{k=1}^\infty (-1)^k C_k \right]^2
\]

\( C_1, \ldots, C_k \) finite subset

\[
\omega^2(C_1, \ldots, C_k) = \frac{\sum_{n=1}^\infty n^2 C_n^2}{\sum_{j=1}^\infty C_j^2 + \frac{2m}{M} \left[ \sum_{k=1}^\infty (-1)^k C_k \right]^2} \left( \frac{\pi c}{L} \right)^2
\]