System: string of mass density $\mu(x)$ and tension $T(x)$. Instantaneous shape is $y(x,t)$.

Differential KE:

$$dT = \frac{1}{2} \mu(x) \left( \frac{\partial y(x,t)}{\partial t} \right)^2 \, dx$$

Differential PE (relative to $y(x,t) = \text{const.}$):

$$dU = T(x) \, dl = T(x) \left\{ \sqrt{\left( \frac{\partial y}{\partial x} \right)^2 + \left( \frac{\partial y}{\partial t} \right)^2} \right\}$$

Lagrangian density:

$$L = \frac{1}{2} \mu(x) \left( \frac{\partial y}{\partial t} \right)^2 - T(x) \left[ \sqrt{1 + \left( \frac{\partial y}{\partial x} \right)^2} - 1 \right]$$

Assuming $\left| \frac{\partial y}{\partial x} \right| \ll 1$, $L = \frac{1}{2} \mu y_t^2 - \frac{1}{2} T y_x^2 + \ldots$

Recall that for

$$S[y(x,t)] = \int_{t_a}^{t_b} \int_{x_a}^{x_b} L(y, y_t, y_x ; x, t)$$

that

$$\delta S = \int_{t_a}^{t_b} \int_{x_a}^{x_b} \left[ \frac{\partial L}{\partial y} - \frac{\partial}{\partial x} \left( \frac{\partial L}{\partial y_x} \right) - \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial y_t} \right) \right] \delta y$$

$$+ \int_{x_a}^{x_b} \left[ \frac{\partial L}{\partial y_t} \delta y \right]_{t_a}^{t_b} + \int_{t_a}^{t_b} \left[ \frac{\partial L}{\partial y_x} \delta y \right]_{x_a}^{x_b}$$

First let's consider what is necessary in order that
The boundary terms both vanish. The first boundary term vanishes when \( \delta y(x, t_a) = \delta y(x, t_b) = 0 \). The second term vanishes when \( \frac{\partial L}{\partial y_y} \delta y \) vanishes at \( x = x_a, b \) for all times \( t \). For the case \( L = \frac{1}{2} \mu y_t^2 - \frac{1}{2} \tau y_x^2 \), we have \( \delta L/\delta y_x = -\tau y_x \), thus, assuming \( \tau(x, a, b) \neq 0 \), the condition \( y_x \delta y = 0 \) at the end points means either (i) \( y_x = 0 \) or (ii) \( \delta y = 0 \) at each endpoint \( x_a, b \). We then have the EL eqn,

\[
\frac{\partial L}{\partial y} - \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial y_t} \right) - \frac{\partial}{\partial x} \left( \frac{\partial L}{\partial y_x} \right) = 0
\]

which for our case yields

\[
\frac{\partial}{\partial x} \left[ \tau(x) \frac{\partial y}{\partial x} \right] = \mu(x) \frac{\partial^2 y}{\partial t^2}
\]

This equation, plus the spatial boundary conditions, governs the dynamics of the string. The simplest case is when \( \mu(x) = \mu \) and \( \tau(x) = \tau \) are both constants, whence we obtain the Helmholtz equation,

\[
\frac{1}{c^2} y_{tt} = y_{xx} \Rightarrow \left( \frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) y(x, t) = 0
\]

with \( c = (\tau/\mu)^{1/2} \), which has units of velocity. This equation may be solved completely, and for arbitrary boundary conditions.
D'Alembert's solution

Define the variables \( u = x - ct \) and \( v = x + ct \). Then

\[
\frac{\partial}{\partial x} = \frac{\partial u}{\partial x} \frac{\partial}{\partial u} + \frac{\partial v}{\partial x} \frac{\partial}{\partial v} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}
\]

\[
\frac{\partial}{\partial t} = \frac{\partial u}{\partial t} \frac{\partial}{\partial u} + \frac{\partial v}{\partial t} \frac{\partial}{\partial v} = -c \frac{\partial}{\partial u} + c \frac{\partial}{\partial v}
\]

Therefore

\[
\frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} = (\frac{\partial}{\partial u} + \frac{\partial}{\partial v})^2 - (-\frac{\partial}{\partial u} + \frac{\partial}{\partial v})^2
\]

\[= 4 \frac{\partial^2}{\partial u \partial v} = 4 \frac{\partial}{\partial u} \frac{\partial}{\partial v}
\]

Thus

\[
\frac{\partial^2 y}{\partial u \partial v} = 0 \Rightarrow y(u,v) = f(u) + g(v)
\]

with \( f(u) \) and \( g(v) \) arbitrary functions as of yet. So:

\[
y(x,t) = f(x-ct) + g(x+ct)
\]

right-mover  left-mover

Now let's apply some initial conditions:

\[
y(x,0) = f(x) + g(x)
\]
\[
c^{-1} y_t(x,0) = -f'(x) + g'(x)
\]

Taking the spatial derivative of the first equation
yields

\[ y_x(x,0) = f'(x) + g'(x) \]

and thus we have

\[ f'(\xi) = \frac{1}{2} y_x(\xi,0) - \frac{1}{2c} y_t(\xi,0) \]

\[ g'(\xi) = \frac{1}{2} y_x(\xi,0) + \frac{1}{2c} y_t(\xi,0) \]

Now all we need to do is integrate \( \int^x_0 d\xi' \):

\[ f(\xi) = \frac{1}{2} y(\xi,0) - \frac{1}{2c} \int^\xi_0 d\xi' y_t(\xi',0) + C \]

\[ g(\xi) = \frac{1}{2} y(\xi,0) + \frac{1}{2c} \int^\xi_0 d\xi' y_t(\xi',0) - C \]

where \( C = f(0) - \frac{1}{2} y(0,0) = \frac{1}{2} y(0,0) - g(0) \). Thus,

\[ y(x,t) = \frac{1}{2} \left[ y(x-ct,0) + y(x+ct,0) \right] + \frac{1}{2c} \int^x_{x-ct} d\xi y_t(\xi,0) \]

Thus we have a solution for all initial conditions.

**Hamiltonian density**

We define the momentum density as \( g = \partial L/\partial y_t \).

The Hamiltonian density is then \( H = g y_t - L \).

Typically \( L = \frac{1}{2} \mu y_t^2 - U(y,y_x) \), hence \( g = \mu y_t \) and

\[ H = \frac{g^2}{2\mu} + U(y,y_x) \]

Expressed in terms of \( y_t \) rather than \( g \), we have
\[
y(x,t) = \frac{1}{2} \left[ y(x-ct,0) + y(x+ct,0) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} \gamma(x',0) \, dx'
\]

Suppose \( y(x,0) = \frac{1}{\pi} \frac{\gamma}{x^2 + \gamma^2} \), \( y_t(x,0) = 0 \).

Then:

\[
y(x,t) = \frac{\gamma/2\pi}{(x-ct)^2 + \gamma^2} + \frac{\gamma/2\pi}{(x+ct)^2 + \gamma^2}
\]

**Evolution**:

- \( t=0 \)
- \( t=t_1 \)
- \( t=t_2 \gg t_1 \)
the energy density,

\[ E(x,t) = \frac{1}{2} \mu y^2_t + U(y, y_x; x) \]

The equations of motion are

\[- \frac{\partial U}{\partial y} - \mu y_{tt} + \frac{\partial}{\partial x} \left( \frac{\partial U}{\partial y_x} \right) = 0\]

Now note that

\[
\frac{\partial \Sigma}{\partial t} = \mu y_t y_{tt} + \frac{\partial U}{\partial y} y_t + \frac{\partial U}{\partial y_x} y_x t
\]

\[= \mu y_t y_{tt} - \mu y_t y_{tt} + \frac{\partial}{\partial x} \left( \frac{\partial U}{\partial y_x} \right) y_t + \frac{\partial U}{\partial y_x} y_x t\]

\[= \frac{\partial}{\partial x} \left[ \frac{\partial U}{\partial y_x} y_t \right] = - \frac{\partial \Sigma}{\partial x} ; \quad J_{\Sigma} = - \frac{\partial U}{\partial y_x} y_t\]

where \( J_{\Sigma} \) is the energy current along the string.

For the case \( U = \frac{1}{2} \tau y^2_x \), we have \( J_{\Sigma} = -\tau y_x y_t \).

Note that

\[
\frac{\partial \Sigma}{\partial t} + \frac{\partial J_{\Sigma}}{\partial x} = 0 \quad ; \quad [\Sigma] = E L^{-1} \quad ; \quad [J_{\Sigma}] = E T^{-1}
\]

which is the continuity equation for energy. Thus,

\[
\frac{d}{dt} \int_{x_1}^{x_2} dx \ E(x, t) = \int_{x_1}^{x_2} dx \ \frac{\partial J_{\Sigma}(x, t)}{\partial x} = J_{\Sigma}(x_1, t) - J_{\Sigma}(x_2, t)
\]

\[\text{rate in} \quad \text{rate out}\]

\[\rightarrow J_{\Sigma}(x_1, t) \quad x_1 \] \[\rightarrow x_2 \] \[\rightarrow J_{\Sigma}(x_2, t)\]
For $U = \frac{1}{2} \tau y_x^2$ with $\mu(x) = \mu$ and $\tau(x) = \tau$ constant, writing $y(x,t) = f(x-ct) + g(x+ct)$ we find

$$E(x,t) = \tau \left[ f'(x-ct) \right]^2 + \tau \left[ g'(x+ct) \right]^2$$

$$J_E(x,t) = c \tau \left[ f'(x-ct) \right]^2 - c \tau \left[ g'(x+ct) \right]^2$$

which are each sums over right-moving and left-moving contributions.

**Example:** Klein-Gordon system $U(y, y_x) = \frac{1}{2} \tau y_x^2 + \frac{1}{2} \beta y^2$

Then $E = \frac{1}{2} \mu y_t^2 + \frac{1}{2} \tau y_x^2 + \frac{1}{2} \beta y^2$. Eqs of motion:

$$L = \frac{1}{2} \mu y_t^2 - U(y, y_x) =$$

$$- \frac{\partial U}{\partial y} - \mu y_{tt} + \frac{\partial}{\partial x} \left( \frac{\partial U}{\partial y_x} \right) = 0$$

$$- \beta y - \mu y_{tt} + \tau y_{xx} = 0$$

Thus we have

$$\left( \frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) y = m^2 y; \quad m = \sqrt{\beta/\mu}$$

This is not the Helmholtz eqn (it is the KG eqn). D'Alembert's solution does not pertain. Still,

$$J_E = - \frac{\partial U}{\partial y_x} y_t = - \tau y_x y_t$$
Momentum flux density and stress-energy tensor:

\[ \mathcal{E} = \frac{1}{2} \mu y_t^2 + \frac{1}{2} \tau y_x^2 \Rightarrow \frac{\partial \mathcal{E}}{\partial x} = \frac{\partial}{\partial t} (\mu y_t y_x) \]

Thus, with momentum current

\[ J_{\mathcal{E}} = \mathcal{E}, \quad \Pi = -\mu y_t y_x = \frac{J_{\mathcal{E}}}{c^2} \]

we may write

\[ \left( \frac{1}{c^2} \frac{\partial}{\partial t}, \frac{\partial}{\partial x} \right) \begin{pmatrix} c \mathcal{E} & -c \Pi \\ J_{\mathcal{E}} & -J_{\Pi} \end{pmatrix} = 0 \]

\[ \frac{\partial}{\partial x} T_{\mu \nu} \]

or \[ \partial_{\mu} T^{\mu \nu} = 0 \], where \( T^{\mu \nu} \) is the stress-energy tensor. Note that while \( J_{\Pi} \) and \( \Pi = \mu y_t y_x \) have the same dimensions, \( J_{\Pi} \) is the momentum density along the string while \( \Pi \) is the momentum density transverse to the string. General result:

\[ T^{\mu \nu} = \frac{\partial L}{\partial (\partial_\nu y)} \partial_\mu y - \delta^{\mu \nu} L \]

This satisfies \[ \partial_{\mu} T^{\mu \nu} = 0 \] for all \( \nu \).

Electromagnetism:

\[ \mathcal{E} = \frac{1}{8\pi} \left( \mathcal{E}^2 + \mathcal{B}^2 \right) \Rightarrow \]

\[ \frac{\partial \mathcal{E}}{\partial t} = \frac{1}{4\pi} \left( \mathcal{E} \cdot \frac{\partial \mathcal{E}}{\partial t} + \mathcal{B} \cdot \frac{\partial \mathcal{B}}{\partial t} \right) \]

\[ = \frac{1}{4\pi} \mathcal{E} \cdot (c \hat{\nabla} \times \hat{\nabla} - 4\pi \hat{J}) + \frac{1}{4\pi} \hat{B} \cdot (-c \hat{\nabla} \times \hat{E}) \]
\[ = - \vec{E} \cdot \vec{J} - \nabla \cdot \vec{S} \]

where \( \vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B} \) = Poynting vector. The stress-energy tensor is

\[
\mathbf{T}^\mu_{\nu} = \begin{pmatrix}
\mathcal{E} & -c^{-1}S_x & -c^{-1}S_y & -c^{-1}S_z \\
-c^{-1}S_x & \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\
-c^{-1}S_y & \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\
-c^{-1}S_z & \sigma_{zx} & \sigma_{zy} & \sigma_{zz}
\end{pmatrix}
\]

with

\[
\sigma_{ij} = \frac{1}{4\pi} \left\{ -\mathcal{E}E_j - B_iB_j + \frac{1}{2} (\vec{E}^2 + \vec{B}^2) \delta_{ij} \right\}
\]

which is the Maxwell stress tensor. Now

\[ \partial_\mu \mathbf{T}^\mu_{\nu} = 0 \quad \text{and} \quad \partial_\mu = \left( \frac{1}{c} \partial_t, \nabla \right) \]

**Reflection at an interface**

Consider a semi-infinite string with \( x \in [0, \infty) \) and with \( y(x, t) = 0 \forall t \). We write

\[ y(x, t) = f(x - ct) + g(x + ct) \]

and impose the boundary condition at \( x = 0 \):

\[ f(-ct) + g(ct) = 0 \Rightarrow f(\xi) = -g(-\xi) \forall \xi \]

Therefore, we have

\[ y(x, t) = g(ct + x) - g(ct - x) \]

\[ f(x - ct) \]
This is the general solution. Now suppose \( g(s) \) resembles a pulse localized around \( s \approx 0 \). In the distant past, \( t \to -\infty \Rightarrow ct-x \to -\infty \). Hence no contribution from right mover.

How about the left mover? Set \( ct+x \approx 0 \Rightarrow x \approx -ct \in [0, \infty) \). I.e. incoming left mover at \( x \approx -ct \). For \( t \to +\infty \), \( ct+x \to +\infty \Rightarrow \) left mover is gone. \( ct-x \approx 0 \Rightarrow x \approx ct \in [0, \infty] \) i.e. outgoing right mover at \( x \approx ct \). Sketch:

Suppose instead \( y_x(0,t) = 0 \ \forall t \).

From \( SS = \ldots - \frac{\partial^2}{\partial y_x} \delta y \bigg|_0 \) must vanish free

\[ \frac{\partial L}{\partial y_x} = -\tau y_x \Rightarrow y_x(0,t) = 0 \ \forall t \]
Shape of string:

\[ y(x,t) = f(x-ct) + g(x+ct) \]
\[ y_x(x,0) = f'(-ct) + g'(ct) \]

Thus \( f'(\xi) = -g'(-\xi) \). Integrate to get

\[ f(\xi) = g(-\xi) \]

So the shape is

\[ y(x,t) = g(ct+x) + g(ct-x) \]
\[ y_x(x,t) = g'(ct+x) - g'(ct-x) \]
\[ = 0 \text{ when } x = 0 \]

• Mass point on a string:

\[ x=0 \]

\[ x<0 : y(x,t) = f(ct-x) + g(ct+x) \]
\[ x>0 : y(x,t) = h(ct-x) \]

Interpretation: \( f = \text{incident wave} \)
\( g = \text{reflected wave} \)
\( h = \text{transmitted wave} \)
Newton's law for mass at $x=0$:

$$m \ddot{y}(0, t) = \gamma y'(0^+, t) - \gamma y'(0^-, t)$$

Discontinuous $y'(0, t) = y_x(0, t) \Rightarrow$ acceleration of $m$.

Furthermore:

$$y'(0^+, t) = -f'(ct) + g'(ct)$$
$$y'(0^-, t) = h'(ct)$$

Continuity $\Rightarrow y(0^-, t) = y(0^+, t) \Rightarrow$

$$h(ct) = f(ct) + g(ct)$$

Let $\xi = ct \Rightarrow$

$$h(\xi) = f(\xi) + g(\xi)$$

$$f''(\xi) + g''(\xi) = -\frac{2\sqrt{\gamma}}{mc^2} g'(\xi)$$

From these, get $g(\xi)$ and $h(\xi)$ in terms of $f(\xi)$.

Fourier transforms:

$$f(\xi) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \hat{f}(k) e^{ik\xi} \quad , \quad \hat{f}(k) = \int_{-\infty}^{\infty} dx f(x) e^{-ikx}$$

Derivatives wrt $\xi$ replaced by $ik \times \hat{f}(k)$ etc.

Then we have

$$(-k^2 + iQk) \hat{g}(k) = k^2 \hat{f}(k)$$
\[ \hat{h}(k) = \hat{f}(k) + \hat{g}(k) \]

with \( Q = \frac{2\pi}{mc^2} = \frac{2\mu}{m} \); \( [Q] = L^{-1} \).

Solution:

\[ \hat{g}(k) = \hat{r}(k) \hat{f}(k), \quad \hat{h}(k) = \hat{t}(k) \hat{f}(k) \]

with

\[ \hat{r}(k) = -\frac{k}{k - i\alpha}, \quad \hat{t}(k) = -\frac{iQ}{k - i\alpha} \]

Note \( t = 1 + r \) since \( h = f + g \).

Shape of transmitted wave:

\[ h(\xi) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \hat{t}(k) \hat{f}(k) \]

\[ = \int_{-\infty}^{\infty} d\xi' t(\xi - \xi') f(\xi') \]

\[ t(\xi - \xi') = \int_{-\infty}^{\infty} \frac{dk}{2\pi} t(k) e^{ik(\xi - \xi')} \]

and for

\[ \hat{t}(k) = -\frac{iQ}{n - i\alpha} \]

find

\[ t(\xi - \xi') = Q e^{-Q(\xi - \xi')} \Theta(\xi - \xi') \]