Lecture 18 (Dec. 2)

• Canonical perturbation theory

Suppose

\[ H(\tilde{q}, \tilde{p}, t) = H_0(\tilde{q}, \tilde{p}, t) + \epsilon H_1(\tilde{q}, \tilde{p}, t) \]

where \(|\epsilon| < 1\). Let's implement a type-II CT generated by \( S(\tilde{q}, \tilde{p}, t) \) (not intended to signify Hamilton's principal function):

\[ \tilde{H}(\tilde{q}, \tilde{p}, t) = H(\tilde{q}, \tilde{p}, t) + \frac{\partial}{\partial t} S(\tilde{q}, \tilde{p}, t) \]

Expand everything in sight in powers of \( \epsilon \):

\[ q_\sigma = Q_\sigma + \epsilon q_{1, \sigma} + \epsilon^2 q_{2, \sigma} + \ldots \]
\[ p_\sigma = P_\sigma + \epsilon p_{1, \sigma} + \epsilon^2 p_{2, \sigma} + \ldots \]
\[ \tilde{H} = \tilde{H}_0 + \epsilon \tilde{H}_1 + \epsilon^2 \tilde{H}_2 + \ldots \]
\[ S = q_\sigma P_\sigma + \epsilon S_1 + \epsilon^2 S_2 + \ldots \]
\[ \text{identity CT} \]

Then

\[ Q_\sigma = \frac{\partial S}{\partial P_\sigma} = q_\sigma + \epsilon \frac{\partial S_1}{\partial P_\sigma} + \epsilon^2 \frac{\partial S_2}{\partial P_\sigma} + \ldots \]

\[ = q_\sigma + \left( q_{1, \sigma} + \frac{\partial S_1}{\partial P_\sigma} \right) \epsilon + \left( q_{2, \sigma} + \frac{\partial S_2}{\partial P_\sigma} \right) \epsilon^2 + \ldots \]
We also have
\[ p_\sigma = \frac{\partial S}{\partial q_\sigma} = P_\sigma + \varepsilon \frac{\partial S_1}{\partial q_\sigma} + \varepsilon^2 \frac{\partial S_2}{\partial q_\sigma} + \ldots \]
\[ = P_\sigma + \varepsilon P_{1,\sigma} + \varepsilon^2 P_{2,\sigma} + \ldots \]

Thus we conclude, order by order in \( \varepsilon_1 \),
\[ q_{k,\sigma} = -\frac{\partial S_k}{\partial P_\sigma}, \quad p_{k,\sigma} = \frac{\partial S_k}{\partial q_\sigma} \]

Next, expand the Hamiltonian:
\[ \tilde{H}(\vec{q}, \vec{p}, t) = H_0(\vec{q}, \vec{p}, t) + \varepsilon H_1(\vec{q}, \vec{p}, t) + \frac{\partial S}{\partial t} \]
\[ = H_0(\vec{q}, \vec{p}, t) + \frac{\partial H_0}{\partial q_\sigma} (q_\sigma - Q_\sigma) + \frac{\partial H_0}{\partial P_\sigma} (P_\sigma - P_\sigma) + \ldots \]
\[ + \varepsilon H_1(\vec{q}, \vec{p}, t) + \varepsilon \frac{\partial}{\partial t} S_0(\vec{q}, \vec{p}, t) + O(\varepsilon^2) \]
\[ = H_0(\vec{q}, \vec{p}, t) + \left( -\frac{\partial H_0}{\partial q_\sigma} \frac{\partial S_1}{\partial P_\sigma} + \frac{\partial H_0}{\partial P_\sigma} \frac{\partial S_1}{\partial q_\sigma} + \frac{\partial S_1}{\partial t} + H_1 \right) \varepsilon + O(\varepsilon^2) \]

Notice we are writing \( q_\sigma = Q_\sigma + (q_\sigma - Q_\sigma) = Q_\sigma - \varepsilon \frac{\partial S_1}{\partial P_\sigma} + \ldots \)

So, e.g.
\[ S_1(\vec{q}, \vec{p}, t) = S_1(\vec{q}, \vec{p}, t) + (q_\sigma - Q_\sigma) \frac{\partial S_1}{\partial q_\sigma} + \ldots \]
\[ = S_1(\vec{q}, \vec{p}, t) - \frac{\partial S_1(\vec{q}, \vec{p}, t)}{\partial P_\sigma} \frac{\partial S_1(\vec{q}, \vec{p}, t)}{\partial q_\sigma} \varepsilon + O(\varepsilon^2) \]

Thus, we have
\[ \widetilde{H}(\tilde{Q}, \tilde{P}, t) = H_0(\tilde{Q}, \tilde{P}, t) + (H_1 + \{S_1, H_0\} + \frac{\partial S_1}{\partial t}) \varepsilon + O(\varepsilon^2) \]
\[ = \widetilde{H}_0(\tilde{Q}, \tilde{P}, t) + \varepsilon \tilde{H}_1(\tilde{Q}, \tilde{P}, t) + O(\varepsilon^2) \]

We therefore conclude
\[ \tilde{H}_0(\tilde{Q}, \tilde{P}, t) = H_0(\tilde{Q}, \tilde{P}, t) \]
\[ \tilde{H}_1(\tilde{Q}, \tilde{P}, t) = \left[ H_1 + \{S_1, H_0\} + \frac{\partial S_1}{\partial t} \right] \tilde{Q}, \tilde{P}, t \]

We are left with a single equation in two unknowns, i.e. \( \tilde{H}_1 \) and \( S_1 \). The problem is underdetermined. We could at this point demand \( \tilde{H}_1 = 0 \), but this is just one of many possible choices. Similar story in QM:

\[ i\hbar \frac{\partial}{\partial t} |\psi\rangle = (\hat{H}_0 + \varepsilon \hat{H}_1) |\psi\rangle \]

Now define \( |\psi\rangle = e^{i\hat{S}/\hbar} |\chi\rangle \) with \( \hat{S} = \varepsilon \hat{S}_1 + \varepsilon^2 \hat{S}_2 + \ldots \). Then find

\[ i\hbar \frac{\partial}{\partial t} |\chi\rangle = \hat{H}_0 |\chi\rangle + \varepsilon \left( \hat{H}_1 + \frac{1}{i\hbar} [\hat{S}_1, \hat{H}_0] + \frac{\partial \hat{S}_1}{\partial t} \right) |\chi\rangle + \ldots \]
\[ = \hat{H} |\chi\rangle \]

Typically we choose \( \hat{S}_1 \) such that the \( O(\varepsilon) \) term vanish. But this isn't the only possible choice. (Note here the correspondence \( \{A, B\} \leftrightarrow \frac{1}{i\hbar} [\hat{A}, \hat{B}] \).)
• **CPT for n = 1 systems**

Here we demonstrate the implementation of CPT in a general n = 1 system. We will need to deal with resonances when n > 1, which we discuss later on.

We assume \( H(q,p) = H_o(q,p) + \epsilon H_1(q,p) \) is time-independent. Let \((\phi_0, J_0)\) be AAV for \(H_o\), so that

\[
\tilde{H}_o(J_0) = H_o(q(\phi_0, J_0), p(\phi_0, J_0))
\]

We define

\[
\tilde{H}_1(\phi_0, J_0) = H_1(q(\phi_0, J_0), p(\phi_0, J_0))
\]

We assume that \( \tilde{H} = \tilde{H}_o + \epsilon \tilde{H}_1 \) is integrable, which for n = 1 is indeed always the case. [Reminder: \( H(q,p) = E \) means all motion takes place on the one-dimensional level sets of \( H(q,p) \).]

Thus there must be a CT taking \((\phi_0, J_0) \rightarrow (\phi, J)\), where

\[
\tilde{H}(\phi_0(\phi, J), J_0(\phi, J)) = E(J)
\]

We solve by a type-\(\Pi\) CT:

\[
S(\phi_0, J) = \phi_0 J + \epsilon S_1(\phi_0, J) + \epsilon^2 S_2(\phi_0, J) + \ldots
\]

Under CT

Then

\[
J_0 = \frac{\partial S}{\partial \phi_0} = J + \epsilon \frac{\partial S_1}{\partial \phi_0} + \epsilon^2 \frac{\partial S_2}{\partial \phi_0} + \ldots
\]

\[
\phi = \frac{\partial S}{\partial J} = \phi_0 + \epsilon \frac{\partial S_1}{\partial J} + \epsilon^2 \frac{\partial S_2}{\partial J} + \ldots
\]
We also write
\[ E(J) = E_0(J) + \varepsilon E_1(J) + \varepsilon^2 E_2(J) + \ldots \]
\[ = \tilde{H}_0(J_0) + \varepsilon \tilde{H}_1(\phi_0, J_0) \quad \text{(no higher order terms)} \]

Now we expand \( \tilde{H}(\phi_0, J_0) = \tilde{H}(\phi_0, J + (J_0 - J)) \) in powers of \( (J_0 - J) \):
\[
\tilde{H}(\phi_0, J_0) = \tilde{H}_0(J) + \frac{\partial \tilde{H}_0}{\partial J} (J_0 - J) + \frac{1}{2} \frac{\partial^2 \tilde{H}_0}{\partial J^2} (J_0 - J)^2
+ \varepsilon \tilde{H}_1(\phi_0, J) + \varepsilon \frac{\partial \tilde{H}_1}{\partial J} \bigg|_{\phi_0} (J_0 - J) + \ldots
\]

Substitute
\[
J_0 - J = \varepsilon \frac{\partial S_1}{\partial \phi_0} + \varepsilon^2 \frac{\partial S_2}{\partial \phi_0} + \ldots
\]
and collect terms to obtain
\[
\tilde{H}(\phi_0, J_0) = \tilde{H}_0(J) + \left( \tilde{H}_1 + \frac{\partial \tilde{H}_0}{\partial J} \frac{\partial S_1}{\partial \phi_0} \right) \varepsilon
+ \left( \frac{\partial \tilde{H}_0}{\partial J} \frac{\partial S_2}{\partial \phi_0} + \frac{1}{2} \frac{\partial^2 \tilde{H}_0}{\partial J^2} \left( \frac{\partial S_1}{\partial \phi_0} \right)^2 + \frac{\partial \tilde{H}_1}{\partial J} \frac{\partial S_1}{\partial \phi_0} \right) \varepsilon^2 + \ldots
\]
where all terms on the RHS are expressed in terms of \( \phi_0 \) and \( J \). We may now read off:

(0) \[ E_0(J) = \tilde{H}_0(J) \]

(1) \[ E_1(J) = \tilde{H}_1(\phi_0, J) + \frac{\partial \tilde{H}_0}{\partial J} \frac{\partial S_1(\phi_0, J)}{\partial \phi_0} \]

(2) \[ E_2(J) = \frac{\partial \tilde{H}_0}{\partial J} \frac{\partial S_2(\phi_0, J)}{\partial \phi_0} + \frac{1}{2} \frac{\partial^2 \tilde{H}_0}{\partial J^2} \left( \frac{\partial S_1(\phi_0, J)}{\partial \phi_0} \right)^2 + \frac{\partial \tilde{H}_1(\phi_0, J)}{\partial J} \frac{\partial S_1(\phi_0, J)}{\partial \phi_0} \]
But the RHS should be independent of $\phi_0$! How can this be? We use the freedom in the functions $S_k(\phi_0, J)$ to make it so. Let’s see just how this works.

Each of the expressions on the RHSs must be equal to its average over $\phi_0$ if it is to be independent of $\phi_0$:

$$\langle f(\phi_0) \rangle = \int_0^{2\pi} \frac{d\phi_0}{2\pi} f(\phi_0)$$

The averages $\langle \text{RHS}(\phi_0, J) \rangle$ are taken at fixed $J$ and not at fixed $J_0$. We must have that

$$S_k(\phi_0, J) = \sum_{l=\infty}^{\infty} S_{k, l}(J) e^{i l \phi_0}$$

Thus

$$\langle \frac{\partial S_k}{\partial \phi_0} \rangle = \frac{1}{2\pi} \left\{ S_k(2\pi, J) - S_k(0, J) \right\} = 0$$

Now let’s implement this in our hierarchy. Consider the level (1) equation,

$$E_1(J) = \tilde{H}_1(\phi_0, J) + \frac{\partial \tilde{H}_0}{\partial J} \frac{\partial S_1(\phi_0, J)}{\partial \phi_0} \frac{1}{V_0(J)}$$

Taking the average,

$$E_1(J) = \langle \tilde{H}_1(\phi_0, J) \rangle + \frac{\partial \tilde{H}_0}{\partial J} \langle \frac{\partial S_1(\phi_0, J)}{\partial \phi_0} \rangle$$

$$= \langle \tilde{H}_1 \rangle$$

this vanishes
Thus,

\[ \langle \tilde{H}_i \rangle = \tilde{H}_i + \nu_0(J) \frac{\partial S_1}{\partial \phi_0} \Rightarrow \frac{\partial S_1(\phi_0, J)}{\partial \phi_0} = \frac{\langle \tilde{H}_i \rangle - \tilde{H}_i(\phi_0, J)}{\nu_0(J)} \]

If we Fourier decompose

\[ \tilde{H}_i(\phi_0, J) = \sum_{l=-\infty}^{\infty} \tilde{H}_{i,l}(J) e^{il \phi_0} \]

then we obtain

\[ l \neq 0: il S_{1,l}(J) = \tilde{H}_{1,l}(J) \Rightarrow S_{1,l}(J) = -\frac{i}{l} \tilde{H}_{1,l}(J) \]

We are free to set \( S_{1,0}(J) = 0 \) (why?).

Now that we've got the hang of the logic here, let's go to second order:

\[ E_2(J) = \frac{\partial \tilde{H}_0}{\partial J} \frac{\partial S_2(\phi_0, J)}{\partial \phi_0} + \frac{1}{2} \frac{\partial^2 H_0}{\partial J^2} \left( \frac{\partial S_1(\phi_0, J)}{\partial \phi_0} \right)^2 + \frac{\partial \tilde{H}_1(\phi_0, J)}{\partial J} \frac{\partial S_1(\phi_0, J)}{\partial \phi_0} + \frac{\partial \tilde{H}_1(\phi_0, J)}{\partial J} \frac{\partial S_1(\phi_0, J)}{\partial \phi_0} \]

Taking the average,

\[ E_2(J) = \frac{1}{2} \frac{\partial \nu_0}{\partial J} \left( \frac{\langle \tilde{H}_1 \rangle - \tilde{H}_1}{\nu_0} \right)^2 + \left. \frac{\partial \tilde{H}_1}{\partial J} \left( \frac{\langle \tilde{H}_1 \rangle - \tilde{H}_1}{\nu_0} \right) \right\rangle \]

which yields, after some work,

\[ \frac{\partial S_2}{\partial \phi_0} = \frac{1}{\nu_0^2} \left\{ \left. \frac{\partial \tilde{H}_1}{\partial J} \langle \tilde{H}_1 \rangle - \left. \frac{\partial \tilde{H}_1}{\partial J} \tilde{H}_1 \right\rangle - \frac{\partial \tilde{H}_1}{\partial J} \langle \tilde{H}_1 \rangle + \frac{\partial \tilde{H}_1}{\partial J} \tilde{H}_1 \right\} \right\} + \frac{1}{2} \frac{\partial \nu_0}{\partial J} \left( \langle \tilde{H}_1^2 \rangle - 2\langle \tilde{H}_1 \rangle^2 + 2\langle \tilde{H}_1 \rangle \langle \tilde{H}_1 - \tilde{H}_1^2 \rangle \right) \]
and the energy to second order is

\[
E(J) = \tilde{H}_0 + \epsilon \langle \tilde{H}_1 \rangle + \frac{\epsilon^2}{\nu_0} \left\{ \langle \frac{\partial \tilde{H}_1}{\partial J} \rangle \langle \tilde{H}_1 \rangle - \langle \frac{\partial \tilde{H}_1}{\partial J} \tilde{H}_1 \rangle \right\} + \frac{1}{2} \frac{\partial^2 \nu_0}{\partial J} \left( \langle \tilde{H}_1^2 \rangle - \langle \tilde{H}_1 \rangle \langle \tilde{H}_1 \rangle \right) + \mathcal{O}(\epsilon^3)
\]

Note that we don't need \( S(\phi, J) \) to obtain \( E(J) \), though of course we do need it to obtain \( (\phi, J_0) \) in terms of \( (\phi, J) \). The perturbed frequencies are \( \nu(J) = \partial E/\partial J \). For the full motion, we need

\[
(\phi, J) \rightarrow (\phi, J_0) \rightarrow (q, p)
\]

- **Example**: quartic oscillator

The Hamiltonian is

\[
H(q, p) = \frac{p^2}{2m} + \frac{1}{2} m \nu_0^2 q^2 + \frac{\alpha}{4} \epsilon q^4
\]

Recall the AAV for the SHO:

- \( J_0 = \frac{p^2}{2m \nu_0} + \frac{1}{2} m \nu_0 q^2 = \frac{H_0}{\nu_0} \)
- \( \phi_0 = \tan^{-1} \left( \frac{m \nu_0 q}{p} \right) \)
- \( q = \left( \frac{2J_0}{m \nu_0} \right)^{1/2} \sin \phi_0 \)
- \( p = \sqrt{2J_0 m \nu_0} \cos \phi_0 \)
Thus, we have

\[ \tilde{H}(\phi_0, J_0) = \nu_0 J_0 + \frac{\alpha}{4} \epsilon \left( \frac{2J_0}{m^2 \nu_0^2} \sin \phi_0 \right)^4 \]

\[ = \nu_0 J_0 + \epsilon \left( \frac{\alpha}{m^2 \nu_0^2} \right) J_0^2 \sin^4 \phi_0 \]

\[ \tilde{H}_0(J_0) \quad \tilde{H}_1(\phi_0, J_0) \]

We therefore have

\[ E_1(J) = \langle \tilde{H}_1(\phi_0, J) \rangle \]

\[ = \frac{\alpha J^2}{m^2 \nu_0^2} \int_0^{2\pi} \frac{d\phi_0}{2\pi} \sin^4 \phi_0 = \frac{3\alpha J^2}{8m^2 \nu_0^2} \]

The frequency, to order \( \epsilon \), is then

\[ \nu(J) = \frac{\partial}{\partial J} (E_0 + \epsilon E_1) = \nu_0 + \frac{3\epsilon \alpha J}{4m^2 \nu_0^2} + O(\epsilon^2) \]

To this order, we may replace \( J \) above by \( J_0 = \frac{1}{2} m \nu_0 A^2 \), where \( A \) = amplitude of oscillations. Thus, pendulum

\[ \nu(A) = \nu_0 + \frac{3\epsilon \alpha A^2}{8m^2 \nu_0^2} + O(\epsilon^2) \]

Only for the linear oscillator \( \ddot{q} = -\nu_0^2 q \) is the oscillation frequency independent of the amplitude.

Next, let's work through the CT \( (\phi_0, J_0) \to (\phi, J) \).
We have
\[ \nu_v \frac{\partial S_1}{\partial \phi_0} = \frac{\alpha J^2}{m^2 v_0^2} \left( \frac{3}{8} - \sin^4 \phi_0 \right) \]

\[ \Rightarrow S_1(\phi_0, J) = \frac{\alpha J^2}{8 m^2 v_0^3} \left( 3 + 2 \sin^2 \phi_0 \right) \sin \phi_0 \cos \phi_0 \]

and
\[ \phi = \phi_0 + \epsilon \frac{\partial S_1}{\partial J} + \mathcal{O}(\epsilon^2) \]

\[ = \phi_0 + \frac{\epsilon \alpha J^2}{4 m^2 v_0^3} \left( 3 + 2 \sin^2 \phi_0 \right) \sin \phi_0 \cos \phi_0 + \mathcal{O}(\epsilon^2) \]

\[ J_0 = J + \epsilon \frac{\partial S_1}{\partial \phi_0} \]

\[ = J + \frac{\epsilon \alpha J^2}{8 m^2 v_0^3} \left( 4 \cos(2\phi_0) - \cos(4\phi_0) \right) + \mathcal{O}(\epsilon^2) \]

To lowest nontrivial order we may invert to obtain

\[ J = J_0 - \frac{\epsilon \alpha J_0^2}{8 m^2 v_0^3} \left( 4 \cos(2\phi_0) - \cos(4\phi_0) \right) + \mathcal{O}(\epsilon^2) \]

With \( q = (2J_0/mv_0)^{1/2} \sin \phi_0 \) and \( p = (2m v_0 J_0)^{1/2} \cos \phi_0 \), we can obtain \((q,p)\) in terms of \((\phi,J)\).

- \( n > 1 \): degeneracies and resonances

Generalizing the CPT formalism to \( n > 1 \) is straightforward. We have \( S = S(\phi_0, J) \), so with \( \alpha \in \{1, \ldots, n\} \),

\[ S = S(\phi_0, J) \]
\[
J_0^\alpha = \frac{\partial S}{\partial \phi_0^\alpha} = J^\alpha + \varepsilon \frac{\partial S_1}{\partial \phi_0^\alpha} + \varepsilon^2 \frac{\partial S_2}{\partial \phi_0^\alpha} + \ldots
\]

\[
\phi^\alpha = \frac{\partial S}{\partial J^\alpha} = \phi_0^\alpha + \varepsilon \frac{\partial S_1}{\partial J^\alpha} + \varepsilon^2 \frac{\partial S_2}{\partial J^\alpha} + \ldots
\]

and

\[
E_0(\vec{J}) = \tilde{H}_0(\vec{J})
\]

\[
E_1(\vec{J}) = \tilde{H}_1(\vec{\phi}_0, \vec{J}) + \nu_0^\alpha(\vec{J}) \frac{\partial S_1(\vec{\phi}_0, \vec{J})}{\partial \phi_0^\alpha}
\]

\[
E_2(\vec{J}) = \nu_0^\alpha(\vec{J}) \frac{\partial S_2(\vec{\phi}_0, \vec{J})}{\partial \phi_0^\alpha} + \frac{1}{2} \nu_0^\alpha(\vec{J}) \frac{\partial S_1(\vec{\phi}_0, \vec{J})}{\partial J^\beta} \frac{\partial S_1(\vec{\phi}_0, \vec{J})}{\partial J^\alpha} + \frac{\partial \tilde{H}_1(\vec{\phi}_0, \vec{J})}{\partial J^\alpha} \frac{\partial S_1(\vec{\phi}_0, \vec{J})}{\partial J^\alpha}
\]

where \( \nu_0^\alpha(\vec{J}) = \partial \tilde{H}_0(\vec{J}) / \partial J^\alpha \). Now we average:

\[
\langle f(\vec{\phi}_0, \vec{J}) \rangle = \int_0^{2\pi} \frac{d\phi_0^1}{2\pi} \ldots \int_0^{2\pi} \frac{d\phi_0^n}{2\pi} f(\vec{\phi}_0, \vec{J})
\]

The equation for \( S_1(\vec{\phi}_0, \vec{J}) \) is

\[
\nu_0^\alpha \frac{\partial S_1(\vec{\phi}_0, \vec{J})}{\partial \phi_0^\alpha} = \langle \tilde{H}_1(\vec{\phi}_0, \vec{J}) \rangle - \tilde{H}_1(\vec{\phi}_0, \vec{J})
\]

\[
= - \sum_{l \in \mathbb{Z}^n} V_{l}^i(\vec{\phi}) e^{i \vec{\ell} \cdot \vec{\phi}_0}
\]

where \( V_{l}^i(\vec{\phi}) = \tilde{H}_{1, l}(\vec{J}) \), i.e. \( \tilde{H}_1(\vec{\phi}_0, \vec{J}) = \sum_{l} V_{l}^i(\vec{\phi}) e^{i \vec{\ell} \cdot \vec{\phi}_0} \)
The prime on the sum means $\hat{\ell} = (0, 0, \ldots, 0)$ is excluded. The solution is

$$S_1(\phi_0, \vec{J}) = -i \sum_{\ell \in \mathbb{Z}^n} \frac{V_i(\vec{J})}{\ell \cdot \vec{J}} e^{i \ell \cdot \phi_0}$$

When the resonance condition

$$\ell \cdot \vec{J}_0(\vec{J}) = 0$$

pertsains (with $\ell \neq 0$), the denominator vanishes and CPT breaks down. One can always find such an $\ell$ whenever two or more of the frequencies $\nu_0^\alpha(\vec{J})$ have a rational ratio. Suppose for example that $\nu_0^2(\vec{J})/\nu_0^1(\vec{J}) = r/s$ with $r, s \in \mathbb{Z}$ relatively prime. Then $r \nu_0' = s \nu_0^2$ and with $\ell = (r, -s, 0, \ldots, 0)$, we have $\ell \cdot \vec{J}_0 = 0$. Even if all the frequency ratios are irrational, for large enough $|\ell|$ we can make $|\ell \cdot \vec{J}_0|$ as small (but finite) as we please. In §15.9, we'll see how any given resonance may be removed canonically. We're just looking at things the wrong way at the moment.