Consider a two-body central force,

\[ U(r) = -\frac{k}{r} + \frac{4kb^{1/2}}{3r^{3/2}}, \]

where \( k > 0 \) and where \( b > 0 \) has dimensions of length.

(a) Sketch the effective potential \( U_{\text{eff}}(r) \). Prove that there is only one circular orbit.

(b) Find the radius \( r_0 \) of the circular orbit. Is this orbit stable?

(c) For small perturbations about the circular orbit, find the frequency of the perturbation. You may express your answer in terms of quantities such as \( r_0 \) itself.

(d) Find the geometric equation of the shape of the perturbed orbit. Is the perturbed orbit closed?

Note: I found it useful to define a dimensionless length \( s \equiv r/a \) with \( a \equiv \ell^2/\mu k \).

Solution:

(a) We have

\[ U_{\text{eff}}(r) = \frac{\ell^2}{2\mu r^2} - \frac{k}{r} + \frac{4kb^{1/2}}{3r^{3/2}} \]

\[ = \frac{k}{a} \left\{ \frac{1}{2s^2} + \frac{4\zeta}{3s^{3/2}} - \frac{1}{s} \right\} \equiv \frac{k}{a} u(s), \]

where \( a \equiv \ell^2/\mu k \), \( \zeta \equiv (b/a)^{1/2} \), and \( s \equiv r/a \). Setting \( U'_{\text{eff}}(r) = 0 \) then entails \( u'(s) = 0 \), where

\[ u'(s) = -\frac{1}{s^3} - \frac{2\zeta}{s^{5/2}} + \frac{1}{s^2} \]

\[ = \frac{1}{s^2} \left( s - 2\zeta s^{1/2} - 1 \right). \]

Thus \( u'(s) = 0 \) yields a quadratic equation in \( s^{1/2} \), the solution of which is \( s = s_0 \), with

\[ s_0^{1/2} = \zeta \pm \sqrt{\zeta^2 + 1}. \]

Since \( s_0^{1/2} > 0 \), we must take the + root, and there is only one circular orbit.

(b) From the relation \( r = as \), we now have

\[ r_0 = \frac{\ell^2}{\mu k} \left( \zeta + \sqrt{\zeta^2 + 1} \right)^2. \]
Note that for the Kepler problem, $\zeta = 0$ and thus $r_0 = \frac{a}{\mu k}$, as we have previously derived.

Clearly the extremum at $s = s_0$ is a global minimum for $u(s)$. As $s \to 0$ we have $u(s) \to \infty$ since the angular momentum barrier dominates. As $s \to \infty$, the Kepler $-1/s$ part of the potential dominates and $u(s)$ approaches zero from below. By the intermediate value theorem, then, all local minima occur for some finite value of $s$, and we have shown that there is a unique extremum, which must correspond to a global minimum to the effective potential. Therefore the circular orbit is stable.

(c) We have

$$\ddot{\eta} = -\frac{1}{m} \frac{d^2}{dr_0^2} U_{\text{eff}}(r_0) \eta = -\frac{k}{a^3} u''(s_0) \eta .$$

Now

$$u''(s) = \frac{3}{s^4} + \frac{5\zeta}{s^{7/2}} - \frac{2}{s^3}$$

$$= -\frac{2}{s^4} (s - \frac{5}{2} \zeta s^{1/2} - \frac{3}{2})$$

and since $s_0 - 2\zeta s_0^{1/2} - 1 = 0$, we have

$$u''(s_0) = \frac{1}{s_0^4} \left(\zeta s_0^{1/2} + 1\right) .$$

This tells us that the oscillation frequency of the perturbed circular orbit is

$$\omega^2 = \frac{k}{\mu a^3} \cdot \frac{\zeta s_0^{1/2} + 1}{s_0^4} .$$

(d) Since $\ell = \mu r^2 \dot{\phi}$ is constant, for a circular orbit we have

$$\frac{d}{d\phi} = \frac{\mu r_0^2}{\ell} \frac{d}{dt} .$$

For almost circular orbits, where $r = r_0 + \eta$, we may apply this to any function which is proportional to the perturbation $\eta$ itself, since any differences arising by writing the formally correct expression,

$$\frac{d}{d\phi} = \frac{\mu (r_0 + \eta)^2}{\ell} \frac{d}{dt}$$

would be higher order in $\eta$. Thus we may write

$$\frac{d^2}{d\phi^2} = -\left(\frac{\mu r_0^2}{\ell}\right)^2 \omega^2 \eta$$

$$= -\frac{k}{\mu a^3} \left(\frac{\mu a^2 s_0^2}{\ell}\right)^2 \left(\frac{\zeta s_0^{1/2}}{s_0^2} + 1\right) \eta = -\left(1 + \zeta s_0^{1/2}\right) \eta .$$
Thus, $\eta'' = -\beta^2 \phi$ where

$$\beta = \sqrt{1 + \zeta s_0^{1/2}} = \sqrt{1 + \zeta^2 + \zeta \sqrt{1 + \zeta^2}}.$$

In order for the perturbed orbit to be closed, we must have $\beta = \frac{p}{q} \in \mathbb{Q}$ a rational number. Obviously this is generically not the case. It can happen only accidentally, at special values of the angular momentum $\ell$, where

$$a = \frac{b}{\zeta} = \left(\frac{2\beta^2 - 1}{(1 - \beta^2)^2}\right) b,$$

with $\beta = \frac{p}{q}$.

[2] Consider three masses $m_1 = m$, $m_2 = 2m$, and $m_3 = 3m$ connected by two springs $k_{12} = k$ (unstretched length $4a$) and $k_{23} = 2k$ (unstretched length $a$). The motion is frictionless and along a horizontal line (i.e. gravity does not enter this problem).

(a) Choose as generalized coordinates the positions $x_{1,2,3}$ of the masses. Write the Lagrangian.

(b) Find the canonical momenta $p_{1,2,3}$ and the forces $F_{1,2,3}$.

(c) Find the $T$ and $V$ matrices.

(d) Find the characteristic polynomial $P(\omega^2) = \text{det}(\omega^2 T - V)$. This expression can be reduced to one involving a single parameter by writing $\tilde{P}(\lambda) \equiv P(\omega^2)/m^3$ where $\lambda \equiv \omega^2/\omega_0^2$, with $\omega_0^2 \equiv k/m$. Show that $\tilde{P}(\lambda)$ is a cubic which factorizes into a product of $\lambda$ and a quadratic $a\lambda^2 + b\lambda + c$.

(e) Solve for the nonzero roots $\lambda_{\pm}$. The corresponding eigenfrequencies are then given by $\omega_{\pm} = \sqrt{\lambda_{\pm} \omega_0}$.

(f) Find expressions for three normal mode eigenvectors. (This means that you can express their components in terms of the values for $\lambda_{\pm}$.) You don’t have to normalize them.

(g) Show that your zero mode agrees with the conclusions from Noether’s theorem.

**Solution:**

(a) The Lagrangian is

$$L = \frac{1}{2}m(\dot{x}_1^2 + 2\dot{x}_2^2 + 3\dot{x}_3^2) - \frac{1}{2}k(x_2 - x_1 - 4a)^2 - k(x_3 - x_2 - a)^2.$$

(b) Clearly

$$p_1 = \frac{\partial L}{\partial \dot{x}_1} = mx_1, \quad p_2 = \frac{\partial L}{\partial \dot{x}_2} = 2m\dot{x}_2, \quad p_3 = \frac{\partial L}{\partial \dot{x}_3} = 3m\dot{x}_3.$$
as well as

\[ F_1 = \frac{\partial L}{\partial x_1} = -k(x_1 + 4a - x_2) \]
\[ F_2 = \frac{\partial L}{\partial x_2} = -k(x_2 - x_1 - 4a) - 2k(x_2 + a - x_3) \]
\[ F_3 = \frac{\partial L}{\partial x_3} = -2k(x_3 - x_2 - a) \].

(c) The equilibrium configuration, where all the forces vanish, is when \( x_2 = x_1 + 4a \) and \( x_3 = x_2 + a = x_1 + 5a \), with \( x_1 \) arbitrary (this corresponds to a continuous symmetry and corresponding conservation law as per Noether’s theorem, as we shall see). Let’s write \( x_1 \equiv \eta_1 \), \( x_2 \equiv 4a + \eta_2 \), \( x_3 \equiv 5a + \eta_3 \).

Then

\[ L = \frac{1}{2}m(\dot{\eta}_1^2 + 2\dot{\eta}_2^2 + 3\dot{\eta}_3^2) - \frac{1}{2}k(\eta_2 - \eta_1)^2 - k(\eta_3 - \eta_2)^2 \].

The \( T \) and \( V \) matrices are

\[ T_{\sigma\sigma'} = \frac{\partial^2 T}{\partial \eta_{\sigma} \partial \eta_{\sigma'}} \bigg|_{\eta=0} = \begin{pmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & 3m \end{pmatrix} \], \quad \quad V_{\sigma\sigma'} = \frac{\partial^2 U}{\partial \eta_{\sigma} \partial \eta_{\sigma'}} \bigg|_{\eta=0} = \begin{pmatrix} k & -k & 0 \\ -k & 3k & -2k \\ 0 & -2k & 2k \end{pmatrix} \].

(d) Let \( \omega_0^2 \equiv k/m \) and \( \lambda \equiv \omega^2/\omega_0^2 \). Define \( \bar{P}(\lambda) \equiv \det(\omega^2 T - V)/m^3 \). Then

\[ \bar{P}(\lambda) = \det \begin{pmatrix} \lambda - 1 & 1 & 0 \\ 1 & 2\lambda - 3 & 2 \\ 0 & 2 & 3\lambda - 2 \end{pmatrix} \]
\[ = (\lambda - 1)[(2\lambda - 3)(3\lambda - 2) - 4] + 2 - 3\lambda \]
\[ = (\lambda - 1)(6\lambda^2 - 13\lambda + 2) + 2 - 3\lambda \]
\[ = 6\lambda^3 - 13\lambda^2 + 2\lambda - 6\lambda^2 + 13\lambda - 2 + 2 - 3\lambda \]
\[ = 6\lambda(\lambda^2 - \frac{19}{6}\lambda + 2) \].

(e) Solving \( \bar{P}(\lambda) = 0 \), we obtain the roots

\( \lambda_1 = 0 \), \( \lambda_{2,3} = \frac{19 \pm \sqrt{73}}{12} \),

which says

\( \omega_1^2 = 0 \), \( \omega_{2,3}^2 = \frac{19 \pm \sqrt{73}}{12} \omega_0^2 \).
(f) To find the normal more eigenvectors, we solve
\[
\begin{pmatrix}
\lambda_j - 1 & 1 & 0 \\
1 & 2\lambda_j - 3 & 2 \\
0 & 2 & 3\lambda_j - 2
\end{pmatrix}
\begin{pmatrix}
\psi^{(j)}_1 \\
\psi^{(j)}_2 \\
\psi^{(j)}_3
\end{pmatrix} = 0
\]
for each mode index \( j \). For \( j = 1 \), where \( \lambda_j = 0 \), we have
\[
\begin{pmatrix}
-1 & 1 & 0 \\
1 & -3 & 2 \\
0 & 2 & -2
\end{pmatrix}
\begin{pmatrix}
\psi^{(1)}_1 \\
\psi^{(1)}_2 \\
\psi^{(1)}_3
\end{pmatrix} = 0
\]
It is easy to see that the (unnormalized) solution here is \( \psi^{(1)}_1 = \psi^{(1)}_2 = \psi^{(1)}_3 = 1 \). For the other two solutions, \( j = 2(+) \) and \( j = 3(-) \), we have, from the first and third rows of the equation
\[
\sum_{\sigma^\prime} (\omega_j^2 T - V)_{\sigma\sigma^\prime} A_{\sigma^\prime j} = 0,
\]
where, recall, \( \psi^{(j)}_{\sigma} = \psi^{(j)}_{\sigma^\prime} \). Thus, the unnormalized eigenvectors we seek are
\[
\psi^{(\pm)} = \begin{pmatrix} 3\lambda_\pm - 2 \\ -(\lambda_\pm - 1)(3\lambda_\pm - 2) \\ 2(\lambda_\pm - 1) \end{pmatrix}.
\]
(g) We have \( \xi = A^T T \eta \). Thus,
\[
\xi_1 = \psi^{(1)}_{\sigma} T_{\sigma\sigma^\prime} \eta_{\sigma^\prime} = m\dot{\eta}_1 + 2m\dot{\eta}_2 + 3m\dot{\eta}_3.
\]
This corresponds exactly to the result from Noether’s theorem. The one-parameter invariance is uniform translation, \( \text{viz.} \)
\[
\tilde{x}_\sigma(x, \zeta) = x_\sigma + \zeta,
\]
from which we derive the conserved quantity
\[
P = \sum_{\sigma = 1}^{3} \left. \frac{\partial L}{\partial \dot{x}_\sigma} \frac{\partial \tilde{x}_\sigma}{\partial \zeta} \right|_{\zeta = 0} = m\dot{x}_1 + 2m\dot{x}_2 + 3m\dot{x}_3
\]
\[
= m\dot{\eta}_1 + 2m\dot{\eta}_2 + 3m\dot{\eta}_3,
\]
with \( \dot{P} = 0 \). \( P \) is, of course, the total system momentum along the \( x \) direction. Note that \( P = \dot{\xi}_1 \), hence \( \dot{P} = 0 \) entails \( \dot{\xi}_1 = 0 \), \( i.e. \) \( \xi_1 \) is a zero mode with \( \omega_1^2 = 0 \).