A particle of mass \( m \) moves in one dimension subject to the potential 

\[ U(x) = \frac{k}{\sin^2(x/a)} \]

(a) Obtain an integral expression for Hamilton’s characteristic function.

(b) Under what conditions may action-angle variables be used?

(c) Assuming that action-angle variables are permissible, determine the frequency of oscillation by the action-angle method.

(d) Check your result for the oscillation frequency in the limit of small oscillations.

Consider one-dimensional motion in the potential \( V(x) = -V_0 \text{sech}^2(x/a) \) with \( V_0 > 0 \).

(a) Sketch the potential \( V(x) \). Over what range of energies may action-angle variables be used?

(b) Find the action \( J \) and the Hamiltonian \( H(J) \).

(c) Find the angle variable \( \phi \) in terms of \( x \) and the energy \( E \).

(d) Find the Solution for \( x(t) \) by first solving for the motion of the action-angle variables.

Helpful mathematical identities:

\[
\int_0^{\bar{u}(E)} du \sqrt{E + V_0 \text{sech}^2 u} = \frac{\pi}{2} \left( \sqrt{V_0} - \sqrt{-E} \right) \text{ if } -V_0 < E < 0
\]

\[
\int du \left( E + V_0 \text{sech}^2 u \right)^{-1/2} = \begin{cases} 
(-E)^{-1/2} \sin^{-1} \left( \frac{-E}{\sqrt{V_0+E}} \sinh u \right) & \text{if } -V_0 < E < 0 \\
E^{-1/2} \sinh^{-1} \left( \frac{E}{\sqrt{V_0+E}} \sinh u \right) & \text{if } E > 0
\end{cases}
\]

where \( \bar{u}(E) = \cosh^{-1} \sqrt{V_0/(-E)} \) in the first integral.

A particle of mass \( m \) moves in the potential \( U(q) = A|q| \). The Hamiltonian is thus

\[ H_0(q, p) = \frac{p^2}{2m} + A|q| \]

where \( A \) is a constant.

(a) List all independent conserved quantities.
(b) Show that the action variable $J$ is related to the energy $E$ according to $J = \beta E^{3/2}/A$, where $\beta$ is a constant, involving $m$. Find $\beta$.

(c) Find $q = q(\phi, J)$ in terms of the action-angle variables.

(d) Find $H_0(J)$ and the oscillation frequency $\nu_0(J)$.

(e) The system is now perturbed by a quadratic potential, so that

$$H(q, p) = \frac{p^2}{2m} + A|q| + \epsilon Bq^2,$$

where $\epsilon$ is a small dimensionless parameter. Compute the shift $\Delta\nu$ to lowest nontrivial order in $\epsilon$, in terms of $\nu_0$ and constants.

[4] Consider the nonlinear oscillator described by the Hamiltonian

$$H(q, p) = \frac{p^2}{2m} + \frac{1}{2}kq^2 + \frac{1}{4}\epsilon aq^4 + \frac{1}{4}\epsilon b p^4,$$

where $\epsilon$ is small.

(a) Find the perturbed frequencies $\nu(J)$ to lowest nontrivial order in $\epsilon$.

(b) Find the perturbed frequencies $\nu(A)$ to lowest nontrivial order in $\epsilon$, where $A$ is the amplitude of the $q$ motion.

(c) Find the relationships $\phi = \phi(\phi_0, J_0)$ and $J = J(\phi_0, J_0)$ to lowest nontrivial order in $\epsilon$. 