PHYSICS 200A : CLASSICAL MECHANICS
SOLUTION SET #3

[1] Two particles of masses $m_1$ and $m_2$ attract each other according to the logarithmic potential $U(r) = U_0 \ln(r/a)$.

(a) Write down and sketch the effective potential $U_{\text{eff}}(r)$.

(b) Find the radius $r_0$ and period $\tau_0$ of a circular orbit.

(c) For small deviations about a circular orbit, write $r(t) = r_0 + \eta(t)$. Derive the equation of motion for the deviation $\eta(t)$ and solve this equation assuming $\eta$ is small. (What do we mean by “small”?)

(d) What is the geometric equation of the perturbation $\eta(\phi)$? Is the perturbed orbit closed? Why or why not?

Solution:

(a) We have

$$U_{\text{eff}}(r) = \frac{\ell^2}{2\mu r^2} + U_0 \ln\left(\frac{r}{a}\right).$$

(b) To find $r_0$, we set $U'_{\text{eff}}(r_0) = 0$. This gives

$$r_0 = \frac{\ell}{\sqrt{\mu U_0}},$$

and we then may write

$$U_{\text{eff}}(r) = U_0 \left[\frac{1}{2} \zeta^{-2} + \ln \zeta\right] + C,$$

where $\zeta \equiv r/r_0$ and $C = U_0 \ln(r_0/a)$ is a constant. The period of the circular orbit is computed as

$$\ell = \mu r_0^2 \dot{\phi} = \mu r_0^2 \cdot \frac{2\pi}{\tau_0} \implies \tau_0 = \frac{2\pi \mu r_0^2}{\ell} = \frac{2\pi \ell}{U_0}.$$

(c) With $r = r_0 + \eta$ we have

$$\mu \ddot{\eta} = -U''_{\text{eff}}(r_0) \eta + O(\eta^2),$$

where

$$U''_{\text{eff}}(r_0) = \frac{3\ell^2}{\mu r_0^4} \cdot \frac{U_0}{r_0^2} = \frac{2U_0}{r_0^2} = \frac{2\mu U_0^2}{\ell^2}.$$

The solution

$$\eta(t) = \eta_0 \cos(\omega t + \delta_\eta),$$
Figure 1: The effective potential $U_{\text{eff}}(r)$, here shown with $a = r_0$.

where $\eta_0$ and $\delta_\eta$ are constants, and

$$\omega = \sqrt{\frac{U''_{\text{eff}}(r_0)}{\mu}} = \frac{\sqrt{2}U_0}{\ell}.$$  

(d) To find the geometric equation of the perturbed orbit, we can write

$$\frac{d\phi}{dt} = \frac{\ell}{\mu r^2(t)} = \frac{U_0}{\ell} + \mathcal{O}(\eta),$$

hence

$$\frac{d^2\eta}{d\phi^2} = \frac{\ell^2}{U_0^2} \frac{d^2\eta}{dt^2} + \mathcal{O}(\eta^2) = 2\eta + \ldots,$$

the solution of which is

$$\eta(\phi) = \eta_0 \cos(\beta \phi + \delta_\phi),$$

where $\beta = \ell \omega/U_0 = \sqrt{2}$ and $\delta_\phi$ is a constant. The perturbed orbit is bounded but not closed, since $\beta$ is not a rational number. This means that the apsides of the orbit will precess.

[2] A particle moves in a central force $\vec{f}(\vec{r}) = \vec{f}(r)$.

(a) Using the equation for the shape of the orbit, derive the force law under which the shape of the orbit is $r(\phi) = a/\cos(\phi - \phi_0)$. Explain why your answer makes excellent sense.
(b) What force law will result in the shape $r(\phi) = 2b/\phi^2$? Is the force attractive or repulsive? Sketch the orbit over the interval $\phi \in (0, \infty)$.

**Solution:**

(a) The geometric equation of the shape of the orbit is

\[
\frac{d^2s}{d\phi^2} + s = -\frac{\mu}{\ell^2 s^2} f(s^{-1}) ,
\]

where $f(r)$ is the force. Setting $s(\phi) = a^{-1} \cos(\phi - \phi_0)$, we have $s'' + s = 0$, hence $f = 0$, i.e. no force. This makes good sense, since $r(\phi) = a/\cos(\phi - \phi_0)$ is the equation of a straight line, i.e. $\hat{n} \cdot \mathbf{r} = a$, where $\hat{n} = \cos \phi_0 \hat{x} + \sin \phi_0 \hat{y}$.

(b) With $s(\phi) = \phi^2/2b$ we have $s'' = 1/b$ and therefore

\[
f = -\frac{\ell^2 s^2}{\mu} (s + b^{-1}) = -\frac{A}{r^3} - \frac{B}{r^2} ,
\]

with

\[
A = \frac{\ell^2}{\mu} , \quad B = \frac{\ell^2}{\mu b} .
\]

Put another way, the force law $f(r) = -Ar^{-3} - Br^{-2}$ will yield a spiral orbit $r(\phi) = B\phi^2/2A$ for $\ell = \sqrt{\mu A}$. In general, though, this force law results in the geometric orbit equation

\[
s'' + \left(1 - \frac{\mu A}{\ell^2}\right) s = \frac{\mu B}{\ell^2} ,
\]
the solution of which is

\[ s(\phi) = \frac{\mu B}{\ell^2 - \mu A} \cdot \left\{ 1 + \varepsilon \cos \beta (\phi - \phi_0) \right\}, \]

with

\[ \beta = \left( 1 - \frac{\mu A}{\ell^2} \right)^{1/2}, \]

where \( \varepsilon \) and \( \phi_0 \) are arbitrary. For \( \ell^2 > \mu A \) and \( B > 0 \), the orbit is a precessing ellipse.

[3] Evil space aliens send a probe into our solar system to observe the earth. The probe orbits the sun with its perihelion at distance \( r_p = \frac{3}{4} a_\oplus \), where its velocity is \( v_p = \frac{4}{3} v_\oplus \). \( (a_\oplus \) and \( v_\oplus \) are the orbital radius and velocity of the earth, respectively.) The probe’s orbit is coplanar with that of the earth, and you may neglect the interaction of the probe with all bodies other than the sun.

(a) Compute the eccentricity of the probe’s orbit.

(b) Compute the probe’s distance from the sun at aphelion, and its velocity at aphelion.

(c) Write down the geometric equation for the probe’s orbit.

(d) Let perihelion occur when the azimuthal angle is \( \phi = \pi \). At what value of \( \phi \) does the probe cross the earth’s orbit?

(e) Compute the period of the probe’s orbit.

Solution :

Let’s consider the general case where \( r_p = \alpha a_\oplus \) and \( v_p = \beta v_\oplus \). The energy of the probe’s orbit is given by

\[ E = \frac{1}{2} m v_p^2 - \frac{GMm}{r_p} = \frac{1}{2} m v_a^2 - \frac{GMm}{r_a}, \]

where \( m \) is the mass of the probe. The probe’s angular momentum is

\[ \ell = m r_p v_p = m r_a v_a = \alpha \beta m a_\oplus v_\oplus. \]

It is helpful to express all velocities in terms of \( v_\oplus \) and all distances in units of \( a_\oplus \). To this end, we write

\[ GM = a_\oplus v_\oplus^2, \]

where \( M \) is the solar mass. Using the angular momentum equation to eliminate \( v_a \) in terms of \( r_a \), we obtain the equation

\[ \frac{\alpha^2 \beta^2}{2 \rho^2} - \frac{1}{\rho} + \frac{1}{\alpha} - \frac{\beta^2}{2} = 0, \]
where \( \rho \equiv r_a/a_\oplus \). This is a quadratic equation for \( \rho \), the solutions of which are \( \rho = \alpha \) (trivial) and

\[
\rho = \frac{\alpha^2 \beta^2}{2 - \alpha \beta^2} .
\]

If \( \rho > \alpha \) then \( r_a \) corresponds to aphelion and \( r_p \) to perihelion. For \( \alpha = \frac{3}{4} \) and \( \beta = \frac{4}{3} \), we have \( \rho = \frac{3}{2} \).

(a) The eccentricity of the probe’s orbit is

\[
\varepsilon = \frac{r_a - r_p}{r_a + r_p} = \alpha \beta^2 - 1 .
\]

For \( \alpha = \frac{3}{4} \) and \( \beta = \frac{4}{3} \), we find \( \varepsilon = \frac{1}{3} \).

(b) The distance at aphelion is \( r_a = \rho a_\oplus = \frac{3}{2} \) AU. The probe’s velocity at aphelion is then

\[
v_a = \alpha \beta v_\oplus / \rho = (2 - \alpha \beta^2) v_\oplus / \alpha \beta = \frac{2}{3} v_\oplus .
\]

(c) The geometric equation of the probe’s orbit is

\[
r(\phi) = \frac{a (1 - \varepsilon^2)}{1 - \varepsilon \cos \phi} ,
\]

where

\[
a = \frac{1}{2} (r_a + r_p) = \frac{\alpha a_\oplus}{2 - \alpha \beta^2} , \quad \varepsilon = \frac{r_a - r_p}{r_a + r_p} = \alpha \beta^2 - 1 .
\]

(d) The probe intersects the earth’s orbit when \( r(\phi) = a_\oplus \). For our probe, \( a = \frac{9}{8} a_\oplus \) and \( \varepsilon = \frac{1}{3} \), so \( a (1 - \varepsilon^2) = a_\oplus \), yielding \( \cos \phi = 0 \), meaning \( \phi = \pm \frac{1}{2} \pi \). See fig. ??.
We have
\[ \frac{a^3}{\tau^2} = \frac{GM}{4\pi^2} = \frac{a_\oplus v^2_\oplus}{4\pi^2}. \]
Thus,
\[ \frac{a^3}{a^3_\oplus} = \left( \frac{v_\oplus}{2\pi a_\oplus} \right)^2 = \left( \frac{\tau}{\tau_\oplus} \right)^2 \quad \implies \quad \tau = \tau_\oplus \cdot \left( \frac{a}{a_\oplus} \right)^{3/2}. \]

With \( a = \frac{9}{8} a_\oplus \), we obtain \( \tau = 1.193 \tau_\oplus \), with \( \tau_\oplus = 1 \text{ yr.} \)

[4] Two objects of masses \( m_1 \) and \( m_2 \) move under the influence of a central potential \( U = k |r_1 - r_2|^{1/4} \).

(a) Sketch the effective potential \( U_{\text{eff}}(r) \) and the phase curves for the radial motion. Identify for which energies the motion is bounded.

(b) What is the radius \( r_0 \) of the circular orbit? Is it stable or unstable? Why?

(c) For small perturbations about a circular orbit, the radial coordinate oscillates between two values. Suppose we compare two systems, with \( \ell'/\ell = 2 \), but \( \mu' = \mu \) and \( k' = k \). What is the ratio \( \omega'/\omega \) of their frequencies of small radial oscillations?

(d) Find the equation of the shape of the slightly perturbed circular orbit: \( r(\phi) = r_0 + \eta(\phi) \). That is, find \( \eta(\phi) \). Sketch the shape of the orbit.

(e) What value of \( n \) would result in a perturbed orbit shaped like that in fig. ??

![Figure 4: Closed precession in a central potential \( U(r) = kr^n \).](image)

**Solution:**

The effective potential is
\[ U_{\text{eff}}(r) = \frac{\ell^2}{2\mu r^2} + kr^n \quad (1) \]
with \( n = \frac{1}{4} \). In sketching the effective potential, I have rendered it in dimensionless form,

\[
U_{\text{eff}}(r) = E_0 U_{\text{eff}}(r/r_0),
\]

where \( r_0 = (\ell^2/nk\mu)^{-(n+2)} \) and \( E_0 = (\frac{1}{2} + \frac{1}{n})\ell^2/\mu r_0^2 \), which are obtained from the results of part (b). One then finds

\[
U_{\text{eff}}(x) = \frac{nx^{-2} + 2x^n}{n + 2}.
\]

Although it is not obvious from the detailed sketch in fig. ??, the effective potential does diverge, albeit slowly, for \( r \to \infty \). Clearly it also diverges for \( r \to 0 \). Thus, the relative coordinate motion is bounded for all energies; the allowed energies are \( E \geq E_0 \).

\( \ell^2 \)

\( \mu r_0^2 \)

\( nkr_0^{n-1} \)

\( 0 \)

\( \frac{\ell^2}{\mu r_0^2} + nkr_0^{n-1} = 0 \).

\( 4 \)

\( 7 \)

Figure 5: The effective \( U_{\text{eff}}(r) = E_0 U_{\text{eff}}(r/r_0) \), where \( r_0 \) and \( E_0 \) are the radius and energy of the circular orbit.

(b) For the general power law potential \( U(r) = kr^n \), with \( nk > 0 \) (attractive force), setting \( U'_{\text{eff}}(r_0) = 0 \) yields

\[
-\frac{\ell^2}{\mu r_0^2} + nkr_0^{n-1} = 0.
\]
Thus,

\[ r_0 = \left( \frac{\ell^2}{nk\mu} \right)^{1/2} = \left( \frac{4\ell^2}{k\mu} \right)^{1/3}. \]  

(5)

The orbit \( r(t) = r_0 \) is stable because the effective potential has a local minimum at \( r = r_0 \), i.e. \( U''_{\text{eff}}(r_0) > 0 \). This is obvious from inspection of the graph of \( U_{\text{eff}}(r) \) but can also be computed explicitly:

\[ U''_{\text{eff}}(r_0) = \frac{3\ell^2}{\mu r_0^3} + n(n-1)kr_0^n \]
\[ = (n+2) \frac{\ell^2}{\mu r_0^3}. \]  

(6)

Thus, provided \( n > -2 \) we have \( U''_{\text{eff}}(r_0) > 0 \).

(c) From the radial coordinate equation \( \mu \ddot{r} = -U'_{\text{eff}}(r) \), we expand \( r = r_0 + \eta \) and find

\[ \mu \ddot{\eta} = -U''_{\text{eff}}(r_0) \eta + O(\eta^2). \]

(7)

The radial oscillation frequency is then

\[ \omega = (n+2)^{1/2} \frac{\ell}{\mu r_0^{3/2}} = (n+2)^{1/2} n \frac{\ell^2}{\mu} k \frac{4}{n+2} \mu \frac{n}{n+2} \ell \frac{n-2}{n+2}. \]  

(8)

The \( \ell \) dependence is what is key here. Clearly

\[ \frac{\omega'}{\omega} = \left( \frac{\ell'}{\ell} \right) \frac{n-2}{n+2}. \]  

(9)

In our case, with \( n = \frac{1}{4} \), we have \( \omega \propto \ell^{-7/9} \) and thus

\[ \frac{\omega'}{\omega} = 2^{-7/9}. \]  

(10)

\[ \text{Figure 6: Radial oscillations with } \beta = \frac{3}{2}. \]
(d) We have that $\eta(\phi) = \eta_0 \cos(\beta \phi + \delta_0)$, with

$$\beta = \frac{\omega}{\phi} = \frac{\mu r_0^2}{\ell} \cdot \omega = \sqrt{n + 2}.$$  \hspace{1cm} (11)

With $n = \frac{1}{4}$, we have $\beta = \frac{3}{2}$. Thus, the radial coordinate makes three oscillations for every two rotations. The situation is depicted in fig. ??.

(e) Clearly $\beta = \sqrt{n + 2} = 4$, in order that $\eta(\phi) = \eta_0 \cos(\beta \phi + \delta_0)$ executes four complete periods over the interval $\phi \in [0, 2\pi]$. This means $n = 14$.

[5] A symmetric top with one fixed point in a gravitational field moves with its symmetry axis nearly vertical ($\theta \ll 1$) and $p_\phi = p_\psi$.

(a) Expand the effective potential through terms of order $\theta^4$.

(b) If $p_\phi^2 > 4I_1 Mg\ell$, show that $U_{\text{eff}}(\theta)$ has a minimum at $\theta = 0$. Sketch $U_{\text{eff}}(\theta)$ for small $\theta$. Prove that the frequency of small oscillations about this configuration is given by

$$\Omega^2 = \frac{p_\phi^2 - 4I_1 Mg\ell}{4I_1^2}.$$  

(c) If $p_\phi^2$ is slightly smaller than $4I_1 Mg\ell$, show that $U_{\text{eff}}(\theta)$ has a maximum at $\theta = 0$ and a minimum at some finite value $\theta^*$. Find $\theta^*$, and sketch $U_{\text{eff}}(\theta)$ for small $\theta$, and find the frequency of small oscillations about $\theta = \theta^*$.

**Solution:**

The Lagrangian is

$$L = \frac{1}{2}I_1 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{1}{2}I_3 (\dot{\phi} \cos \theta + \dot{\psi})^2 - Mg\ell \cos \theta.$$  

Since $L$ is cyclic in $\phi$ and $\psi$, their conjugate momenta are conserved:

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = I_1 \dot{\phi} \sin^2 \theta + I_3 (\dot{\phi} \cos \theta + \dot{\psi}) \cos \theta$$

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3 \cos \theta \dot{\phi} + I_3 \dot{\psi}.$$  

Solving for $\dot{\phi}$ and $\dot{\psi}$, we have

$$\dot{\phi} = \frac{p_\phi - p_\psi \cos \theta}{I_1 \sin^2 \theta}$$

$$\dot{\psi} = \frac{p_\psi}{I_3} - \frac{(p_\phi - p_\psi \cos \theta) \cos \theta}{I_1 \sin^2 \theta}.$$
We then have
\[ E = T + U = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{(p_\phi - p_\psi \cos \theta)^2}{2 I_1 \sin^2 \theta} + \frac{p_\psi^2}{2 I_3} + Mg\ell \cos \theta , \]
as in §8.6.2 of the class notes. The energy \( E \) is conserved. We are given \( p_\phi = p_\psi \).

(a) Expanding,
\[ \frac{1 - \cos \theta}{\sin \theta} = \tan \left( \frac{1}{2} \theta \right) = \frac{1}{2} \theta + \frac{1}{24} \theta^3 + \ldots , \]
hence
\[ U_{\text{eff}}(\theta) = \frac{p_\psi^2}{2I_3} + Mg\ell + \left( \frac{p_\psi^2}{4I_1} - Mg\ell \right) \frac{\theta^2}{2} + \left( \frac{p_\psi^2}{2I_1} + Mg\ell \right) \frac{\theta^4}{24} + \mathcal{O}(\theta^6) . \]

(b) The coefficient of the quadratic term is always positive. The coefficient of the quadratic term is positive if \( p_\psi^2 > 4Mg\ell I_1 \), in which case the minimum of \( U_{\text{eff}}(\theta) \) lies at \( \theta = 0 \). Setting \( \dot{E} = 0 \) gives
\[ I_1 \ddot{\theta} = -\frac{\partial U_{\text{eff}}}{\partial \theta} , \]
from which we read off the frequency of small oscillations,
\[ \Omega^2 = \frac{U''_{\text{eff}}(0)}{I_1} = \frac{p_\psi^2 - 4Mg\ell I_1}{4I_1^2} . \]

(c) For \( p_\psi^2 < 4Mg\ell I_1 \), the coefficient of the quadratic term is negative. The effective potential then has a local maximum at \( \theta = 0 \), and a local minimum at \( \theta = \pm \theta^* \), where
\[ \theta^* = \sqrt{3} \cdot \sqrt[3]{\frac{4Mg\ell I_1 - p_\psi^2}{2Mg\ell I_1 + p_\psi^2}} . \]
The small oscillation frequency is
\[ \Omega^2 = \frac{U''_{\text{eff}}(\theta^*)}{I_1} . \]

In general, when
\[ U_{\text{eff}} = \frac{1}{2} A \theta^2 + \frac{1}{12} B \theta^4 + \ldots , \]
we have, ignoring the higher order terms,
\[ U'_{\text{eff}}(\theta^*) = 0 \quad \Rightarrow \quad (\theta^*_{\text{min}})^2 = \begin{cases} 0 & \text{if } A > 0 \\ 3|A|/B & \text{if } A < 0 , \end{cases} \]
and thus

\[ U''_{\text{eff}}(\theta^*) = \begin{cases} 
A & \text{if } A > 0 \\
2 |A| & \text{if } A < 0
\end{cases} \]

Thus, the frequency of small oscillations is given by

\[ \Omega^2 = \frac{4Mg\ell I_1 - P^2_0}{2I_1^2} \]