Lecture VII  Waves: Stability and Structure

→ Recall D'Alembert's Paradox

Drag Force = Difference in Momentum Flux thru ends

\[ F_d = \int_{A_1} \left( p_1 + \rho v_1^2 \right) \, dA_1 - \int_{A_2} \left( p_2 + \rho v_2^2 \right) \, dA_2 \]

For ideal fluid, upstream/downstream symmetry of flow

\[ p_1 = p_2 \]
\[ v_1 = v_2 \]

\[ \therefore \quad F_d = 0 \]

i.e., effect of flow is enhanced inertia (induced mass)
For viscous fluid (i.e., no slip boundary condition) \( \rightarrow \) upstream/downstream asymmetry
\( \rightarrow \) wake

\( \Rightarrow \) Drag occurs

Aim is to understand structure and dynamics of wake.

\( \Rightarrow \) Recall: No-slip B.C.'s
\( \Rightarrow \) Separation occurs

\( \Rightarrow \) Potential flow outside wake
\( \Rightarrow \) Hole fills in to form wake
\( \Rightarrow \) Line of separation
- why is rotational/vortical for Re > Recri?
- how does hole fill up?

- Separation is unstable
- Kelvin-Helmholtz Instability

\[
\text{KH} \quad \Rightarrow \quad \text{Free energy} \Rightarrow \text{DV}
\]

- Simplification: interface
\[
\begin{align*}
\text{V}_x &= V & \text{for } \theta < \theta_0 \\
\text{V}_x &= 0 & \text{for } \theta_2 \geq \theta_0
\end{align*}
\]

- \( \omega = \sqrt{\text{DV}_x / \text{DZ}} \) except interface

- Can treat as potential flow in regions \( \theta_0 \) and match at interface
- Interface ripples \( \Rightarrow \) dynamic b.c.
Physical ideas:

1. $\text{inflated}$
2. high $U_j$, low $p$
3. $\text{zero } U_j$, high $p$

$\frac{\partial U}{\partial x} \text{ perturbation}$

$\text{ripple cut face}$

$\rho + \frac{\partial U^2}{2} = \text{const.}$

$\Delta U < 0$ - ripple brings in dense fluid
- flow in region drops

$\Delta U > 0$ - ripple brings in rare fluid
- flow on $\infty$ increases

$\Delta p > 0$ - Bernoulli $\Rightarrow$ pressure increases

$\Delta p$ - pressure decreases
\[ \frac{\partial P}{\partial r} > 0 \Rightarrow \frac{\partial V}{\partial r} < 0 \text{ further, } \\
\frac{\partial P}{\partial r} < 0 \Rightarrow \frac{\partial V}{\partial r} > 0 \text{ further, } \\
\text{reinforcing azimuthal perturbation!} \]

N.B. k-H instability driven viscous mixing via turbulence, mixing billows etc.

To calculate:

\[ \frac{\partial}{\partial z} \left( \rho \frac{\partial \phi}{\partial z} \right) - \rho \frac{\partial^2 \phi}{\partial t^2} = \text{const} \]

\[ \frac{\partial^2 \phi}{\partial t^2} = \text{assumed equilibrium} \]

\[ \mathbf{D} \cdot \mathbf{V} = 0 \]

\[ \mathbf{V} = \mathbf{D} \phi \]

\[ \omega = 0, \text{ except interface} \]

\[ \mathbf{D} \phi = 0 \]

\[ (\Delta^2 - k^2) \phi = 0 \]

wave along interface (symmetry)

\[ \phi = \sum \phi_n e^{ihx} \left( e^{-i\frac{n\pi}{2}} e^{-i\frac{n\pi}{4}} \right) \]

decay away from interface
Matching conditions

→ Pressure continuity
\[ \bar{p}_1 (0^+) = \bar{p}_2 (0^-) \]

→ \( \phi \) continuity
\[ \frac{\partial \phi}{\partial z} = \frac{\partial \phi}{\partial z} \]
\[ \text{e.g.,} \quad -\rho = \rho \frac{\partial \phi}{\partial z} + \frac{\sigma (\partial \phi)^2}{2} \]

\[ \frac{\partial \phi}{\partial z} \]

\[ \partial \frac{\partial \phi}{\partial z} - \kappa^2 \phi = 0 \]

and
\[ \int_{0^-}^{0^+} \partial \frac{\partial \phi}{\partial z} = 0 \implies \partial \phi \bigg|_{0^-} - \partial \phi \bigg|_{0^+} = 0 \]

Now,
\[ \partial_+ \bar{u} + \bar{u} \cdot \nabla \bar{u} = -\frac{\partial \bar{p}}{\partial \phi} \]
\[ \rho_1 \left( \partial_+ \bar{v}_{z_1} + \bar{v} \partial_{x_1} \bar{v}_{z_1} \right) = -\partial_z \bar{p}_1 \]
\[ \partial_2 \left( \partial_+ \bar{v}_{z_2} \right) = -\partial_z \bar{p}_2 \]
$$\vec{V}_{z_1} = -\epsilon k \vec{P}_1 \over \rho_1 (\xi \nu - \omega)$$
$$\vec{V}_{z_2} = i k \vec{P}_2 \over -\rho_2 \omega$$

Now, dynamic boundary:

$$\vec{M} = \vec{M}(x,t) \rightarrow \text{displacement}$$

$$\frac{d\vec{M}}{dt} = \vec{V}_{z_1}$$

and

$$\partial_t \vec{\psi} + \nabla \partial_x \vec{\psi} = \frac{d\vec{\psi}}{dt}$$

$$-\epsilon (\omega - k \nu) \vec{M}_n = \vec{V}_{z_2}, n$$

$$-\epsilon (\omega - k \nu) \vec{M}_n = -\epsilon k \vec{P}_1 \over \rho_1 (\xi \nu - \omega)$$

$$\vec{P}_1 = -\rho_1 (\xi \nu - \omega)^2 \vec{M}_n$$
\[ \tilde{P}_2 = \frac{\alpha_2 \omega^2}{k} \]

and \( \tilde{P}_1 = \tilde{P}_2 \Rightarrow \)

\[ -\frac{\alpha_1 (kv - \omega)^2}{k} = \frac{\alpha_2 \omega^2}{k} \]

and finally,

\[ \omega = kv \left( \frac{\alpha_1 + i (\alpha_1 \alpha_2)^{1/2}}{\alpha_1 + \alpha_2} \right) \]

\[ \Rightarrow \gamma = kv (\alpha_1 \alpha_2) \]

\[ \Rightarrow \omega_{real} = kv \left[ \frac{\alpha_1}{(\alpha_1 + \alpha_2)} \right] \]

\[ \Rightarrow \text{no exchange of stabilities here.} \]

\[ \Rightarrow \alpha_1 = \alpha_2 \quad \gamma \sim kv \]
→ what happens?

→ vortex roll-up, follows

(See Felskovich
E = 2.3 F2.4)

→ vortex streets etc.

→ N.B. Vorticity concentrated as layer of interface.

→ More generally:

\[
\gamma^2 = \frac{\rho_0 \alpha_0 \kappa (v_2 - v_0)^2}{(\rho_0 + \rho_2)^2} + \frac{(\alpha_1 - \alpha_2)(\gamma H)}{\rho_0 + \rho_2}
\]

\[
- \frac{T H^3}{(\rho_0 + \rho_2)} + \text{Rayleigh-Taylor}
\]

\[
\text{Surface Tension}
\]

\[
\rightarrow \text{(Capillarity)}
\]

→ Threshold wind to excite waves
Surface Tension

- An important property of interfaces is surface tension. i.e. Force due to decrease in surface area of interface.

- Familiar from droplets, capillary waves, etc.

\[ dF = -P_1 \, dv - P_2 \, (-dv) + \tau \, dA \]

\( + \) change in free energy

\( - \) expands

\( + \) contracts

Isothermal displacement of

\( \text{1 expands into 2} \)
\[ \text{d}V = \text{d}A \text{d}V \]

\[ \text{displacement} \]

\[ \text{d}A = \int \int \text{d}x \text{d}y \left( 1 + \left( \frac{\partial y}{\partial x} \right)^2 + \left( \frac{\partial y}{\partial y} \right)^2 \right)^{1/2} \]

\[ = \int \int \text{d}x \text{d}y \]

i.e. displacement expands area

\[ x, y \text{ parametrize surface} \]

So, for small slope:

\[ \text{d}A = \int \int \text{d}x \text{d}y \pm \left[ \left( \frac{\partial y}{\partial x} \right)^2 + \left( \frac{\partial y}{\partial y} \right)^2 \right] \]

\[ = \int \int \text{d}x \text{d}y \left( -V \text{d}n \right) \text{d}y \]

\[ \text{d}F = \left[ \rho \left( \rho - \rho_i \right) - \mathbf{V} \cdot \mathbf{n} \right] \]
criterion for equilibrium:

\[ p_2 - p_1 = \sqrt{D^2} \]

\[ dF = (p_2 - p_1) dA d\theta + \frac{1}{2} dA \]

More generally:

Now consider (i.e. not "weakly curved" interface)

\[ ds = (R_0 + dM) d\theta \]

\[ ds = d\theta_0 \left( 1 + \frac{dM}{R_1} \right) \]

Radius curvature of interface as shown

In general surface parametrized by 2 radii of curvature, \( R_1, R_2 \) (Gauss):

\[ dA = \int d\theta_1 d\theta_2 \left( 1 + \frac{dM}{R_1} \right) \left( 1 + \frac{dM}{R_2} \right) \]

\[ = \int d\theta_1 d\theta_2 \left( \frac{1}{R_1} + \frac{1}{R_2} \right) dA \]
\[
\begin{align*}
\frac{\partial F}{\partial \text{d}A_0} &= \int_{A_0} \left[ (p_1 - p_2) + \frac{1}{2} \nabla \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \right] \text{d}A_0 \\
\n\text{for equilibrium with interface (general)}
\end{align*}
\]

\[
\nabla \left( \frac{1}{R_1} + \frac{1}{R_2} \right) = p_1 - p_2
\]

Laplace's Law

\[
\frac{\partial F}{\partial \text{d}A_0} = \nabla \left( \frac{1}{R_1} + \frac{1}{R_2} \right) = p_1 - p_2
\]

\[\text{Laplace's Law}\]

\[
\begin{align*}
\text{N.B.:} \\
\text{Given 2-phase equilibrium (separate domains), can use Laplace law to estimate droplet size for immiscible liquids}
\end{align*}
\]

\[
\begin{align*}
\text{O.E. } p_1 > p_2, \quad R &= \sqrt{\frac{p_1 - p_2}{\gamma}} \\
\text{For SW, } R &\sim \frac{p_1 - p_2}{\rho_1 V_2} \quad \rho_1 V_2
\end{align*}
\]

\[
\omega^2 \star \mathbf{g} = k g \left( \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} \right) + \frac{\gamma}{\rho_1} \frac{V_2^3}{(\rho_1 + \rho_2)}
\]
For \( \theta_4 \gg \theta_2 \)

\[ \omega^2 = g h + \frac{\sigma}{\rho} \frac{h^2}{L} \]  

\text{Gravity wave} \quad \text{Capillary wave}

- Cross-over at few cm.  
- Capillary effects important at \( \leq 5 \) cm.
Wake Structure

- Physics ideas: wake flow created by response to separation
- Link: Drag - asymmetry - wake flow
- Width: laminar, turbulent scales
- Deficit and punchline re: N-S vs Euler

Wakes

body

\[ u \]

region of wake limited in angular extent

region behind body of departures from potential flow. Wake rotational.

- Wake breaks upstream downstream symmetry of ideal flow

- Wake is consequence of a body experiencing drag.

Message: A little viscosity forces a global adjustment in flow structure
Note:

- In general, wake multi-component Kelvin – waver. Bangs wake. center-line.

due screw bubbler

K. E. Kelvin b.c.

(at friction)

Here consider spherical cow of wake problems.

\[ F \sim \rho u^2 R^2 \]

sphere.

so \( F \sim \rho u^2 R^2 \)

\[ \Rightarrow \text{no surface effects} \]

\[ \Rightarrow \text{How calculate wake structure?} \]

Force of Drag = \( \frac{\text{Rate of Net Momentum Loss}}{\text{from Flow}} \)
\[ \text{F}_t \sim C_{DPA} U^2 \]

Stokes drag coefficient

\( C_D \)

\( \Gamma / \text{Re} \)

\( \Gamma / \text{Re} \)

\( \text{in laminar wake, flow not turbulent, but inertia relevant} \)

Further:

- Distance behind body
  \( x >> R_y \rightarrow \text{wake} \)

- If body speed \( U_x \), then in frame where body stationary

\[ U \rightarrow \frac{-U_x}{V + U_x} \]

\( V_x < 0 \) (steeper on wake)

\( V \) differs zero on limited region
How limited is signal propagation as function of distance? Only.

How does wake form?

- No slip boundary condition
- Fluid slows down past body
- Separation discontinuity
- Fluid exits hole
- Viscosity smoothes out discontinuity

Wake is turbulent wake:
- Turbulent mixing smooths discontinuity faster than viscous mixing.

Boundary of wake is traced by fluid particles:
- Passing close to body
- Scattered by diffusion and turbulent mixing.
Now, to calculate rate of loss of momentum from flow return to D'Alembert construction, but with asymmetry

\[
\begin{align*}
\text{Drag} &= \text{difference in flux out flux} \\
\text{Momentum Flux} \quad \mathcal{F} &= \mathcal{F}_{\text{Tot}} \\
\Pi_{xx} &= P_0 + \rho U^2 \\
\Pi_{xx}(x) &= P_0 + P' + \rho U^2 + \rho U V_x^2 + \text{th. o.}
\end{align*}
\]
\[ F_x = - \int \partial_\theta q \Omega V_x \]

Now, can take conical symmetry \( \Theta \)

\[ F_x = - \pi w(x)^2 \Omega \Omega V_x \]

What is width? \( v \to 0 \)

\( F_x > 0 \)

Now, need \( w(x) \) to get \( V_x \)!

\( \rightarrow \) Observe

- problem now reduced to one of scale.
- water self-similar:
  \[ w \sim x^y, \quad x \] 
- waters can be laminar or turbulent
c) Laminar, \( \frac{UR}{\nu} > 1 \)

but not \( > \lambda \)

\[
\frac{\partial u}{\partial t} + u \cdot \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = -\frac{\partial p}{\partial x}
\]

\[
\frac{\partial u}{\partial t} + u \cdot \frac{\partial v}{\partial x} - \nu \frac{\partial^2 v}{\partial x^2} = -\frac{\partial p}{\partial y}
\]

\[
\frac{\partial u}{\partial x} = 0 \quad \text{narrow wave}
\]

\[
U \frac{\partial u}{\partial z} - \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} \right) = -\frac{\partial p}{\partial x}
\]

\[
u \frac{\partial u}{\partial x} - \nu \frac{\partial^2 v}{\partial z^2} = -\frac{\partial p}{\partial y}
\]

\( \delta x \sim 1/x \) downstream distance

\( \delta z \sim 1/w \) scale

\[
\text{Obvious: } v = \frac{1}{(2\nu)^{1/2}} \exp \left[ -\frac{y^2}{2\nu x} \right]
\]
\[
\begin{align*}
\left( \frac{v_x}{w} - \frac{v}{w^2} \right) v_x & \sim -\frac{p}{w} \\
\left( \frac{v_x}{w} - \frac{v}{w^2} \right) v_x & \sim -\frac{p}{w} \\
D \cdot \nabla v & = 0 \quad \Rightarrow \quad \frac{v_x}{w} \sim \frac{v_y}{w}
\end{align*}
\]

as \( p \) negligible (will show) \( \Rightarrow \)

\[
\begin{align*}
\frac{u}{x} & \sim \frac{v}{w^2} \\
W & \sim \left( \frac{v_x}{u} \right)^{1/2} \\
& \Rightarrow \text{diffusive spreading of momentum by} \quad \frac{x}{u} \\
& \Rightarrow \sim (xt)^{1/2} \text{ with } t \sim x/u
\end{align*}
\]

\[
\begin{align*}
W & \sim \left( \frac{v_x}{u} \right)^{1/2} \left( \frac{v_x}{u} \right)^{1/2} \\
\frac{W}{R} & \sim \left( \frac{x}{R} \right)^{1/2} \left( \frac{v_x}{u} \right)^{1/2} \\
\frac{W}{R} & \sim \left( \frac{x}{R} \right)^{1/2} \frac{1}{Re^{1/2}}
\end{align*}
\]

\( \text{akin Bl = 0.008} \) \( \checkmark \)
\[ \nabla \cdot E \Rightarrow \nabla \cdot \mathbf{v} \nabla \frac{P}{\rho w} \sim \frac{v}{w^2} \]

\[ \frac{v_x}{x} \sim \frac{v}{w} \]

\[ P \sim \rho v \frac{v_x}{x} \]

\[ \frac{P}{\rho x} \sim \frac{v}{x} \frac{v_x}{x^2} \]

but eqn. \( \left( \frac{4}{x} - \frac{v}{w^2} \right) \frac{v_x}{x} \sim -\frac{P}{\rho x} \)

\[ o\left(\frac{y}{w^2}\right) \quad o\left(\frac{y}{x^2}\right) \]

so drop \( P \sqrt{x} \)

\[ N \Rightarrow \frac{P}{\rho w} \sim \frac{v}{w^2} v_z \]
Some Observations re: Waves

\[ F_x = -\rho U \int \nu_x \, dy \, dz \]

\[ Q = \rho \int \nu_x \, dy \, dz \quad \rightarrow \text{Wake deficit} \]

\[ \text{net mass flow thru wake area} \]

\[ Q \text{ is } x \text{ independent:} \]

\[ i.e. \quad Q \sim \frac{F_x}{U} + \text{const} \]

but, if encircle body:
have:

\[ \oint \mathbf{v} \cdot d\mathbf{a} = 0 \quad \text{tot} \]

\[ \text{c.e. no water leak} \]

but

\[ 0 = \int \mathbf{v} \cdot d\mathbf{a} + \int \mathbf{v} \cdot d\mathbf{q} \]

\[ \text{wave} \quad \text{pot flow} \]

\[ \oint \mathbf{v} \cdot d\mathbf{a} = Q \]

\[ Q \sim v_x A \]

\[ Q \text{ Finite} \Rightarrow \]

\[ V_x \sim \frac{1}{A} \sim \frac{1}{r^2} \]

so

\[ \int \mathbf{v} \cdot d\mathbf{q} = -Q \]

\[ \text{pot flow} \]

\[ V_x A \sim -Q \]

\[ V_x \sim \frac{1}{r^2} \quad \text{potential flow} \]

\[ \text{monopole} \]
- Euler equations
  \( v \sim 1/r^3 \rightarrow \text{dipole} \)

But for \( N-S \) eqn.

\( v \sim 1/r^2 \rightarrow \text{monopole} \)

- Global adjustment of potential
- Flow outside wake induced by viscosity and the wake.

Message: A little \( v \) forces a global adjustment in flow structure.
by analogy with H.T. gases
\[ x \frac{\partial v}{\partial x} = -\frac{\partial P}{\partial x} \]
\[ v = v_{\text{mix}} \]

(iii) Turbulent Wakes
\[ Re \approx \frac{UR}{v} \gg 1 \]

\[ \frac{\partial}{\partial x} \left( \frac{U v^2}{x} \right) \rightarrow \frac{v_x}{x} \]

\[ \frac{\partial}{\partial x} \left( \frac{U v^2}{x} \right) \rightarrow \frac{v_x}{x} \]

\[ \frac{\partial}{\partial x} \left( \frac{U v^2}{x} \right) \rightarrow \frac{v_x}{x} \]

[wave speeds by advection, not diffusion]

\[ \bar{v}_y \sim \text{turbulent velocity} \]

Take wake turbulence isotropic:

\[ \bar{v}_x \sim \bar{v}_y \]

\[ W \sim x \bar{v}_x / U \]

but from drag:

\[ -\bar{v}_x = Fa / \rho w^2 \]

\[ \Rightarrow \]
\[ W \sim X \frac{F_i}{\rho u^2} \sim X \left( \frac{F_i}{\rho u^2} \right) \]

\[ W^3 \sim F_i \frac{x}{\rho u^2} \]

\[ W \sim \left( \frac{F_i}{\rho u^2} \right)^{1/3} X^{1/3} \]

\[ \sim (cR^2)^{1/3} X^{1/3} \]

then, comparing widths:

**laminar:** \[ \frac{w}{R} \sim \left( \frac{x}{R} \right)^{1/2} Re^{-3/2} \]

\[ Re \sim \frac{UR}{\nu} \]

**turbulent:** \[ \frac{w}{R} \sim \left( \frac{x}{R} \right)^{1/3} C_0^{1/3} \]

Interestingly, laminar wake expands with downstream length more rapidly!
Why?

- Turbulence can relax TV behind object (due separation) rapidly and faster than \( V \). Thus, surrounding flow penetrates the closed water region more rapidly, less wake expansion.

Also observe: Wake Re drops with

\[
Re \sim \frac{u_f V_y}{\nu} ~ \frac{u_f V_x}{\nu} ~ \frac{F_x}{\nu^2 u_f R}\]

**y direction wake flow Re**

(spr)

\[
Re \sim \frac{F_y}{\rho u_w v} ~ \frac{u_f}{\nu} ~ \frac{C_n}{(C_m R)^{1/3}}
\]

\[
C_n \sim 1 ~ \frac{(u_f)}{\nu} \frac{(R/x)^{1/3}}
\]
\[ \text{Re}(x) \sim \text{Re}_c (R/x_x)^{1/3} \]

and \( \text{Re}(x) \rightarrow \infty \) at

\[ x \sim R (\text{Re}_c)^{-3} \]

distance behind hoot where turbulent wake transitions to
laminar.

i.e. akin to: transition from turbulent mixing to viscous mixing.

N.B. [In wake, vertical / rotational region can expand into rotational region but never reverse!]

[We would really violate H-Thm...]
Later discussion

Walker - supplement

Revisit turbulent wake using turbulent viscosity, i.e.

\[ W \sim \left( \frac{\nu x}{u^3} \right)^{1/2} \quad (\nu \to 0) \]

\[ \rightarrow \left( \frac{D_T x}{u^3} \right)^{1/2} \]

\[ \nu_T \sim W \tilde{\nu} \] i.e. is width of turbulent wake set by turbulent diffusion, following Blasius law.

\[ \sim W \left( \frac{E_{fr}}{\rho u \tilde{w}^2} \right) \]

\[ \tilde{\nu} \sim \frac{E_{fr}}{\rho u \tilde{w}} \sim \text{const} / W \]

\[ \rightarrow \]

\[ W \sim \left( \frac{E_{fr} x}{\rho u^3 \tilde{w}} \right)^{1/2} \]

\[ W^{3/2} \sim \left( \frac{E_{fr}}{\rho u^2} \right)^{1/2} x^{1/2} \sim (\nu R^2)^{1/2} x^{1/2} \]

\[ W \sim (C_0)^{1/3} R^{7/3} x^{1/3} \sim C_0^2 R x^{1/2} \]
\[ \frac{w}{R} \sim C_0 \frac{1}{3} (x/R)^{1/3} \] 

\[ \text{gives } \sqrt{ } \]

Now, \( Q \sim \sigma w \)

\[ \sim \left( \frac{\sigma w^3}{w} \right) \]

\[ \sim \frac{\sigma w w^2}{\sigma w w} \]

\[ \sim \frac{Q}{w} \sim \frac{Q}{R} \left( \frac{x}{R} \right)^{1/3} \]

1. Point is that turbulent viscosity mixing drops down-stream relatively 
   to constant viscosity mixing.

- Follows from \( \sigma w \sim \frac{Q}{w} \)

- Explains why turbulent wake spreads more slowly than laminar wake.