Engineering Flows: Models and Mixing Length Theory

- Law at Wall and
  Pronstal Mixing Length Theory
- Another Look at Wakes
- Heat Transfer: Laminar and
  Turbulent
All three make heavy use of

- scale invariance

- analogy of turbulent mixing with mixing/diffusion in gases.

\[ \text{Mixing Length} \]
\[ \text{Theory} \]
\[ \text{V}_{\text{Th}} \rightarrow \sqrt{V} \text{ i.e. } V_{\alpha} \]
\[ V_{\text{mfp}} \rightarrow \times \text{ i.e. distance from well} \]

N.B. - Useful in concert with phenomenology

- no real physical resemblance of diffusive mixing to turbulent mixing (recall Richardson).
Turbulent Pipe Flow

Till now → homogeneous flow in a periodic box → cascade in scale space (Kolmogorov) 1941

Now → inhomogeneous flow in a pipe
→ momentum transport in a turbulent boundary layer (Prandtl) 1931, 82

Consider turbulent pipe flow:
\[ Re \gg 1 \times 10^6 \]

- \( u \) on wall
- \( \text{core} \)
- logarithmic profile (inertial sublayer) (virtually all high Re experiments → "profile consistency")
- linear profile (viscous sublayer)

Common features of pipe flow:
- linear → logarithmic \( U(x) \) profile near boundary
- logarithmic profile persists over a broad range of Re
  \[ Re = \frac{2Ua}{v} \]
Logarithmic profile "universal" \( \left( \frac{\text{Prandtl}}{\text{Law of the Wall}} \right) \)
- resistance increases with increasing \( Re \)
- pressure drop/length \( \lambda = \frac{2a \Delta P \ell}{U^2} \) (\( \rightarrow \) \( \text{TKE} \))
- \( \frac{1}{2} \rho U^2 \) \( \rightarrow \) mean flow energy

Resistance curve
- laminar branch
- turbulent branch
- turbulent resistance curve universal

\text{What is going on? Physics of resistance?}
- no slip boundary condition
- \( U = U(x) \rightarrow d \)
  \( x \rightarrow 0 \)

\( \Rightarrow \) momentum flux
\( \Rightarrow \) to wall

\( 2 \nu + u \cdot \nabla u \rightarrow 0 \)
\( u^2 \nu = - \frac{\Delta P}{\ell} \)
→ momentum flux to wall → stress on the wall

Wall stress must balance pressure drop for steady flow

\[ \tau, \sigma, \text{ some } \dot{u} \]

Wall stress: \[ \frac{\partial u^2}{u_f} \]

\[ u_f = \text{ friction velocity} \]

\[ \rho U_f^2 2\pi a l = \Delta P \pi a^2 \]

\[ A = 2\pi a l \]

\[ \Delta l \rightarrow \]

\[ \Delta \Delta P \rightarrow \]

Pressure drop

Stationary \[ \Rightarrow \rho U_f^2 (2\pi a l) = (\Delta P) \pi a^2 \]

\[ U_f = \left( \frac{\Delta P}{2\rho} \left( \frac{a}{\ell} \right) \right)^{1/2} \]

Friction velocity
\[ U_\tau = \text{friction velocity} = \text{"typical" velocity of turbulence in turbulent pipe} \]

- think of as energy containing layer.

Deriving the inertial sublayer profile:

1. **Dimensional reasoning**

   in pipe flow inertial sublayer, have

   3 dimensional parameters: \( \rho, \nu, \frac{L}{L} \)

   density, stress, distance

   

   Key Assumption: Scale invariance

   on scale \( R_s = \frac{\nu}{U_\tau} < x < L \)

   *Universality of logarithmic profile motivated*

   *Scale invariance assumption*

   now, seek velocity gradient \( \frac{dU}{dx} \)

   \( \frac{dU}{dx} = U_\tau, x, \rho \)
The simplest form for $\frac{dU}{dx}$ is:

\[
\frac{dU}{dx} = \frac{U_0}{x}
\]

\[\Rightarrow U = \frac{U_0}{K} \ln\left(\frac{x}{x_0}\right)
= \frac{U_0}{K} \ln x + \text{const.}
\]

\[\Rightarrow \text{logarithmic profile (consequence of scale)}
\text{invariance in pipe flow)}
\]

\[\Rightarrow K = 4 \quad \text{universal constant} \Rightarrow \text{Von-Karman constant}
\]

\[x_0 \Rightarrow \text{width of viscous sublayer} \sim \sqrt{U_0}
\]

ii) Physical Reasoning

stationary flow \Rightarrow

momentum flux to wall = pressure drop
Mixing Length Theory initiated by Boussinesq.

\[ \langle \tilde{v}_x \tilde{v}_z \rangle = \frac{\partial}{\partial x} \]

Reynolds stress

\[ \langle \tilde{v}_x \tilde{v}_z \rangle = \rho = \frac{\partial}{\partial x} \ 	ext{momentum flux} \]

Now to calculate \( \langle \tilde{v}_x \tilde{v}_z \rangle \):

- take velocity fluctuation as generated by mixing of \( U(x) \), so

\[ \frac{\partial U}{\partial x} \]

"mixing length"

\[ \frac{\partial U}{\partial x} \]

\( V_2 \) results from mixing of mean profile \( U \)

analogous to Chapman-Enskog expansion, i.e.
here, scale invariance $\sim x$

mixing length set by distance from wall

$<v_x v_z> \approx \frac{\partial u}{\partial x}$

$\approx u_x \frac{\partial u}{\partial x}$

$u_x \rightarrow$ drag coefficient

$Y_t = u_x x$ $\rightarrow$ "eddy viscosity"

"turbulent viscosity" $\rightarrow$ key concept

$\Rightarrow$ rate of turbulent diffusion of momentum

then momentum balance $\Rightarrow$

$u_x \frac{\partial u}{\partial x} = u_x^2$

$\Rightarrow$

$u = \frac{u_x}{H} \ln \left( \frac{x}{x_0} \right)$ $\rightarrow$ logarithmic profile

$\Rightarrow$ law of the wall
FAQ

Some comments:

- as in k-w, clear phenomenology critical to guiding the approximations & scale invariance

"Mixing length theory always works ... provided you know the mixing length..."

- why a single value of velocity, i.e. U*

Consistent with mixing length hypothesis, velocity fluctuations generated by mixing of mean flow gradient, i.e.

\[ \nabla \cdot \frac{\partial U}{\partial x} \sim \frac{\partial \frac{\partial U}{\partial x}}{\partial x} \]

\[ \sim \frac{\partial U}{\partial x} \frac{\partial u}{\partial x} \]

- consistent with logarithmic profile
- scale invariance
What happens at wall?

- Viscous sublayer / cut-off of inertial layer?
  - when \( \nu^* < r \)
  - \( u_\tau x \leq r \)
  - \( x \leq r / u_\tau \equiv x_0 \)
  - Viscous sublayer scale.

In viscous sublayer flow linear:

\[
\frac{\partial u}{\partial x} = u_\tau^2
\]

\[
\therefore \quad u = \frac{u_\tau^2 x}{r}
\]

- Note effect of turbulence is to:
  - Flatten profile
  - Higher transport at fixed wall stress
  - Reduce central velocity
  - Limit \( Q \) (quality factor)
Matching, for constant:

\[ \chi_0 = \frac{y}{u_f} \quad \text{so} \]

\[ u = \frac{u_f}{k} \ln \left( \frac{u_f}{k} x \right) \]

Note: Flow on upper and sub-layer is turbulent, but mixing layer affected by dissipation range scale \( \Rightarrow \) linear profile

Now - turbulent dissipation \( \overline{} \)

Consider \( \nabla \overline{\text{S}} \overline{\text{E}} \) :

\[ \frac{\partial \overline{\text{E}}}{\partial t} + \overline{\text{v}} \cdot \nabla \overline{\text{E}} + \langle \overline{\nabla} \cdot \overline{\text{v}} \rangle \overline{\text{E}} + \overline{\text{v}} \cdot \frac{\partial}{\partial x} \overline{\text{E}} \]

\[ = - \overline{\nabla} \rho + \overline{\text{v}} \cdot \nabla \overline{\text{E}} \]

\( \overline{\text{v}} \) and \( \overline{\rho} \) \( \Rightarrow \)

\[ \frac{\partial}{\partial t} \langle \overline{\text{E}} \rangle + \langle \overline{\text{v}} \cdot \overline{\nabla} \overline{\text{E}} \rangle + \langle \overline{\text{v}} \rangle \langle \overline{\nabla} \cdot \overline{\text{E}} \rangle + \langle \overline{\text{v}} \rangle \langle \overline{\text{v}} \rangle \langle \overline{\text{v}} \rangle \langle \overline{\text{v}} \rangle \]

\[ \frac{\partial}{\partial x} \begin{pmatrix} \overline{\text{v}}_x \overline{\text{v}}_z \overline{\text{v}}_y \overline{\text{v}}_z \end{pmatrix} \overline{\text{E}} + \langle \overline{\text{v}}_x \rangle \langle \overline{\text{v}}_z \rangle \overline{\text{E}} - \frac{\partial}{\partial x} \left( \sqrt{\overline{\rho}} \right) - \overline{\text{v}} \langle \overline{\text{v}} \rangle \overline{\text{E}} \]
For net energy budget:
\[
\frac{\partial \overline{E}}{\partial t} + \frac{\partial}{\partial x} \left( \overline{u_x \overline{v_z}} \right) = -\overline{\frac{\partial u_x}{\partial x} \overline{v_z}} - \nu \left\langle (\overline{v_z})^2 \right\rangle
\]

\(\overline{u_x \overline{v_z}}\) : input to fluctuations by relaxation of mean shear flow (Reynolds work)
\(\overline{\frac{\partial u_x}{\partial x} \overline{v_z}}\) : reduction energy
\(\nu \left\langle (\overline{v_z})^2 \right\rangle\) : [dissipation of fluctuation, energy by viscosity]

\(\overline{u_x \overline{v_z}}\) can define:
\[
\overline{E} = \overline{u_x \overline{v_z}} \frac{\partial \overline{u}}{\partial x}
\]

\(\frac{\partial \overline{u_x \overline{v_z}}}{\partial x}\) : turbulent dissipation rate

and using mixing length theory:
\[
\overline{u_x \overline{v_z}} = u_{+} \times \frac{\partial \overline{u}}{\partial x}
\]
\[
\Delta = \left( u_{+} \right) \left( \frac{\partial \overline{u}}{\partial x} \right)^2 = \gamma \left( \frac{\partial \overline{u}}{\partial x} \right)^2
\]

\(\Delta\) : rate of "heating" by turbulent relaxation
Obviously: \[ \nabla \cdot \vec{v} = \nabla \cdot (2\mathbf{u}) \]

small scale dissipation

and

\[ \varepsilon = \left( \frac{\partial x}{\partial z} \right) \left( \frac{\partial x}{\partial z} \right) \]

\[ = \frac{U_0}{x} \]

\[ \Rightarrow \text{dissipation rate as function} \]

i.e.

\[ \varepsilon = \frac{V_0}{l} \]

\[ V_0 \rightarrow U_0 \]

\[ l \rightarrow x \]

\[ \Rightarrow \varepsilon \text{ finite as } x \rightarrow \infty \] (i.e. viscous sublayer gradient diverges then)

Additional References:
- S. B. Pope, "Turbulent Flow"
- H. Tennekes and J. Lumley, "A First Course in Turbulence"

and of course,
Landau & Lifshitz.
Now interesting to tabulate comparison between Pipe Flow and $k$-$\varepsilon$ Problem.

**Pipe Flow (Prandtl)**

- Scales: $a$, $x$, $v/u_\tau$
- Invariance: $X \to$ real space

**K41 (Kolmogorov)**

- Scales: $\ell_0$, $\ell_1$, $\ell_2$
- Invariance: $l \to$ scale space

### Inertial Sublayer

- Length: $u_\tau^2 = v_\tau \frac{\partial y}{\partial x}$
- Energy: $\varepsilon = \frac{u(\varepsilon)^2}{\ell_\ell}$

### Viscous Sublayer

- Energy dissipation rate:
  - $\varepsilon' = u(\varepsilon) \frac{\partial v}{\partial x}$
  - $1/v(\varepsilon) = \frac{\ell_\ell}{\ell_1}$

### Universal Profile

- Velocity profile:
  - $u(x) = u_\tau \frac{\ell_1 - x}{\ell_1}$
  - $\beta = \frac{u(\varepsilon)}{\ell_1}$

### Universal Spectral Scaling

- Dissipation:
  - $\varepsilon = v_\tau$
  - $x_\varepsilon = \frac{\varepsilon}{u_\tau}$
  - $l_\ell = \varepsilon^{1/4}$
\[
\begin{align*}
\text{Resistance} & \quad \text{Loss of Flow} \\
\text{Practical Issues} &
\end{align*}
\]
taking \( Re = 2\nu U / \nu \)

Can rewrite friction law as:

\[
\frac{1}{\sqrt{\lambda}} = 0.88 \ln \left( Re \sqrt{\lambda} \right) - 8.5
\]

\[
Re = \frac{2\nu U}{\nu}
\]

\[
\lambda = \frac{2\pi \Delta P / L}{\frac{1}{2} \rho U^2}
\]

- good fit to pipe flow data
Additional Cases

Turbulent Wakes Thermal Boundary Layers

Here:

- turbulent wakes, complete wake story
  - behind
  - scaling
  - eddy mixing

- Thermal BL / Heat Transfer
  - shock, set up, types
    -
  - heat transfer problem
    - heat transfer coeff
    - Nu
    - laminar/turbulent
  - cite to temp fluctuation turbulence -
    pressure scalar
References: Boundary layers, wakes, heat transfer.

Landau & Lifshitz: excellent "physicist style" treatment of these "engineering" subjects.

V. Kracou: Good summary, many examples.

H. Tennekes, J. Lumley: Basic discussion, good first course.

R1. \textit{Wake} - \textit{Simple Physics} \\

\textbf{Wake is:}

- region of departure from potential flow behind object
  moving thru water and experiencing drag

\begin{itemize}
  \item body
  \item \textbf{wake}
\end{itemize}

- work is inexorably coupled to drag

- message of wake:
  \begin{itemize}
    \item\{A little more force a global adjustment of flow structure\}
  \end{itemize}

- drag - \begin{itemize}
  \item thinking on frame where object at rest, drag results from loss of flow momentum to object.
\end{itemize}
Which spreads faster downstream, laminar or turbulent wall?

\( \text{(ii) Turbulent Waves} \quad \frac{\text{Re} \sim UR/v}{\text{Re} \gg 1} \)

\[
\begin{align*}
\frac{\partial \rho u v}{\partial y} + \frac{\partial (\rho u^2)}{\partial x} - \frac{-\partial P}{\partial x} &= 0 \\
\downarrow \\
\frac{\partial \rho u^2}{\partial x} &= \frac{\rho \frac{\partial u}{\partial x}}{U_x} \\
\frac{\partial \rho u^2}{\partial x} &= \frac{\rho \frac{\partial u}{\partial x}}{U_x} \\
\end{align*}
\]

\[W \sim \frac{u_x x}{U_x} \]

\[\bar{u_y} \sim \text{turbulent velocity} \]

Take wall turbulence isotropic

\[ \bar{u_x} \sim \bar{u_y} \]

\[W \sim \frac{\bar{u}_x X}{U_x} \]

but from drag:

\[ \bar{u}_x \sim \frac{F_x}{\rho u w^2} \]
\[ W \sim X \frac{F_d}{p u^2} \sim X \left( \frac{F_d}{p u^2} \right) \]

\[ W^3 \sim \frac{F_d}{p u^2} X \]

\[ W \sim \left( \frac{F_d}{p u^2} \right)^{1/3} \times X \]

\[ \sim \left( C_0 R^2 \right)^{1/3} \times X^{1/3} \]

Then, comparing widths:

Laminar: \[ \frac{w}{R} \sim \left( \frac{X}{R} \right)^{1/2} \frac{Re}{Re \sim UR/R} \]

Turbulent: \[ \frac{w}{R} \sim \left( \frac{X}{R} \right)^{1/3} C_0^{1/3} \]

Interestingly, laminar wake expands with downstream tenth more rapidly.
Why?

- turbulence can relax TV behind object (due separation) rapidly and faster than V. Thus surrounding flow penetrates the dead water region more rapidly.

Also observe: Wake Re drops with

\[ Re \approx \frac{w_\infty V_x}{\nu} = \frac{w_\infty}{\nu} \frac{V_x}{\nu} \]

y direction: Wake flow Re

\[ \text{Re} = \frac{F_y}{\rho w V} \]

- \[ C_{n,1} \approx \left( \frac{u \nu}{\overline{V}} \right) \left( \frac{R}{x} \right)^{1/3} \]
\[ \text{Re}(x) \sim \text{Re}_c \left( \frac{R}{x} \right)^{\frac{1}{3}} \]

and \( \text{Re}(x) \to 0 \) as \( x \to R \left( \text{Re}_c \right)^{\frac{3}{2}} \)

Distance behind root where turbulent wake transitions to laminar.

i.e., thin lid: transition from turbulent mixing to viscous mixing

N.B. In wobbling vertical/rotational region can expand into \( H \)-rotational region, but never reverse!

i.e., would really violate \( H \)-Thm...
Wakes - Supplement

Revisit turbulent wake using turbulent viscosity, i.e.

\[ W \sim \left( \frac{nu}{u} \right)^{1/2} \quad (n \to 0) \]

\[ \sim \left( \frac{\Delta x}{nu} \right)^{1/2} \]

i.e. with width of turbulent wake set by turbulent diffusion following Blasius law.

\[ \text{but} \quad \Omega_1 \sim \bar{W} \quad \text{as turbulent viscosity at mixing length level} \]

\[ \sim \bar{W} \left( \frac{E_f}{\rho u w^2} \right) \]

\[ \sim \frac{E_f}{\rho u w} \sim \text{const}/w \]

\[ \uparrow \]

\[ W \sim \left( \frac{E_f}{\rho u^2 w} \right)^{1/2} \]

\[ w^{3/2} \sim \left( \frac{E_f}{\rho u^2} \right)^{1/2} \times 1^{1/2} \sim (E_R^2)^{1/2} \times 1^{1/2} \]

\[ W \sim \left( C_0 \right)^{1/3} R^{2/3} X^{1/3} \sim C_0 R X^{1/2} \]
\[ \frac{w}{R} \sim c_0^{1/3} (x/R)^{1/3} \]  \\

Now, \( D_T \sim c w \)

\[ \sim \frac{\langle \nabla w \rangle}{w} \]

\[ \sim \frac{c w w^2}{\partial w} \sim \frac{c}{w} \sim \frac{c}{\sqrt{R}} (x/R)^{1/3} \]

The point is that turbulent viscosity mixing drops downstream relative to constant viscous mixing.

- Follows from \( \nabla w \sim \frac{c}{w} \)

- Explains why turbulent wake spreads more slowly than laminar wake.
Thermal Boundary Layer + Heat Transfer

Consider stationary flow + heat conduction

\[ \nabla^2 T + \nabla \cdot D \sigma = \nabla^2 T \]

\[ \alpha = \frac{\kappa}{\rho c_p} \]

\[ \rho \nabla \cdot \mathbf{u} = -\frac{DP}{\rho} + \sum r \nabla v + \sum \sigma \]

So: dimensionless #

\[ \to \text{Re, as usual} \]

\[ \to \text{Pr} = \frac{\nu}{\kappa} \]

n.b. \to if buoyant

\[ \text{Ra} = \frac{g \alpha (\rho_f - \rho)}{\kappa c_p L^3} \]

Reynolds #.

Now generic problems.
$T - T_0 \quad = \quad F \left( \frac{U}{U_0}, \frac{U_0}{v}, \text{Re} \right)$

$\frac{V}{U} \quad = \quad F \left( \frac{V}{U}, \text{Re} \right)$

$\text{scaling of result}$
Further ways of keeping score:
- if concerned with cooling body to surface heat flux of body:

\[ h = \alpha = q / (T - T_a) \]

- Flow \( T \) to body \( T \)

More:
- Heat transfer coefficient
- Effectiveness as \( \varphi = -h \Delta T / \dot{Q} \) \( \Rightarrow \) h is strongly tied to boundary layer dynamics

- Dimensionless ratio:

\[ N = \frac{h \varphi}{\Delta T} \]

\( \text{Nu} \text{seq} \# \)

= \[ N = f(Re, Pr) \text{ for Blot heat transfer} \]
N.B.: Note trade-offs in cooling problem, e.g., resistance of pipe heat transfer.

\[ u \rightarrow T_0 \rightarrow T \rightarrow T_1 \]

How does \( Nu \) scale in laminar BL?

\[ z = -\frac{K \Delta T}{j u} \]

\[ \sim \frac{K (T_1 - T_2)}{\delta} \rightarrow \text{boundary layer thickness} \]

\[ \delta \sim \frac{1}{(Re)^{1/2}} \]

How effective is laminar flow in cooling?

\[ \text{Surface heat flux} \]

but we know for laminar BL

\[ C = \frac{1}{(Re)^{1/2}} \]

for Pr = 1.
\[ \frac{Nu}{K} \sim \left( \frac{2}{T_i - T_o} \right)^{1/2} \]

\[ \sim \frac{H}{K} \left( T_i - T_o \right) \frac{P}{\sigma} \]

\[ \sim \sqrt{Re} \]

\[ \frac{Nu}{\sqrt{Re}} \sim f(Re) \rightarrow \text{Nusselt number} \]

\[ h \sim \frac{H}{\sqrt{Re}} \]

\[ \rightarrow \text{heat transfer coeff.} \]

\[ \sim \text{(note size scaling)} \]

\[ \sim \text{note Cp importance!} \]

2. Turbulent B.L.

\[ T \]

\[ \text{Sufficient to calculate temp field in flow.} \]
\[ \frac{g}{\rho} = -z \frac{dT}{dy} \]

\[ \text{therm., eddy v.s.} \]

\[ k_T = \rho C_p U_T \frac{V_T}{y} \]

\[ \delta \]

\[ \frac{dT}{dy} = \frac{1}{\rho C_p U_T} \frac{1}{y} \]

\[ T = \frac{2}{\rho C_p U_T} \ln \left( \frac{y}{y_o} \right) + f(P) \]

additional, driddle const. may enter.

\[ N = \frac{K_T}{K} \frac{V_T}{y} \]

\[ y_o = \frac{V}{U_T} \frac{1}{\rho} \]

\[ \rho \approx 1 \]
A turbulent flow is characterized by temperature fluctuations.

Production: \[ \frac{\partial}{\partial t} \frac{\partial T}{\partial x} = \frac{\partial}{\partial t} \left( \frac{T^2}{\nu} \right) \]

\[ \sim \left( \frac{1}{\nu} \right)^{1/3} \frac{V}{L} \]

\[ \frac{\partial}{\partial t} \left( \frac{u^2}{\nu} \right) \approx \left( \frac{1}{\nu} \right)^{1/3} \left( \frac{v}{L} \right)^{1/3} \]

in scaling for turbulent fluctuations.

But? \[ Pr \ll 1 \rightarrow \text{how reconcile discussions?} \]

one field may see other smooth? \[ TBC \]